Chapter 9 Overview: Parametric and Polar Coordinates

As we saw briefly last year, there are axis systems other than the Cartesian System for graphing (vector coordinates, polar coordinates, rectangular coordinates--for Complex Numbers—and others). The Calculus should apply to them as well.

Previously, we saw the relationship between parametric motion and vectors. We will explore this further and investigate how derivatives relate the concepts of tangent line slopes, increasing/decreasing, extremes, and concavity to vector-function (parametric) graphs.

Polar coordinates are, at their base, drastically different from Cartesian or Vector coordinates. We will review the basics we learned last year about the conversions between polar and Cartesian coordinates and look at graphing in polar mode, with an emphasis on the calculator. As with vector-functions, we will investigate how derivatives relate the concepts of tangent line slopes, increasing/decreasing, extremes, and concavity. Further, we will see how integrals relate to area in polar mode.

Finally, we will revisit motion in parametric mode and see how motion could be described in polar mode.

There are two contexts within which we will consider parametric and polar functions.

I. Motion
II. Graphing

9.1: Parametric Motion

Vocabulary:
1. ***Parameter*** – dummy variable that determines *x*- and *y*-coordinates independent of one another
2. ***Parametric Motion*** – movement that occurs in a plane
3. ***Vector*** – a directed line segment. As such, it can be used to represent anything that has magnitude and direction (e.g., force, velocity, etc.)
4. ***Magnitude*** – the length/size of a vector
5. ***Direction of a Vector*** – the standard position angle when a vector is placed with its tail at the origin

In Chapter 3, we considered motion in a parametric context. At the time, we only had the derivative as a tool for analysis, but now we have the integral also.

Recall from earlier:

Remember:

Position = $\langle x(t), y(t)\rangle$

Velocity = $\langle x'(t), y'(t)\rangle$

Acceleration = $\langle x''(t), y''(t)\rangle$

Speed = $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

Therefore:

Position = $\left\langle \int x'(t)dt, \int y'(t)dt \right\rangle$

Velocity = $\left\langle \int x''(t)dt, \int y''(t)dt \right\rangle$

Ex 1 Find the velocity vector for the particle whose position is described by $\langle t^2 - 2t, t^2 + 1\rangle$. At $t = 3$, find the speed of the particle.

First, notice that this problem has a slightly different notation than we are used to. You may remember that *chevrons* (the angled brackets) are one of the ways that we write vectors. Since this is a position vector, we really have the following:

$$x = t^2 - 2t \text{ and } y = t^2 + 1$$

Taking the derivatives, we get

$$\frac{dx}{dt} = 2t - 2 \text{ and } \frac{dy}{dt} = 2t$$

Since the derivatives of x and y are actually velocities, we can simply write the velocity vector using chevrons.

$$\langle 2t - 2,\ 2t \rangle$$

Of course, we could still write vectors using the $x\vec{i} + y\vec{j}$ form, but this can become cumbersome when x and y are equations in terms of t.

In general for this particle, $\text{speed} = \sqrt{(2t-2)^2 + (2t)^2}$ and

at $t = 3$, $\text{speed} = \sqrt{(2(3)-2)^2 + (2(3))^2} = \sqrt{52}$

Ex 2 At $t = 3$, find the speed of the particle whose velocity vector is $\langle 2t - 2,\ 2t \rangle$.

You may recall that speed is the magnitude of velocity, and we found magnitude of vectors last year using the Pythagorean Theorem. We can simply apply those two pieces of information to conclude the following:

$$speed = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$\text{speed} = \sqrt{(2t-2)^2 + (2t)^2}$ for this particular velocity vector, and

at $t = 3$, $\text{speed} = \sqrt{(2(3)-2)^2 + (2(3))^2} = \sqrt{52}$

When dealing with derivatives of parametric equations, it is important to realize we have gotten a little sloppy by talking about "THE derivative"--as if there were only one. It was not that important before because, in Cartesian mode, *x* was the independent variable and *y* was the dependent variable. We did not deal with derivatives in terms of different variables within the same problem. That comes back to haunt us now.

There is still the issue of displacement vs. distance traveled. We recall:

Key Idea #1:

Displacement = $\int_a^b v(t)dt = \begin{cases} \int_a^b x'(t)dt \\ \int_a^b y'(t)dt \end{cases}$ **Distance traveled** = $\int_a^b |v(t)|dt$

Note that $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$, therefore: distance traveled

$$= \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2}\ dt$$

OBJECTIVES

Find the position of an object in motion in two dimensions from its velocity.
Find the arc length of a curve expressed in parametric mode.

Key Idea #2: Position can be found two ways

a) Indefinite integration of the velocity equation, with an initial value, will yield the position equation. Position can be determined by substituting the time value.

b) Definite integration (by calculator) will yield displacement. Adding displacement to the initial value will yields the position

Ex 2 Given that an object in motion has $v(t)=\left\langle t^2+2,\, t^3-1\right\rangle$ and the initial position $\langle -1, 3\rangle$ (that is, $t=0$), find the position at $t=2$.

$$v(t)=\left\langle t^2+2,\, t^3-1\right\rangle \rightarrow \begin{cases} x'(t)=t^2+2 \\ y'(t)=t^3-1 \end{cases}$$

Considering these separately,

$$x'(t)=t^2+2 \rightarrow x(t)=\int\left(t^2+2\right)\, dt=\frac{1}{3}t^3+2t+c_1$$
$$x(0)=-1 \rightarrow -1=\frac{1}{3}(0)^3+2(0)+c_1 \rightarrow c_1=-1$$

and

$$y'(t)=t^3-1 \rightarrow y(t)=\int\left(t^3-1\right)\, dt=\frac{1}{4}t^4-t+c_2$$
$$y(0)=3 \rightarrow 3=\frac{1}{4}0^4-0+c_2 \rightarrow c_2=3$$

So, $x(t)=\frac{1}{3}t^3+2t-1$ and $y(t)=\frac{1}{4}t^4-t+3$ and

$$\left\langle x(2),\, y(2)\right\rangle=\left\langle \frac{17}{3},\, 5\right\rangle$$

It should be mentioned that the fact that c_1 and c_2 equaled the initial values is due to the equations being polynomials and the initial $t = 0$. The constants would not equal the initial values if this had not been true.

Ex 3 Given that an object in motion has $v(t)=\left\langle \sin^3 t, \cos^2 t\right\rangle$ and the initial position $\langle -1, 3\rangle$, find the position at $t=2$.

It this case, we cannot take the anti-derivatives of the velocity, because these functions cannot be integrated by any techniques we have. We can find the definite integrals by out calculator, though, and these would be displacements from the initial point.

$$x(t) = -1 + \int_0^2 \sin^3 t \; dt = .059$$
$$y(t) = 3 + \int_0^2 \cos^2 t \; dt = 3.811$$

Ex 4 BC 2015 #2

Again, velocity is a vector, so displacement is a vector also, as we saw in the last two examples. But $|\mathrm{v}(\mathrm{t})|$ is the speed and $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$. The distance traveled—which, in most cases, is the Arc Length of the curve--is:

Arc Length/Distance Formulae:

Function: $$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dt$$

Parametric: $$L = \int_\alpha^\beta \sqrt{(x'(t))^2 + (y'(t))^2} \; dt$$

Note that the distance traveled is not always equal to the Arc Length. The particle can trace over the part or all of the path traveled.

Ex 5 Find the distance traveled by the object in Ex 2.

$$\text{Distance} = \int_0^2 \sqrt{(\sin^3 t)^2 + (\cos^2 t)^2} \; dt$$
$$= 1.651$$

Ex 6 Find the arc length of $x = \cos t$ and $y = \sin t$ for $t \in [0, 4\pi]$ and compare it to the distance traveled.

This problem highlights the difference between Arc Length and Distance.

$$\begin{aligned} \text{D} &= \int_0^{4\pi} \sqrt{(-\sin t)^2 + (\cos(t))^2}\, dt \\ &= \int_0^{4\pi} 1\, dt \\ &= t\Big]_0^{4\pi} \\ &= 4\pi \end{aligned}$$

But we saw that that this curve is the unit circle, with the motion traced over the circle twice. The distance traveled is 4π, but the arc length is actually half this distance, or 2π. This should make sense, since the circumference of the Unit Circle is 2π.

9.1: Free Response Homework

Find (a) the velocity vector, (b) the speed, and (c) acceleration vector for each of the following parametric equations.

1. $x(t)=\sqrt{t},\ y(t)=\cos\ t$

2. $x(t)=5\sin t,\ y(t)=t^2$

3. $x(t)=t^2-5t,\ y(t)=-4t^2+1$

4. $x=\sec\ \theta,\ y=\tan\ \theta$

Find the position at $t=5$ of an object moving according to the given equation with the given initial position.

5. $x'(t)=t-t^2,\ y'(t)=\frac{5}{3}t^{2/3},\ x(1)=-2,\ y(1)=3$

6. $x'(t)=3-t^2,\ y'(t)=t^{1/2}-t^{-1/2},\ x(0)=-2,\ y(0)=3$

7. $x'(t)=e^{-t},\ y'(t)=3e^{3t},\ x(-2)=0,\ y(-2)=-1$

8. $x'(t)=t\cos t^2,\ y'(t)=3\sin t,\ x(1)=-2,\ y(1)=3$

9. $x'(t)=t\ln t,\ y'(t)=\tan^3 t,\ x(0)=-3,\ y(0)=1$

10. $x'(t)=\cos e^t,\ y'(t)=\cot^2(t+1),\ x(3)=0,\ y(3)=0$

Find the distance traveled by each object below. Determine if the distance is equal to the arc length.

11. $x(t)=t-t^2,\ y(t)=\frac{4}{3}t^{3/2},\ t\in[1,2]$

12. $x(t)=t\sin t,\ y(t)=t\cos t,\ t\in[-3,3]$

13. $x(t)=3t-t^3,\ y(t)=3t^2,\ t\in[0,\pi]$

14. $x(t)=e^t\sin t,\ y(t)=e^t\cos t,\ t\in[0,\pi]$

15. $x(t)=\cos^2 t,\ y(t)=\cos t,\ t\in[0, 4\pi]$

16. $x(t)=\sqrt{t},\ y(t)=\cos\ t,\ t\in[0, 6]$

AP Handout

17. BC 2005B #1

18. BC 2002 #3

19. BC 2008B #1

20. BC 2009 #3

9.1: Multiple Choice Homework

1. If f is a vector valued function defined by $f(t)=(\sin 2t, e^{-t})$, then $f''(t)=$

a) $2\cos 2t - e^{-t}$

b) $-4\sin 2t + e^{-t}$

c) $(2\cos 2t, -e^{-t})$

d) $(\sin 2t, e^{-t})$

e) $(-4\sin 2t,\ e^{-t})$

---

2. A particle is moving such that at any time $t > 0$ its x-coordinate is $4t+t^2$ and its y-coordinate is $\frac{1}{3t+1}$. The acceleration vector of the particle at time $t = 1$ is

a) $\left\langle 2, \frac{1}{32}\right\rangle$

b) $\left\langle 2, \frac{9}{32}\right\rangle$

c) $\left\langle 5, \frac{1}{4}\right\rangle$

d) $\left\langle 6, -\frac{3}{16}\right\rangle$

e) $\left\langle 6, -\frac{1}{16}\right\rangle$

---

3. A particle position is given by $\left\{ \begin{array}{c} x(t) = e^{t} \\ y(t) = e^{-t} \end{array} \right\}$. The speed at t = 1 is

a) 2.693 b) 2.743 c) 3.086 d) 3.844 e) 7.542

---

4. At time $t \geq 0$, a particle moving in the *xy*-plane has velocity vector given by $v(t) = \langle t^2, 5t \rangle$. What is the acceleration vector of the particle at time $t = 3$?

a) $\left\langle 9, \frac{45}{2} \right\rangle$ b) $\langle 6, 5 \rangle$ c) $\langle 2, 0 \rangle$

d) $\sqrt{306}$ e) $\sqrt{61}$

---

5. A particle moves on a plane curve so that at any time $t > 0$ its velocity vector is given by the equations $v_x = t^2 - t$ and $v_y = 2t - 1$. The acceleration vector of the particle at $t = 1$ is

a) $(0, 1)$ b) $(0, -1)$ c) $(2, 2)$

d) $(-1, 1)$ e) $(1, 2)$

---

6. A particle moves on a plane so that its position vector is $p(t) = \left\langle \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 7, \frac{2}{3}t^3 + \frac{3}{2}t^2 - 2t + \pi^6 \right\rangle$ is at rest when

a) $t = 1$ only b) $t = \frac{1}{2}$ only c) $t = -2$ only

d) $t = 1, \frac{1}{2}$ e) $t = 1, \frac{1}{2}, -2$

---

7. If is a vector-valued function defined by $f(t)=\left(e^{-t},\cos t\right)$, then $f''(t)=$

a) $-e^{-t}+\sin t$ b) $-e^{-t}-\cos t$ c) $\left(-e^{-t},-\sin t\right)$

d) $\left(e^{-t},\cos t\right)$ e) $\left(e^{-t},-\cos t\right)$

---

8. A particle moving in the xy-plane with its x coordinate given by $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+\frac{1}{2}t^2-1$ and its y coordinate given by $y(t)=\frac{1}{2}t^2-t+1$. When the particle is moving up it is also

a) moving right
b) moving left
c) at rest
d) cannot be determined
e) does not exist

---

9. The position of a particle moving along a line is given by $s(t)=2t^3-24t^2+90t+7$ for $t\geq 0$. For what values of t is the speed of the particle increasing?

a) $3<t<4$ only b) $t>4$ only c) $t>5$ only

d) $0<t<3$ and $t>5$ e) $3<t<4$ and $t>5$

---

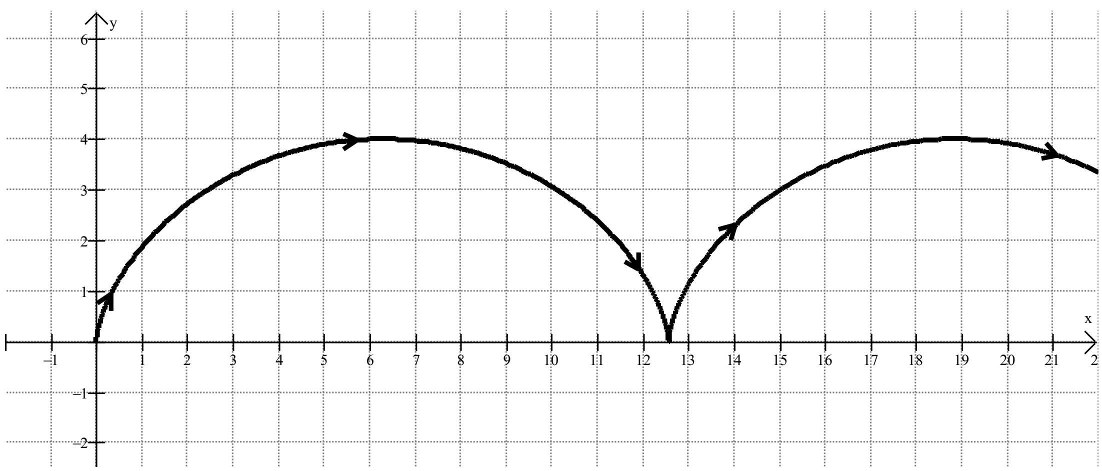
10. A particle is moving such that at any time $t>0$ its x-coordinate is $t-t^2$ and its y-coordinate is $(3t-2)^3$. The acceleration vector of the particle at time $t=1$ is

a) (0,1) b) (–1,9) c) (–1,54) d) (–2,9) e) (-2, 54)

9.2: Parametric Graphing

Parametric mode allows one to graph non-functions. The parameter can often be eliminated in order to graph a function, but some functions cannot be expressed in Cartesian coordinates. The most well-known example is the cycloid, which is created by rolling a circle along the x-axis and tracing the path traveled by a point on the circle.

Ex 1 Graph $\begin{cases} x(t) = 2(t - \sin t) \\ y(t) = 2(1 - \cos t) \end{cases}$ on your calculator.



Remember

1. Radian mode in Calculus (degrees in Physics).
2. We have to be concerned about the *t*-values (and the window) when graphing parametrics. Unless otherwise given, we need to include negative values of *t*.

In the second context for parametrics, we will be asked to find the slopes of tangent lines and toe determine concavity for a function described in parametric mode.

Key Ideas

1. $\frac{dy}{dx}$ is still the slope of the tangent line and $\frac{d^2y}{dx^2}$ still determines concavity.
2. $\frac{dy}{dt}$ is different from $\frac{dx}{dt}$ is different from $\frac{dy}{dx}$.

As we saw, given parametric equations $\langle x(t), y(t)\rangle$, it is easy to find $\langle x'(t), y'(t)\rangle$. But $\langle x'(t), y'(t)\rangle$ is $\left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$, which is the velocity vector. How do we get $\frac{dy}{dx}$ instead of $\frac{dy}{dt}$, so we can talk about slopes and increasing or decreasing intervals? This is where the advantages of Liebnitz notation become apparent. Since Liebnitz notations work the same as fractions,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ and } \frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$$

OBJECTIVES

Graph relations in parametric mode.
Find the slope of a tangent line to a curve in parametric mode.
Find the concavity of a curve in parametric mode.
Find the arc length of a curve in parametric mode.
Graph relations in parametric mode.
Eliminate the parameter to identify the function form of a parametric.

Ex 2 Find the equation of the line tangent to $x(t)=t^2,\ y(t)=t^3-t$ at $t=2$.

As we remember, finding the equation of any line requires having a point and a slope. $t=2$ is the parameter, not the point (*x, y*). The point comes from substituting $t=2$ into $x(t)=t^2,\ y(t)=t^3-t$:

$$x(2)=4$$
$$y(2)=2^3-2=6$$
$$(x,y)=(4,6)$$

The slope of the tangent line, of course, would come from substituting $t=2$ into $\frac{dy}{dx}$:

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{3t^2-1}{2t}$$
$$\left.\frac{dy}{dx}\right|_{t=2}=\frac{dy/dt}{dx/dt}=\frac{3(2)^2-1}{2(2)}=\frac{11}{4}$$

Therefore, the equation of the tangent line is

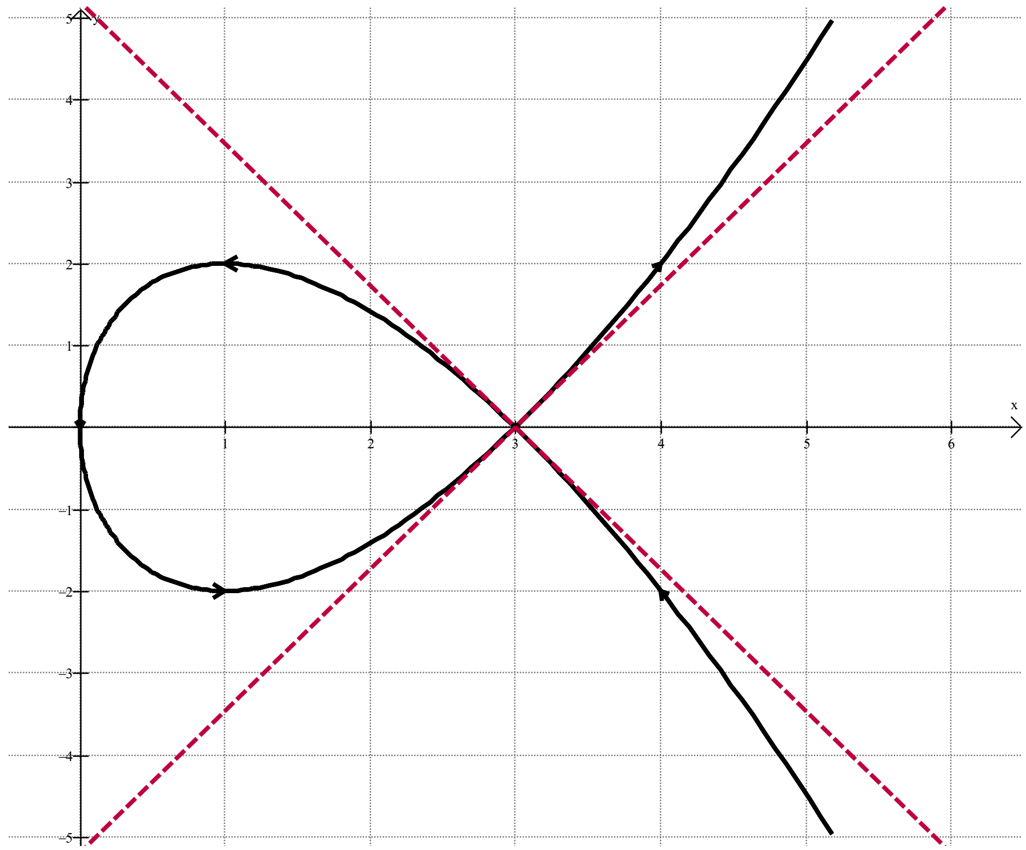
$$y-6=\frac{11}{4}(x-4)$$

Ex 3 Find the equation of the line tangent to $x(t)=t^2,\ y(t)=t^3-3t$ at $(3,0)$.

This time, we have the point, but not the parameter. We can find t by setting the point equal to the equations:

$$x(t)=t^2=3\Rightarrow t=\pm\sqrt{3}$$
$$y(t)=t^3-3t=0\Rightarrow t=0,\ \pm\sqrt{3}$$

$t = 0$ gives the correct y-value but not the right x-value, so $t = \pm\sqrt{3}$ are the correct parameters. Yes, there are two parameters, meaning there are two tangent lines involved. A quick look at the calculator shows why:



$$\frac{dy}{dx} = \frac{3t^2 - 3}{2t} \Rightarrow \begin{cases} t = \sqrt{3} \Rightarrow \dfrac{dy}{dx} = \dfrac{3(3) - 3}{2\sqrt{3}} = 1.732 \\ t = -\sqrt{3} \Rightarrow \dfrac{dy}{dx} = \dfrac{3(3) - 3}{-2\sqrt{3}} = -1.732 \end{cases}$$

The two tangent lines are

$$y = 1.732(x - 3) \text{ and } y = -1.732(x - 3)$$

Ex 4 Find the points on the curve defined by $x(t)=t^2$ and $y(t)=t^3-3t$ where the tangent line is either vertical or horizontal.

From the last example, we know $\frac{dy}{dx}=\frac{3t^2-3}{2t}$. The tangent line would be horizontal where $\frac{dy}{dx}=0$; that is, where $3t^2-3=0$, or $t=\pm 1$. This is the parameter, though, not the point.

$$t=1\Rightarrow(x,y)=(1,-2)$$

$$t=-1\Rightarrow(x,y)=(1,2)$$

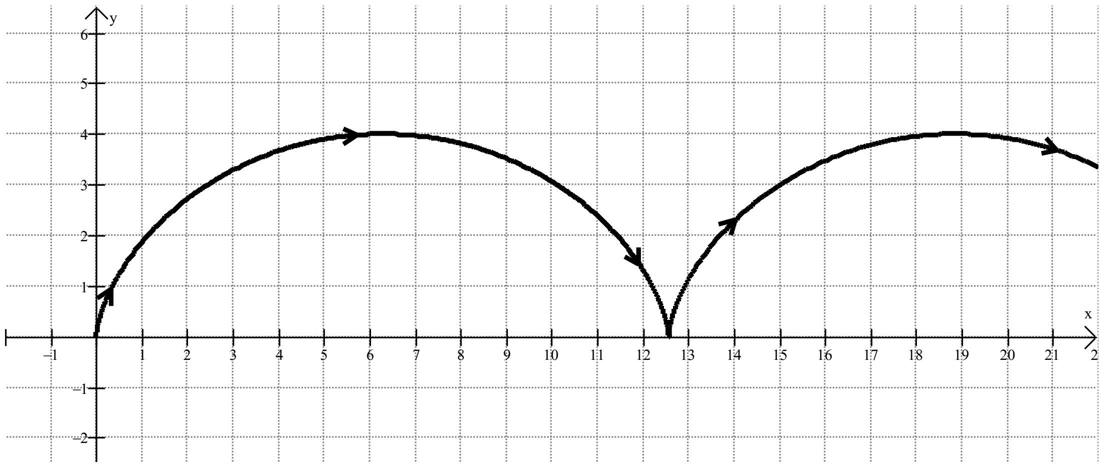
The tangent line would be vertical where $\frac{dy}{dx}=\text{dne}$; that is, $2t=0$.

$$t=0\Rightarrow(x,y)=(0,0)$$

As we saw earlier, parametrics allow one to graph non-functions. The parameter can often be eliminated in order to graph a function, but some functions cannot be expressed in Cartesian coordinates. The most well-known example is the cycloid, which is created by rolling a circle along the x-axis and tracing the path traveled by a point on the circle.

Ex 5: BC 2006 #3

Ex 6 Show, algebraically, that the cycloid $\begin{Bmatrix} x(t) = 2(t - \sin t) \\ y(t) = 2(1 - \cos t) \end{Bmatrix}$ in Ex 1 is concave down on $t \in (0, 2\pi)$.



Notice that, in the *x*-equation, the *t* is both inside and outside the trig function. We would not be able to isolate *t* so as to eliminate the parameter, despite the fact that the graph clearly shows a function (that is, it passes the vertical line test.)

Concavity is determined by $\frac{d^2y}{dx^2}$. To find this, we need $\frac{dy}{dx}$:

$$\begin{Bmatrix} x(t) = 2(t - \sin t) \\ y(t) = 2(1 - \cos t) \end{Bmatrix} \Rightarrow \begin{Bmatrix} x'(t) = 2(1 - \cos t) \\ y'(t) = 2\sin t \end{Bmatrix} \Rightarrow \frac{dy}{dx} = \frac{\sin t}{1 - \cos t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

$$= \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{2\sin t}$$

$$= \frac{\frac{(1-\cos t)\cos t - \sin t(\sin t)}{(1-\cos t)^2}}{2(1-\cos t)}$$

$$= \frac{\cos t - \cos^2 t - \sin^2 t}{2(1-\cos t)(1-\cos t)^2}$$

$$= \frac{\cos t - 1}{2(1-\cos t)^3}$$

$$= \frac{-1}{2(1-\cos t)^2}$$

Clearly, $\frac{d^2y}{dx^2}$ is always negative (since the numerator is a negative constant and the denominator is always non-negative). Therefore, the cycloid is always concave down.

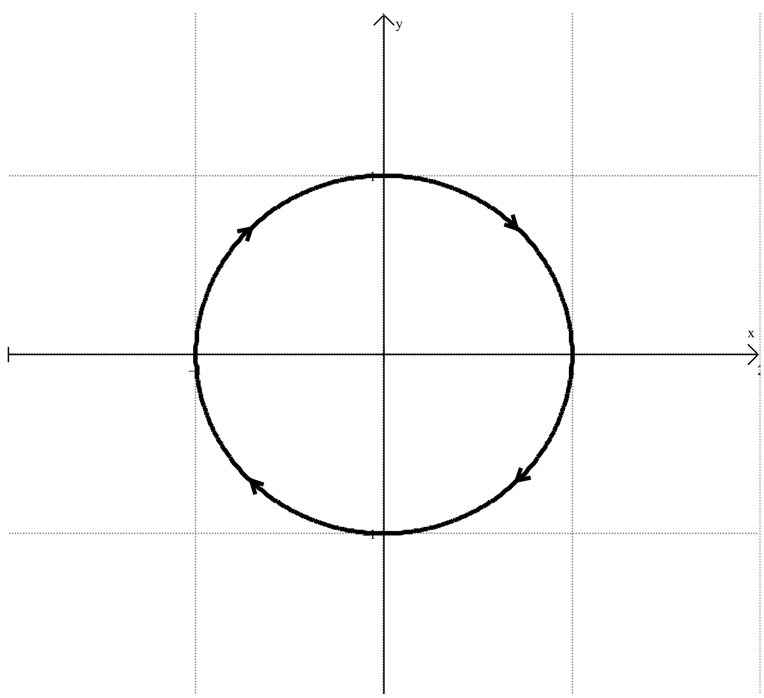
“Eliminating the parameter” refers to the process of getting rid of the variable that is defining our x and y. Because we are more familiar with function mode, sometimes it is easier to convert parametric into function mode. We do this by isolating t in either equation and substituting into the other equation.

Ex 7 Eliminate the parameter for the parametric curve, by the equations $x = t^2 - 2t$ and $y = t + 1$.

$$y = t + 1$$
$$t = y - 1$$
$$x = t^2 - 2t$$
$$x = (y-1)^2 - 2(y-1)$$
$$x = y^2 - 4y + 3$$

So the parametric equation is a parabola opening to the right, as we suspected from our sketch.

Ex 8 Sketch the curve represented by the parametric equations $x = \sin t$ and $y = \cos t$ for $t \in [0, 2\pi]$.



Notice that the parameter, t, in this case represents an angle in radians. Finding values for x and y, we can see that we trace out a circle starting at the point (0,1) performing 1 complete revolution clockwise.

Ex 9 Eliminate the parameter in example 3.

The way to eliminate the parameter when trig functions are involved is to use the Pythagorean identity: $\sin^2 t + \cos^2 t = 1$. Since $x = \sin t$ and $y = \cos t$ in example 5,

$$\sin^2 t + \cos^2 t = 1$$
$$x^2 + y^2 = 1$$

It should make sense that we got the equation of the unit circle give the graph we have.

9.2 Free Response Homework

Find the equation of the line tangent to the given curve at the given point.

1. $x(t)=e^{t}$ and $y(t)=(t-1)^{2}$ at (1,1)

2. $x(t)=2t^{2}+1$ and $y(t)=\frac{1}{3}t^{3}-3$ at $t=3$

3. $x(t)=e^{\sqrt{t}}$ and $y(t)=t-\text{Ln } t^{2}$ at t=1

4. $x(t)=t\sin t$ and $y(t)=t\cos t$ at $(0,-\pi)$

Find $\frac{dy}{dx}$ and $\frac{d^{2}y}{dx^{2}}$.

5. $x(t)=t^{4}-1$ and $y(t)=t^{2}-t^{3}$

6. $x(t)=1+\tan t$ and $y(t)=\cos 2t$

7. $x(t)=e^{-t}$ and $y(t)=te^{2t}$

8. $x(t)=1+t^{2}$ and $y(t)=t \text{ Ln } t$

Find the points (a) where the tangent line is vertical and (b) where the tangent line is horizontal.

9. $x(t)=t(t^{2}-3)$ and $y(t)=3(t^{2}-3)$

10. $x(t)=\frac{3t}{t^{3}+1}$ and $y(t)=\frac{3t^{2}}{t^{3}+1}$

Sketch each of the following parametric curves (with a calculator).

11. $x=\sqrt{t}$ and $y=\cos t$ for $t\in[0,6]$

12. $x=\cos t$ and $y=\sqrt{t}$ for $t\in[0,6]$

13. $x=t^{2}-5t$ and $y=-4t^{2}+1$ for $t\in[0,10]$

14. $x = 5\sin\ t$ and $y = t^2$ for $t \in \left[-\pi, \pi\right]$

15. $x(t) = t^3 - 2t$ and $y(t) = t^2 - t$ for $t \in [-2\pi,\ 2\pi]$

16. $x(t) = 4\cot t$ and $y(t) = 4\sin^2 t$ for $t \in [-2\pi,\ 2\pi]$

17. $x(t) = \sin(t + \sin t)$ and $y(t) = \cos(t + \cos t)$ for $t \in [-2\pi,\ 2\pi]$

18. $x(t) = 3\sin t + 3t\cos t$ and $y(t) = 3\cos t + 3t\sin t$ for $t \in [-2\pi,\ 2\pi]$

Eliminate the parameter for the following parametric equations

19. $x(t) = \sqrt{t}$ and $y(t) = \cos\ t$

20. $x(t) = e^t$ and $y(t) = e^{-t}$

21. $x(t) = \ln\ t$ and $y(t) = \sqrt{t}$

22. $x(t) = \sec t$ and $y(t) = \tan t$

AP Handout

23. BC 2000 #4

24. BC 2006 #3

25. BC 2010 #3

26. BC 2011 #1

27. BC 2012 #2

9.2: Multiple Choice Homework

1. In the *xy* plane, the graph of the parametric equations $x=2t-7$ and $y=-3t+2$, for $-2\le t\le 2$, is a line segment with a slope of

a) $-\frac{2}{3}$ b) $-\frac{3}{2}$ c) 2 d) -3 e) 13

---

2. A curve in the *xy* plane is defined by $\left\{\begin{array}{c} x=t^3+t \\ y=t^4+2t^2 \end{array}\right\}$. An equation of the tangent line to the curve at the point where $t=1$ is

a) $y=2x$
b) $y=8x$
c) $y=2x-1$
d) $y=4x-5$
e) $y=8x+13$

---

3. The equation of the line tangent to $\left\{\begin{array}{c} x=e^t \\ y=t^2+6t \end{array}\right\}$ at $t=0$ is

a) $y-1=6x-6$ b) $y=6e(x-1)$ c) $y-1=6x$

d) $y=6x-6$ e) $y=\frac{6(x-1)}{e}$

---

4. A particle moves in the *xy*-plane so that its position is given by $\left\langle 3t^2-19t,\, e^{2t-7}\right\rangle$. The slope of the tangent line at $t=4$ is

a) $-\frac{e}{28}$ b) $-\frac{28}{e}$ c) $\frac{e}{5}$ d) $\frac{2e}{5}$ e) $\frac{5}{2e}$

---

5. Give the length of the curve determined by $\left\{ \begin{array}{c} x = 2\cos t \\ y = 2\sin t \end{array} \right\}$ on $t \in [0, 2]$.

a) 2 b) 4 c) 6 d) 8 e) 10

---

6. A curve is described by the parametric equations $x = t^2 + 2t$ and $y = t^3 + t^2$. An equation of the line tangent to the curve at the point determined by $t = 1$ is

a) $2x - 3y = 0$ b) $4x - 5y = 2$ c) $4x - y = 10$

d) $5x - 4y = 7$ e) $5x - y = 13$

---

7. If $x(t) = t^2$ and $y(t) = t^3$, then $\frac{d^2y}{dx^2}$ at $t = 3$ is

a) $\frac{1}{16}$ b) $\frac{3}{2}$ c) $\frac{3}{4}$ d) $\frac{1}{4}$ e) $\frac{9}{4}$

---

8. If $x(t) = 3 - t^2$ and $y(t) = \ln(t^2 + 1)$, then $\frac{d^2y}{dx^2}$ at $t = 1$ is

a) $-\frac{1}{4}$ b) $-\frac{1}{8}$ c) 0 d) $\frac{1}{8}$ e) $\frac{1}{4}$

9.3: Intro to AP: Parametric Coordinate FRQs

Key Ideas:

Know the formulas.
Position can be found two ways:

- Indefinite integration of the velocity equation, with an initial value, will yield the position equation. Position can be determined by substituting the time value.
- Definite integration (by calculator) will yield displacement. Adding displacement to the initial value will yields the position

$\frac{dy}{dx}$ is still the slope of the tangent line and $\frac{d^2y}{dx^2}$ still determines concavity.
$\frac{dy}{dt}$ is different from $\frac{dx}{dt}$ is different from $\frac{dy}{dx}$.

Common Sub-Topics:

- Given velocity vectors, find position and/or the acceleration
- Speed
- Find the time a particle reaches a particular speed
- Slope of a tangent line
- Total distance

Summary of Key Phases

When = solve for t
Where = solve for position
Which direction = is the velocity positive or is the velocity negative
Speeding up or slowing down = are the velocity and acceleration in the same direction or opposite (do they have the same sign or not)

Formulas:

Position = $\langle x(t), y(t)\rangle$

Velocity = $\langle x'(t), y'(t)\rangle$

Acceleration = $\langle x''(t), y''(t)\rangle$

Position = $\langle \int x'(t)dt, \int y'(t)dt\rangle$

Velocity = $\langle \int x''(t)dt, \int y''(t)dt\rangle$

Speed = $|v(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$

Position = $\textit{Initial Value} + \int_a^b v(t)dt = \begin{cases} x(a) + \int_a^b x'(t)dt \\ y(a) + \int_a^b y'(t)dt \end{cases}$

Displacement = $\int_a^b v(t)dt = \begin{cases} \int_a^b x'(t)dt \\ \int_a^b y'(t)dt \end{cases}$

Distance traveled $= \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

Arc Length: $\mathrm{L} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{d^2y}{dx^2} = \frac{d/dt\left(\frac{dy}{dx}\right)}{dx/dt}$$

OBJECTIVES

- Graph relations in parametric mode.
- Find the slope of a tangent line to a curve in parametric mode.
- Find the concavity of a curve in parametric mode.
- Find the arc length of a curve in parametric mode.
- Graph relations in parametric mode.
- Eliminate the parameter to identify the function form of a parametric.

EX 1 A particle moves in the *xy*-plane so that at time *t*, its position vector is $\langle x(t), y(t)\rangle$. At $t=2$, its position vector is $(1,\ 5)$. It is also known that

$$\langle x'(t), y'(t)\rangle = \langle \frac{\sqrt{t+2}}{e^t}, \sin^2 t\rangle.$$

(a) Is the particle's horizontal movement to the left or the right at $t=2$? Explain your answer. Find the slope of the particle's path at $t=2$.
(b) Find the *x*-coordinate of the particle at $t=4$.
(c) Find the acceleration vector at time $t=4$.
(d) Find the speed at time $t=4$.
(e) Find the total distance traveled by the particle between $t=2$ and $t=4$.

(a) Is the particle's horizontal movement to the left or the right at $t=2$? Explain your answer. Find the slope of the particle's path at $t=2$.

$x'(2) = \frac{\sqrt{2+2}}{e^2} = 2e^{-2} > 0$, therefore, the particle is moving to the right.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin^2 t}{\sqrt{t+2}/e^t} = \frac{e^t \sin^2 t}{\sqrt{t+2}} \rightarrow \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{2}e^2\sin^2 2 = 3.055$$

(b) Find the *x*-coordinate of the particle at $t=4$.

$$x(4) = 1 + \int_2^4 \frac{\sqrt{t+2}}{e^t}dt = 1.253$$

(c) Find the acceleration vector at time $t = 4$.

$$\langle x''(4), y''(4)\rangle = \langle -0.041, 0.989\rangle$$

(d) Find the speed at time $t = 4$.

$$\text{Speed} = |v(4)| = \sqrt{(x'(4))^2 + (y'(4))^2} = \sqrt{(.0448)^2 + (0.5727)^2} = 0.574$$

(e) Find the total distance traveled by the particle between $t = 2$ and $t = 4$.

$$|v(4)| = \int_2^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt = 0.650$$

Ex 2 For $0 \le t \le 2\pi$, a particle is moving such that its position at time t is $\langle x(t), y(t)\rangle$, where $x(t)$ is not explicitly given, but $y(t) = 2\cos t$ and $\dfrac{dx}{dt} = e^{\sin t}$. At $t = 0$, the particle's position is $(0, 2)$

(a) Find the acceleration vector at time $t = 4$. Show the set up for your calculations.

(b) Find the slope of the particle's path at $t = 2$.

(c) For $0 \le t \le 2\pi$, find the first time the speed is equal to 1.3. Show the set up for your calculations.

(d) Find the total distance traveled by the particle between $t = 2$ and $t = 4$.

(a) Acceleration vector at time $t = 4$ is

$$a(x,y) = \langle (\cos t)e^{\sin t},\ -2\sin t\rangle;\ a(4) = \langle -0.307,\ -0.151\rangle$$

(b) Find the slope of the particle's path at $t = 2$.

(c) Speed: $\sqrt{\left(e^{\sin t}\right)^2 + (-2\sin t)^2} = 1.3$. Graph in function mode to solve.

$t = 4.057$

(d) Total distance= $= \int_2^4 \sqrt{\left(e^{2\sin t}\right) + (-2\sin t)^2} = 3.291$ and $t = 4$.

9.3 Free Response Homework

1. A particle moves in the *xy*-plane so that at time t, $-5 \le t \le 15$, its position vector is $\langle t^3 - 6t^2 + 3,\ t^2 - 8t - 1 \rangle$.

(a) At what time is the particle at rest? Justify your answer.

(b) What is the particle's velocity and speed at t=5?

(c) Is the speed at t=5 increasing or decreasing? Justify your answer.

(d) What is the average speed of the particle on $1 \le t \le 9$?

---

2. A particle's velocity $(x(t), y(t))$ at time $0 \le t \le 10$ is described by the parametric equations $x'(t) = \dfrac{t}{\sqrt{t^2+4}}$ and $y'(t) = \dfrac{5-t}{\sqrt{10t - t^2}}$. At $t = 0$, the particle's position is $(2, 0)$.

(a) At what time is the particle at rest? Justify your answer.

(b) What are the particle's speed and velocity at t=5?

(c) Is the speed at t=5 increasing or decreasing? Justify your answer.

(d) What is the total distance traveled by the particle on $1 \le t \le 9$?

---

3. The velocity vector of a particle moving in the *xy*-plane has components given by $\dfrac{dx}{dt} = \sqrt{3t}$ and $\dfrac{dy}{dt} = 2\sin\left(\dfrac{t^3}{3}\right)$ for $0 \le t$. At time $t = 4$, the position of the particle is (1, 5).

(a) Find the acceleration vector of the particle at $t = 1$.

(b) On $0 \le t \le 4$, at what time does the speed of the particle first reach 3.5?

(c) Find the distance traveled on $0 \le t \le 4$.

(d) Is there a point at which the tangent line is vertical? If so, find the coordinates of that point.

---

4. An object moving along a curve in the xy-plane is at position $(x(t), y(t))$ at time t, where

$$\frac{dx}{dt} = \tan^{-1}\left(e^{-t}\right) \text{ and } \frac{dy}{dt} = \cos\left(e^{-t}\right)$$

for $t \geq 0$. At time $t = 1$, the object is at (2, -3).

(a) Write the equation of the tangent line to the curve at (2, -3).

(b) Find the acceleration vector and speed of the object at time $t = 1$.

(c) What is the total distance traveled by the particle on $t \in [-2, \ 3]$?

(d) Find the y-coordinate of the particle at t=0.

---

5. For $t > 0$, a particle is moving such that its position at time t is $\langle x(t), y(t) \rangle$, where $y(t)$ is not explicitly given, but $x(t) = \dfrac{t-2}{t^2+3}$ and its vertical velocity is given by $v_y(t) = 3t^2 - 4t - 4$. At time $t = 1$, the position of the particle is $\left(-\dfrac{1}{4}, 3\right)$.

(a) Find the acceleration vector at time $t = 4$. Show the set up for your calculations.

(b) Find the equation of the line tangent to the particle's path at $t = 1$.

(c) For $t > 0$, find the first time the speed is equal to 1.3. Show the set up for your calculations.

(d) Find the total distance traveled by the particle between $t = 1$ and $t = 4$.

---

6. For $t > 0$, a particle is moving such that its position at time t is $\langle x(t), y(t) \rangle$, where $x(t)$ is not explicitly given, but $v_x(t) = 8 - 2e^{-2t}$ and $y(t) = \frac{6}{t^3}$. At time $t = 1$, the position of the particle is $\langle 6, -3 \rangle$.

(a) Find the acceleration vector at time $t = 4$. Show the set up for your calculations.

(b) Find the slope of the particle's path at $t = 2$.

(c) Is the speed is increasing or decreasing at $t = 7.7$. Show the set up for your calculations.

(d) Find the total distance traveled by the particle between $t = 2$ and $t = 4$.

---

7. For $t > 0$, a particle is moving such that its position at time t is $\langle x(t), y(t) \rangle$, where $x(t)$ is not explicitly given, but $v_x(t) = t \cos 3t^2$ and $y(t) = 3t^2 - \frac{4}{t} - 4$ At time $t = 1$, the position of the particle is $(1, -5)$.

(a) Find the acceleration vector at time $t = 4$. Show the set up for your calculations.

(b) Find the slope of the particle's path at $t = 2$.

(c) Find the position of the particle at $t = 2$.

(d) Find the total distance traveled by the particle between $t = 1$ and $t = 4$.

---

8. Particle P moves along the x-axis such that, for time $t > -6$, its horizontal position is given by $x(t) = 2t^3 - 3t^2 - 72t + 108$, but its vertical position equation is not explicitly given. For time $t > -6$, its vertical velocity is given by $v_y(t) = t^2 - t - 12$. At time $t = 1$, the position of the particle is $(-2, 1)$.

(a) Find the acceleration vector at time $t = 4$. Show the set up for your calculations.

(b) Find the slope of the particle's path at $t = 2$.

(c) For $0 \le t \le 2\pi$, find the first time the speed is equal to 1.3. Show the set up for your calculations.

(d) Find the total distance traveled by the particle between $t = 2$ and $t = 4$.

---

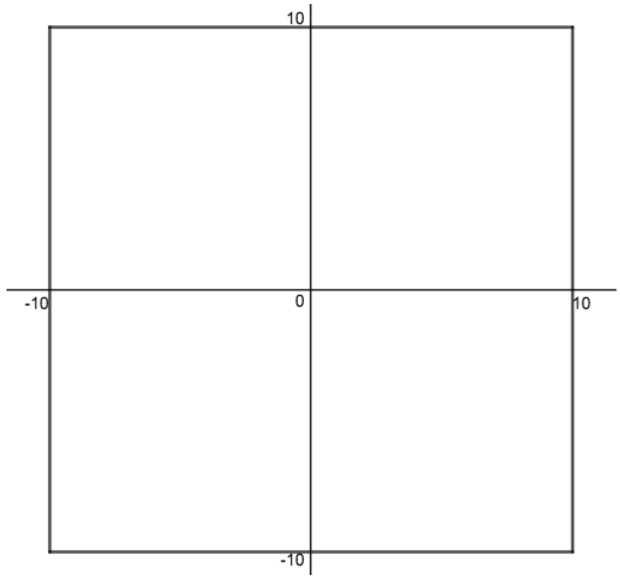
9. A dog named Sparky runs around in his owner's 20 ft x 20 ft backyard. Sparky's owner is a mathematician and has laid an xy-coordinate grid in his backyard, with the origin at the center of the yard. Sparky's position at time t seconds can be modeled in feet by the parametric vector

$$\langle x(t), y(t) \rangle = \langle t\sin(2t), t\cos(3t) \rangle \text{ for } 0 \le t \le 10 \text{seconds.}$$

(a) What is Sparky's speed at $t = 5$ seconds?

(b) How many feet did Sparky run during the first 3 seconds?

(c) On the graph below, draw Sparky's path of motion in the backyard during the time interval $t = 0 \text{to} t = 10$. (Hint: use parametric mode on your calculator and adjust the time window.)



(d) Sparky is not allowed to run into the flower garden, which is a vertical strip in the yard starting at $x = -7$ and going until the left fence ($x = -10$). Sparky is a rebellious dog and enters the flower garden anyway. For how long is he in the flower garden during his 10-second run?

---

10. A fly is buzzing around a 10' by 10' bedroom. For $0 \le t \le 5$ seconds, the horizontal position of the fly is given by $x(t) = t\cos 8\sqrt{t}$, while $y(t)$ is not explicitly given. The vertical velocity is given by $v_y(t) = 20\cos(4t)$ and the fly is in the center of the room $(0, 0)$ at $t = 0$.

(a) What is the position of the fly at $t = 3.6$?

(b) What is the fly's average speed on the interval $0 \le t \le 3.6$?

(c) What is the fly's acceleration vector at time $t = 3.6$?

(d) When is the first time the fly hits the south wall of the room?

---

11. 2021BC # 2

12. 2022BC # 2

13. 2023BC # 2

14. 2024BC # 2

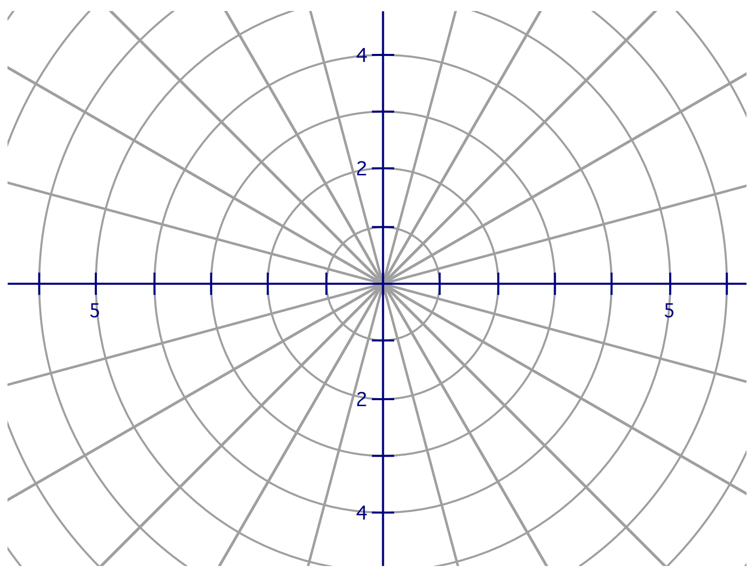
9.4: Polar and Cartesian Graphs

Lat year, we briefly looked at vectors in Polar Form. Here is a quick review:

Vocabulary

Polar Coordinates—Defn: System where points (r, θ) are defined by distance from the origin (pole) and at what angle that distance is measured. [Note that the points are (r, θ). That is, the dependent variable comes first in the pair.]

Because of the definition, a Polar axis system looks like a radar screen:



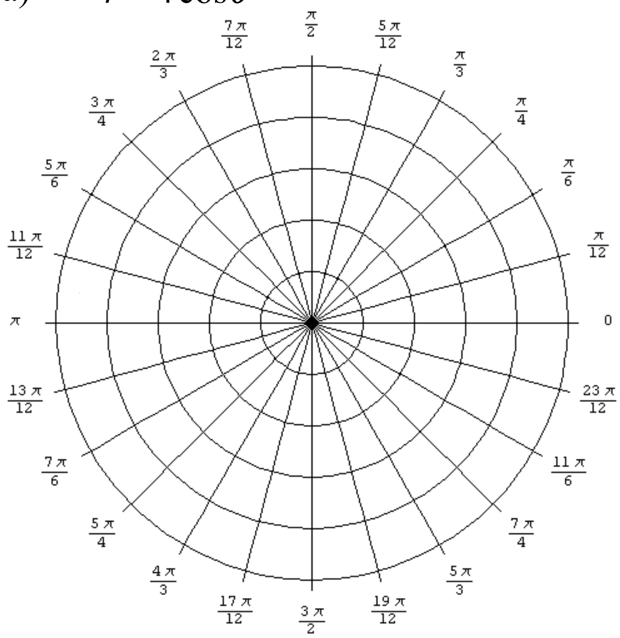
Points on the graph carry coordinates that describe how far from the origin (Pole) and in which direction the point is. One of the confusing things is that the radar screen is not usually drawn because of the time involved so polar graphs are drawn on what looks like—but is not—a Cartesian system.

OBJECTIVES
Graph curves in polar form. Recognize certain polar equations as having particular graphs. Determine the θ which would give a specific *x*- or *y*-value.

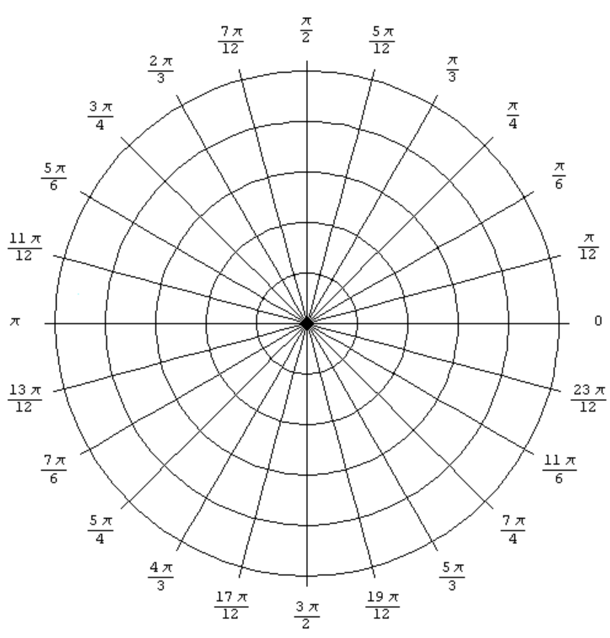
We are so used to the Cartesian System and its perspective of points being measurement horizontally and vertically from the Origin, that sometimes the perspective shift to Polar Coordinates and the idea of distance in a specific direction is difficult. **This is especially true when the *r*-value can now be negative.** While point-wise plotting can be tedious, it can really help with the change in perspective.

EX 1. Plot the following functions:

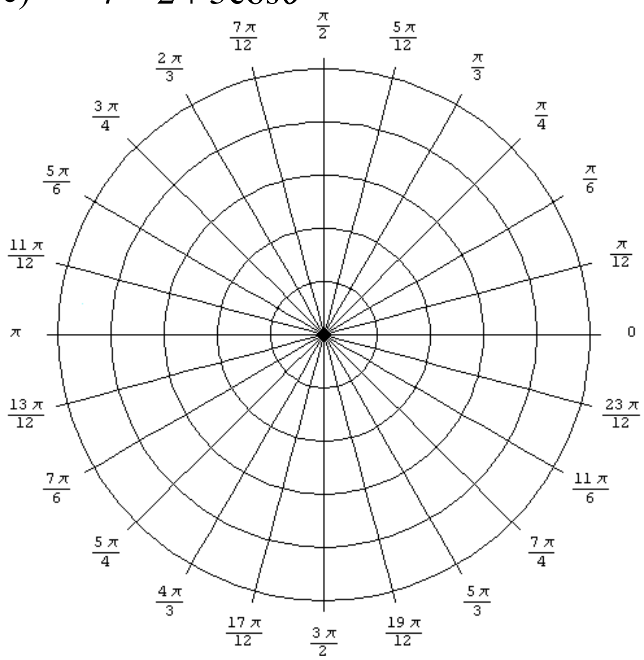
a) $r = 4\cos\theta$



b) $r = 2 + 2\cos\theta$

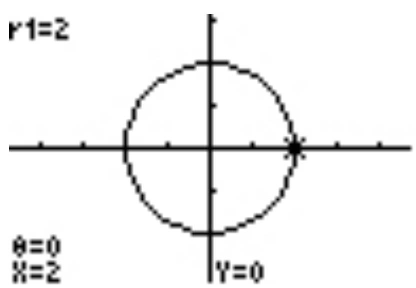


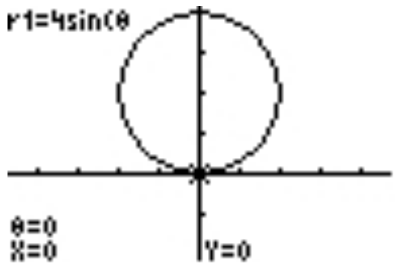
c) $r = 2 + 3\cos\theta$

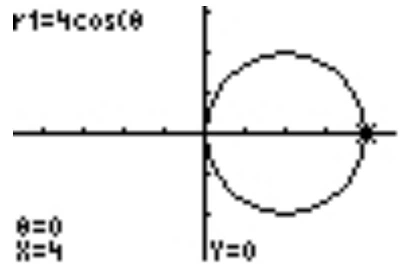


Summary of Standard Pole Graphs

1. Circles: $r = c$, $r = a\sin\theta$ or $r = a\cos\theta$. The first is a circle with radius c and center at the origin. The second and third are circles through the origin with diameter a. $r = a\sin\theta$ has the diameter on the 90° radial line and $r = a\cos\theta$ has the diameter on the 0° radial line.

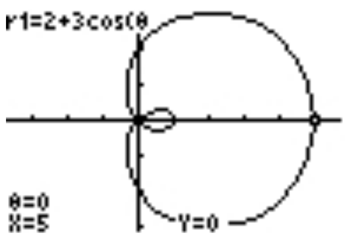


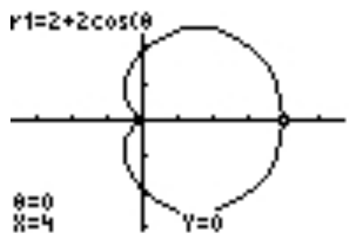


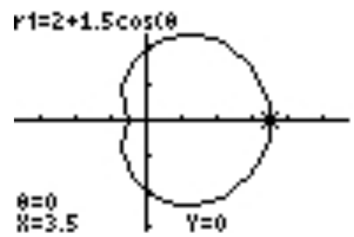


Note that the calculator lists points as it did in parametric mode, with θ determining *x* and *y*.

2. Limacons: $r = c + a\sin\theta$ or $r = c + a\cos\theta$. As with circles, theses have their center-lies on the 90° and 0° radial line, respectively. The relative size of a and c determine if there is a loop or not.

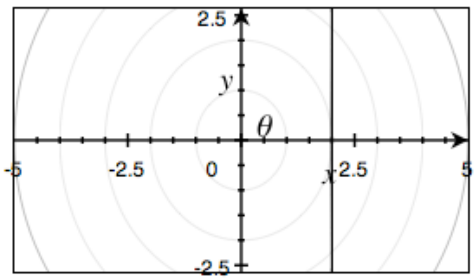




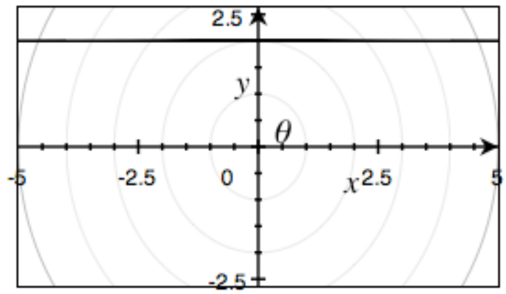


The loop results from *r* being a negative value, because the trig value is negative and *a* is greater than *c*. This is a huge difference from polar vectors. Last year, we said *r* is always positive, and it was…in that context. **Now, *r* can be negative.**

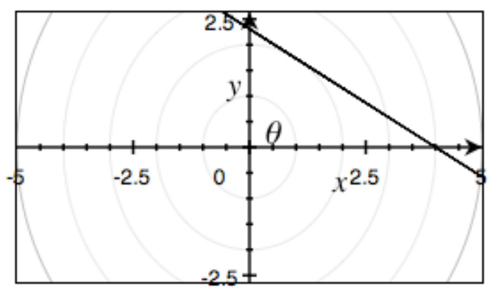
3. Lines: $r = a\csc\theta$, $r = a\sec\theta$, $r = a\sec(\theta - \alpha)$, and $\theta = \alpha$. The first two are lines *a* units from the pole and perpendicular to the 90° and 0° radial line, respectively. The third is for any line a units from the origin where *a* is measured along the α radial line. The last is a line through the pole and lying on the α radial line. These do not come up as often on the AP test as they used to. They are now generally distracter answers in multiple-choice questions.



$$r = 2\sec\theta$$



$$r = 2\csc\theta$$



$$r = 2\sec\left(\theta - \frac{\pi}{3}\right)$$

Other than memorizing the equations above, this section is mostly about familiarizing yourself with polar graphs. The best way to do that is to play with your calculator.

Recall from PreCalculus that we learned formulas to convert between the two systems:

Conversions:

$$x = r\ \cos\ \theta \quad \text{and} \quad y = r\ \sin\ \theta$$

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \pm\cos^{-1}\frac{x}{r}$$

EX 2 Convert $r = 4\sin\theta$ to Cartesian and identify the curve.

$$r = 4\sin\theta \Rightarrow r = 4\left(\frac{y}{r}\right) \Rightarrow r^2 = 4y \Rightarrow x^2 + y^2 = 4y$$

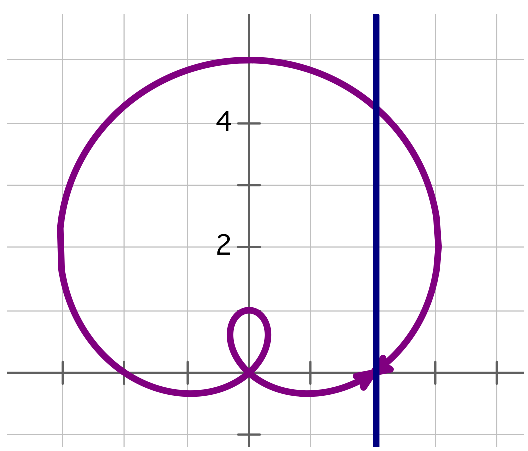
This is a circle.

Ex 3 Convert $3x + 4y = 1$ to Polar form.

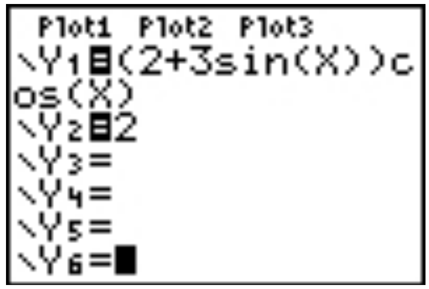
$$3x + 4y = 1 \Rightarrow 3(r\cos\theta) + 4(r\sin\theta) = 1$$
$$\Rightarrow r(3\cos\theta + 4\sin\theta) = 1$$
$$\Rightarrow r = \frac{1}{(3\cos\theta + 4\sin\theta)}$$

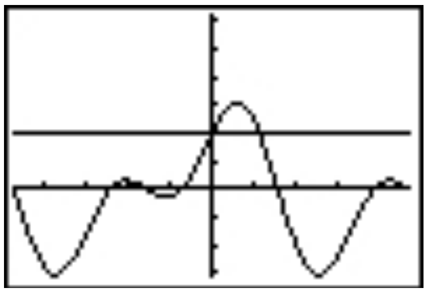
Just as there are specific Cartesian equations for which we are expected to know the shape ($x^2 + y^2 = 1$ is a circle, $y = mx + b$ is a line, etc.), there are certain polar equations that AP expects you to know on sight:

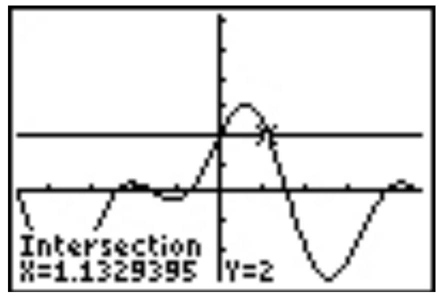
Ex 4 Find the values of θ on $r = 2 + 3\sin\theta$ where $x = 2$.



$$x = r\cos\theta = (2 + 3\sin\theta)\cos\theta = 2$$


Plot1 Plot2 Plot3
\Y1=(2+3sin(X))c
os(X)
\Y2=2
\Y3=
\Y4=
\Y5=
\Y6=




Intersection
X=1.1329395 Y=2

$x = 0, 1.133$

9.4: Free Response Homework

Graph the following polar equations.

1. $r = \sin\left(\frac{1}{2}\theta\right)$

2. $r = 4\sin 3\theta$

3. $r = \sin(4\theta)$

4. $r = 2(1 - \sin\theta)$

5. $r = 4\cos 4\theta$

6. $r^2 = 9\cos 2\theta$

7. $r = 2\cos^2\theta\sin\theta$

8. $r = \dfrac{12}{3 + \sin\theta}$

Convert the following polar equations to Cartesian and identify the curve where possible.

9. $r\sin\theta = 2$

10. $r = \dfrac{1}{1 + 2\cos\theta}$

11. $r^2 = \sin 2\theta$

Convert the following Cartesian equations to polar.

12. $x^2 - y^2 = 1$

13. $x^2 = 4y$

14. $2x - y = 1$

For the following, find the value(s) of θ where the curve has the given Cartesian value.

15. $r = 4\sin\theta$ and $x = -1$

16. $r = 4\sin\theta$ and $y = 3$

17. $r = 4\sin 3\theta$ and $x = 3$

18. $r = 4\sin 3\theta$ and $y = 2$

19. $r = 2\cos^2\theta\sin\theta$ and $x = .4$

20. $r = 2\cos^2\theta\sin\theta$ and $y = .4$

21. $r^2 = 6\cos 2\theta$ and $x = -2$

22. $r^2 = 6\cos 2\theta$ and $y = .4$

9.4: Multiple Choice Homework

1. The Cartesian equation of the polar curve $r = 2\sin\theta + 2\cos\theta$ is

a) $(x-1)^2 + (y-1)^2 = 2$ b) $x^2 + y^2 = 2$ c) $x + y = 4$

d) $x^2 + y^2 = 4$ e) $y^2 - x^2 = 4$

---

2. $r^2 = a^2 \sin 2\theta$ $r = a(1+\cos\theta)$ $r\sin\theta = 3$ $r = 4\cos 3\theta$ $r = \frac{2}{\theta}$

How many of these 5 equations in polar form have graphs that are symmetric with respect to either the x-axis?

a) one b) two c) three d) four e) five

---

3. $r^2 = a^2 \sin 2\theta$ $r = a(1+\cos\theta)$ $r\sin\theta = 3$ $r = 4\cos 3\theta$ $r = \frac{2}{\theta}$

How many of these 5 equations in polar form have graphs that are symmetric with respect to either the y-axis?

a) one b) two c) three d) four e) five

---

4. The graph of the polar equation $r = 4 - 5\sin\theta$ is a

a) Cartioid b) Limacon c) Limacon with a loop

d) Circle e) Line

---

5. The graph of the polar equation $r = 4\sin(\theta - 5)$ is a

a) Cartioid b) Limacon c) Limacon with a loop

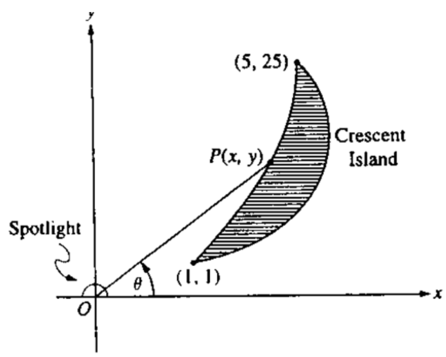
d) Circle e) Line

---

6. The graph of the polar equation $r = 5 - 4\sin\theta$ is a

a) Cartioid b) Limacon c) Limacon with a loop

d) Circle e) Line



7. The figure above shows a spotlight shining on a point $P(x,y)$ on the shoreline of Crescent Island. The spotlight is located at the origin and is rotating. The portion of the shoreline on which the spotlight shine is in the shape of the parabola $y = x^2$ from the point (1, 1) to (5, 25). Let be the angle between the beam of light and the positive *x*-axis. For what values between 0 and $\frac{\pi}{2}$ does the spotlight shine on the shoreline?

a) $.5 \le x \le .785$ b) $.785 \le x \le 1.190$

c) $1.190 \le x \le 1.373$ d) $.785 \le x \le 1.373$

e) $1.373 \le x \le 1.570$

9.5: Polar Coordinates, Arc Length, and Simple Area

OBJECTIVES

Find the arc length of a shape describe in polar coordinates.
Find the area of a shape described in polar coordinates.

Arc Length

There are two main geometric uses for integrals in polar coordinates—Area and Arc Length. The Arc Length is straightforward:

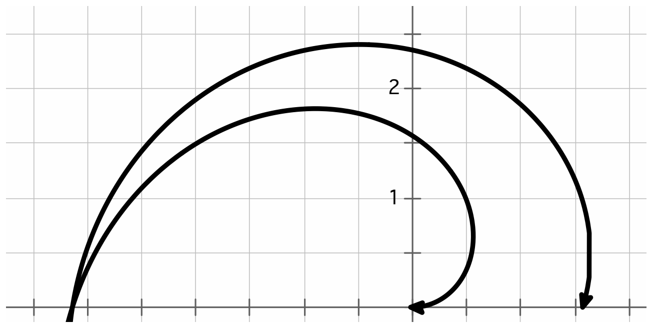
$$\text{L} = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

The Arc Length formula is straight-forward and easy to use.

Ex 4 Find the arc length of $r = 5 + \cos\theta$ on $\theta \in [0, 2\pi]$.

$$\begin{aligned} \text{L} &= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\ &= \int_{0}^{2\pi} \sqrt{(5+\cos\theta)^2 + (-\sin\theta)^2}\, d\theta \\ &= 31.731 \end{aligned}$$

Ex 5 Find the perimeter of the region bounded by $r = \theta$ and $r = \frac{1}{2}(\theta + \pi)$on $\theta \in [0, \pi]$.



$$L_1 = \int_0^{\pi} \sqrt{\theta^2 + 1^2}\, d\theta = 6.110$$

$$L_2 = \int_0^{\pi} \sqrt{\left[\frac{1}{2}(\theta + \pi)\right]^2 + \left(\frac{1}{2}\right)^2}\, d\theta = 7.573$$

$$L_3 = r_2(0) - r_1(0) = \frac{\pi}{2} = 1.573$$

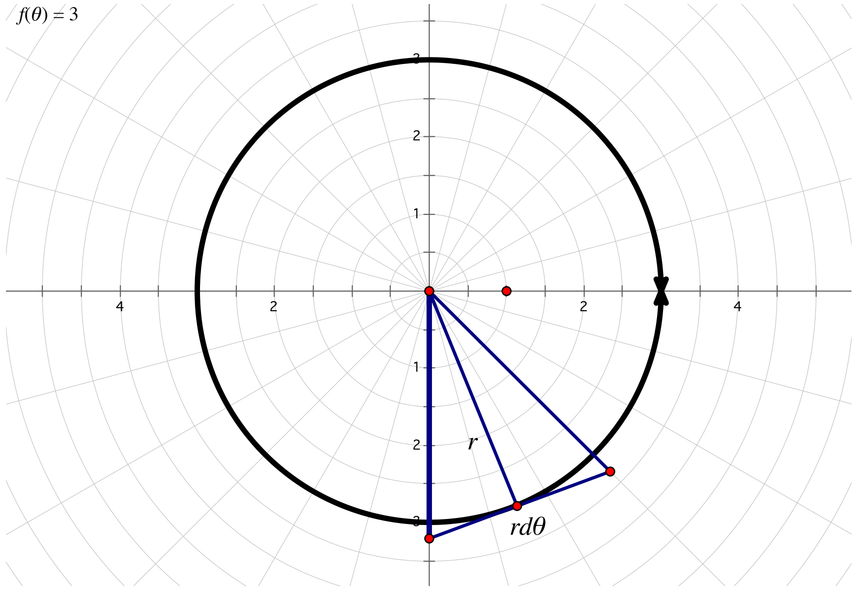
$$P = L_1 + L_2 + L_3 = 15.256$$

Simple Area

Area, on the other hand, appears straightforward but can be tricky. The formula is simple:

$$\text{A} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \; d\theta$$

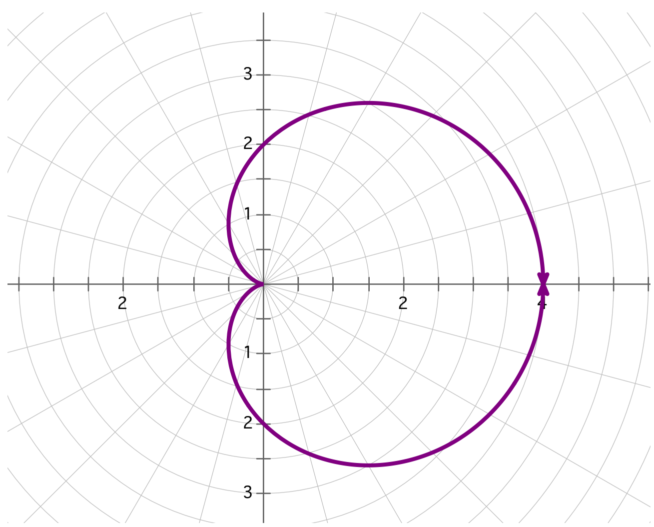
With Cartesian coordinates, we considered area as the sum of Riemann Rectangles. With polar coordinates, we would have triangles.



The coefficient $\frac{1}{2}$ comes area of said triangles, namely, $\frac{1}{2}bh$, where $h=r$ and $b=r\ d\theta$.

Ex 6 Find the area enclosed by the limacon $r=2+2\cos\theta$.

We can see by grapher that the limacon completes its graph on $\theta\in[0,\ 2\pi]$.



$$
\begin{aligned}
A &= \frac{1}{2}\int_0^{2\pi}(2+2\cos\theta)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}\left(2(1+\cos\theta)\right)^2\,d\theta \\
&= \frac{1}{2}\int_0^{2\pi}4(1+\cos\theta)^2\,d\theta \\
&= 2\int_0^{2\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta \\
&= 2\left(\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)\Big|_0^{2\pi} \\
&= 6\pi
\end{aligned}
$$

The real difficulty is in finding the boundaries for the integral if they are not obvious or not given. Sometimes, they can be found by calculator. Other times, they are best found either by the *r*-values or as points of intersection.

Ex 7 Which of the following equations gives the area of the region enclosed by the graph of the polar curve $r = 1+\cos\theta$? (note: This is a non-calculator question.)

(A) $\int_0^{\pi}(1+\cos\theta)d\theta$ (B) $\int_0^{\pi}(1+\cos\theta)^2\,d\theta$ (C) $\int_0^{2\pi}(1+\cos\theta)d\theta$
(D) $\int_0^{2\pi}(1+\cos\theta)^2\,d\theta$ (E) $\frac{1}{2}\int_0^{2\pi}\left(1+\cos^2\theta\right)d\theta$

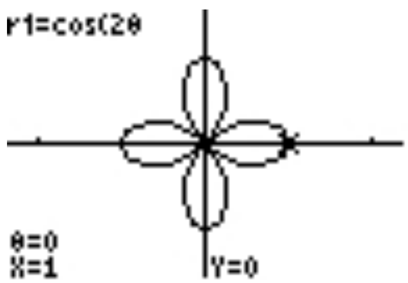
$$A = \int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta = \int_0^{2\pi}\frac{1}{2}(1+\cos\theta)^2\,d\theta$$

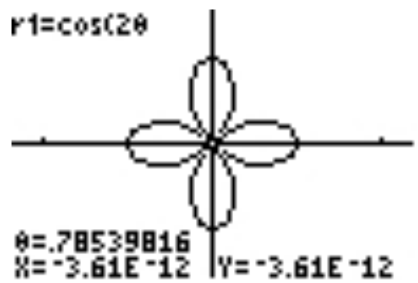
Note that we can eliminate (A), (C) and (E) because the Polar area formula calls for r to be squared. But the formula also calls for $\frac{1}{2}$, which neither (B) nor (D) have. The issue is that, because of the symmetry of the curve, we can integrate half the region and double the result to get the whole area. In other words,

$$\int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta = 2\int_0^{\pi} \frac{1}{2}(1+\cos\theta)^2 \, d\theta$$
$$= \int_0^{\pi} (1+\cos\theta)^2 \, d\theta$$

Ex 8 Find the area enclosed by one leaf of the four-leaf rose $r = \cos 2\theta$.

It is fairly clear from the picture what "one leaf" means, but it is not clear what ANGLES define that area, even after tracing.





Algebraically, each leaf begins and ends at the pole, so we can find the boundaries by setting $r = 0$ and solving for θ:

$$0 = \cos 2\theta$$
$$2\theta = \pm\frac{\pi}{2} \pm 2\pi n$$
$$\theta = \pm\frac{\pi}{4} \pm \pi n$$

Our trace does help us now see that the leaf on the right is bounded by $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$. So

$$
\begin{aligned}
A &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \; d\theta \\
&= \int_{-\pi/4}^{\pi/4} \frac{1}{2} (\cos 2\theta)^2 \; d\theta \\
&= \frac{1}{4} \int_{-\pi/4}^{\pi/4} (\cos 2\theta)^2 \; 2\,d\theta \\
&= \frac{1}{4} \int_{-\pi/2}^{\pi/2} (\cos u)^2 \; du \\
&= \frac{1}{4} \left[\frac{1}{2}(u) - \frac{1}{4} \sin 2u \right]_{-\pi/2}^{\pi/2} \\
&= \pi/8
\end{aligned}
$$

A third twist, which often appears on the multiple choice part of the AP Exam is, using the symmetry of the graph, combine the coefficient with the boundaries. For instance, a question might ask for the set-up of this problem, but $\int_{-\pi/4}^{\pi/4} \frac{1}{2} (\cos 2\theta)^2 \; d\theta$ will not be an option. $\int_0^{\pi/4} (\cos 2\theta)^2 \; d\theta$ might be an option and they want to see if you recognize this is the same integral. Because of the symmetry of the curve, you can find the area of half the leaf $\left(\theta \in \left[0, \frac{\pi}{4} \right] \right)$ and double the integral, which cancels the $\frac{1}{2}$ in the formula.

9.5: Free Response Homework

Find the area enclosed by each of these curves.

1. $r=\sin\left(\frac{1}{2}\theta\right)$

2. $r=4\sin 3\theta$

3. $r=\sin(4\theta)$

4. $r=2(1-\sin\theta)$

5. $r=4\cos 4\theta$

6. $r^2=9\cos 2\theta$

7. $r=2\cos^2\theta\sin\theta$

8. $r=\dfrac{12}{3+\sin\theta}$

Find the area.

9. Inside $r=\theta$ on $\theta\in[0,\pi]$.

10. Inside $r=1+\sin\theta$ on $\theta\in\left[\frac{\pi}{2},\pi\right]$.

11. Enclosed by one leaf of $r=\sin 4\theta$

12. Inside $r^2=4\cos 2\theta$

Find the arc length.

13. $r=5\cos\theta$ on $\theta\in\left[0,\frac{3\pi}{4}\right]$.

14. $r=e^{2\theta}$ on $\theta\in[0,2\pi]$.

15. $r=2^{\theta}$ on $\theta\in[0,2\pi]$.

16. $r=\sqrt{5\cos\theta}$ on $\theta\in\left[0,\frac{\pi}{2}\right]$.

17. $r=1+\sin\theta$ on $\theta\in[0,2\pi]$.

9.5: Multiple Choice Homework

1. The length of the path described by the polar curve $r = 2 + \cos^2\theta$ for $0 \le \theta \le \pi$ is given by

a) $L = \frac{1}{2}\int_0^{\pi}\left(2 + \cos^2\theta\right)^2 d\theta$ b) $L = \int_0^{\pi}\sqrt{1 + \left(2 + \cos^2\theta\right)^2}\, d\theta$

c) $L = \int_0^{\pi}\sqrt{\left(2 + \cos^2\theta\right)^2 + 4\sin^2\theta}\ d\theta$

d) $L = \int_0^{\pi}\sqrt{\left(2 + \cos^2\theta\right)^2 + \sin^2 2\theta}\ d\theta$

e) $L = \int_0^{\pi}\sqrt{\left(2 + \cos^2\theta\right)^2 + 2\sin 2\theta}\ d\theta$

---

2. The length of the path described by the polar curve $r = 3 + 3\sin\theta$ for $0 \le \theta \le \pi$ is given by

a) $\frac{1}{2}\int_0^{\pi}\left(3 + 3\sin\theta\right)^2\ d\theta$

b) $\int_0^{\pi}\sqrt{1 + \left(3 + 3\sin\theta\right)^2}\ d\theta$

c) $\int_0^{\pi}\sqrt{\left(3 + 3\sin\theta\right)^2 + 9\cos^2\theta}\ d\theta$

d) $\int_0^{\pi}\sqrt{\left(3 + 3\sin\theta\right)^2 + 3\cos^2 2\theta}\ d\theta$

e) $\int_0^{\pi}\sqrt{\left(3 + 3\sin\theta\right)^2 + 3\sin\theta}\ d\theta$

---

3. The area inside $r=3+2\cos\theta$ is

a) 9.425 b) 18.850 c) 28.274 d) 34.558 e) 69.115

---

4. The area of one loop of the graph of the polar equation $r=2\sin(3\theta)$ is given by which of the following expressions?

a) $4\int_0^{\frac{\pi}{3}}\sin^2(3\theta)\,d\theta$ b) $2\int_0^{\frac{\pi}{3}}\sin(3\theta)\,d\theta$ c $2\int_0^{\frac{\pi}{3}}\sin^2(3\theta)\,d\theta$

d) $2\int_0^{\frac{2\pi}{3}}\sin^2(3\theta)\,d\theta$ e) $2\int_0^{\frac{2\pi}{3}}\sin(3\theta)\,d\theta$

---

5. The total area enclosed by $r=1+\sin\theta$ is

a) $\frac{\pi}{2}$ b) $\frac{\pi}{2}+1$ c) $\frac{3\pi}{2}$ d) $\frac{3\pi}{2}+1$ e) $\frac{3\pi}{2}-1$

---

6. If $r=f(\theta)$ is continuous for $0<\alpha<\beta<2\pi$, then the area between $r=f(\theta)$, $\theta=\alpha$ and $\theta=\beta$ is given by

a) $\frac{1}{2}\int_\alpha^\beta f(\theta^2)d\theta$ b) $\frac{1}{2}\int_\alpha^\beta f(\theta)d\theta$

c) $\frac{1}{2}\int_\alpha^\beta \theta f(\theta^2)d\theta$ d) $\frac{1}{2}\int_\alpha^\beta \theta f(\theta)d\theta$

e) $\frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta$

---

7. The total area enclosed by $r = 3 + 3\sin\theta$ is

a) 9.425 b) 18.850 c) 39.206

d) 42.412 e) 84.823

---

8. To the nearest whole number, find the arc length of the curve $r = 2\sin^2\frac{1}{2}\theta$ from θ = 0 to θ = π.

a) 3 b) 4 c) 5 d) 6 e) 11

---

9. The area of one loop of the graph of the polar equation $r = 2\sin 3\theta$ is

a) $\frac{\pi}{6}$ b) $\frac{\pi}{4}$ c) $\frac{\pi}{3}$ d) $\frac{\pi}{2}$ e) $\frac{2\pi}{3}$

9.6: Area Involving Two Polar Equations

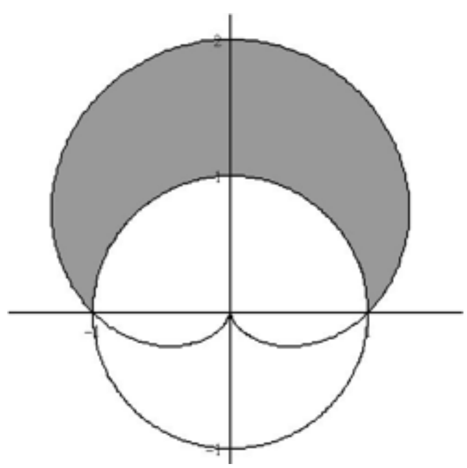
There are two different kinds of figures involving two or more polar curves that will be explored here. **The difference is based on which part of the shape is shaded.**

OBJECTIVES
Find the area of a shape described by two polar coordinate equations.

Area Between Two Polar Curves

As with Cartesian coordinate problems dealing with area between two curves, there can be a region bounded by two polar curves. The shaded region will be inside one curve and outside the other. Instead of "top minus bottom" in Cartesian, it would be "outer minus inner." Better yet, both systems an be combined as "further minus nearer."

Ex 1 Find the area inside $r=1+\sin\theta$ and outside $r=1$.



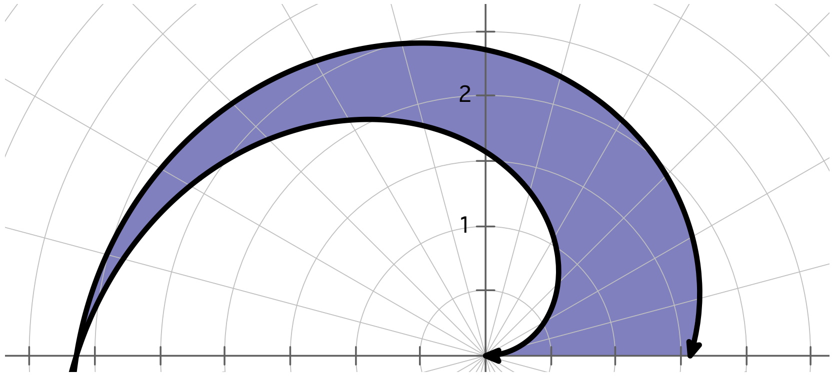
The boundaries would, again, be the problem, but the boundaries are found by setting the equations equal to each other.

$$
\begin{aligned}
1+\sin\theta &= 1 \\
\sin\theta &= 1 \\
\theta &= 0 \pm \pi n \\
\theta &= 0 \text{ and } \pi
\end{aligned}
$$

Just as we can find area between two Cartesian curves by integrating the difference, we can find the area between to polar curves by integrating the difference. Where the Cartesian set-up required we use top – bottom, polar requires further – closer.

$$
\begin{aligned}
\text{A} &= \int_0^{\pi} \frac{1}{2}(1)^2 \, d\theta - \int_0^{\pi} \frac{1}{2}(1+\sin\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{\pi} (1)^2 - (1+\sin\theta)^2 \, d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(1-\left(1+2\sin\theta+\sin^2\theta\right)\right) d\theta \\
&= \frac{1}{2}\int_0^{\pi} \left(\sin^2\theta - 2\sin\theta\right) d\theta \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right) - 2\cos\theta\right]_0^{\pi} \\
&= \frac{1}{2}\left[\left(\frac{1}{2}\pi - \frac{1}{4}\sin 2\pi\right) - 2\cos\pi\right] - \frac{1}{2}\left[0 - 2\cos 0\right] \\
&= \frac{\pi}{4} + 2
\end{aligned}
$$

Ex 2 Let R be the region bounded by $r=\theta$ and $r=\frac{1}{2}(\theta+\pi)$ on $\theta \in [0, \pi]$.



$$A = \frac{1}{2}\int_0^{\pi}\left[\left(\frac{1}{2}(\theta+\pi)\right)^2 - (\theta)^2\right]d\theta$$

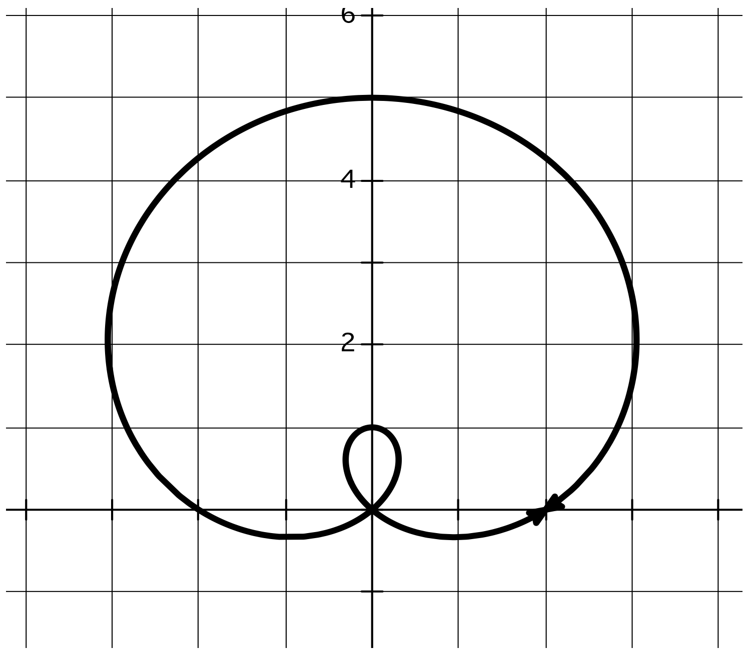
$$= \frac{1}{2}\int_0^{\pi}\left[\left(\frac{1}{4}(\theta^2+2\pi\theta+\pi^2) - \theta^2\right)\right]d\theta$$

$$= \frac{1}{2}\int_0^{\pi}\left[\left(-\frac{3}{4}\theta^2+\frac{1}{2}\pi\theta+\frac{1}{4}\pi^2\right)\right]d\theta$$

$$= \frac{1}{2}\left[-\frac{3}{4}\left(\frac{\theta^3}{3}\right)+\frac{\pi}{2}\left(\frac{\theta^2}{2}\right)+\frac{1}{4}\pi^2\theta\right]_0^{\pi}$$

$$= \frac{\pi^3}{8}$$

Ex 3 Find the area outside the inner loop and inside the outer loop of $r = 2 + 3\sin\theta$.



The first thing to consider is the boundaries. The graph shows that the inner loop starts and ends at $r = 0$. Therefore,

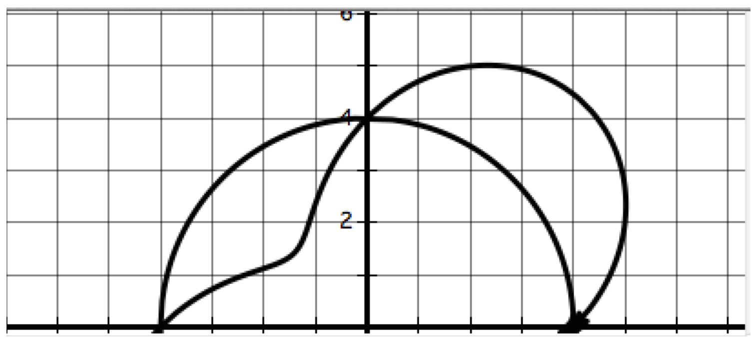
$$0 = 2 + 3\sin\theta$$
$$\sin\theta = -\frac{2}{3}$$
$$\theta = \begin{cases} -.730 \pm 2\pi n \\ 3.871 \pm 2\pi n \end{cases}$$

$$\theta = 3.871, 5.665$$

$$\begin{aligned} A &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \\ &= \frac{1}{2}\int_{-.730}^{3.871} (2 + 3\sin\theta)^2 \, d\theta - \frac{1}{2}\int_{3.871}^{5.665} (2 + 3\sin\theta)^2 \, d\theta \\ &= 1.316 \end{aligned}$$

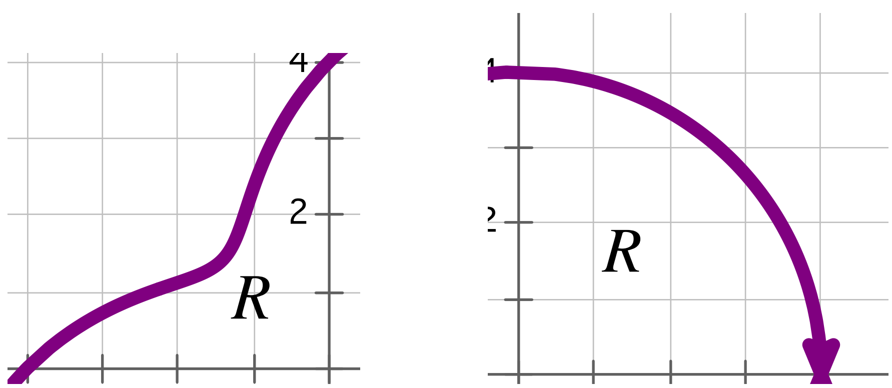
Area Inside Two Polar Curves

EX 4 Let *R* be the region inside the graph of the polar curve $r = 4$ and inside the polar curve $r = 4 + 2\sin 2\theta$ on $0 \le \theta \le \pi$, as shown above.



Find the value of the area of the region *R*.

The key to this kind of probem is to split the picture into two pictures each of which is a region bounded by one curve. Each region can be found and then they can be added.



$$\begin{aligned} 4 + 2\sin 2\theta &= 4 \\ \sin 2\theta &= 0 \\ 2\theta &= 0 \pm \pi n \\ \theta &= 0 \pm \frac{\pi}{2} n \\ \theta &= 0, \frac{\pi}{2}, \pi \end{aligned}$$

So the area formula in each case yields:

$A = \frac{1}{2}\int_{\pi/2}^{\pi} (4 + 2\sin 2\theta)^2 d\theta$ and $A = \frac{1}{2}\int_{0}^{\pi/2} (4)^2 d\theta$

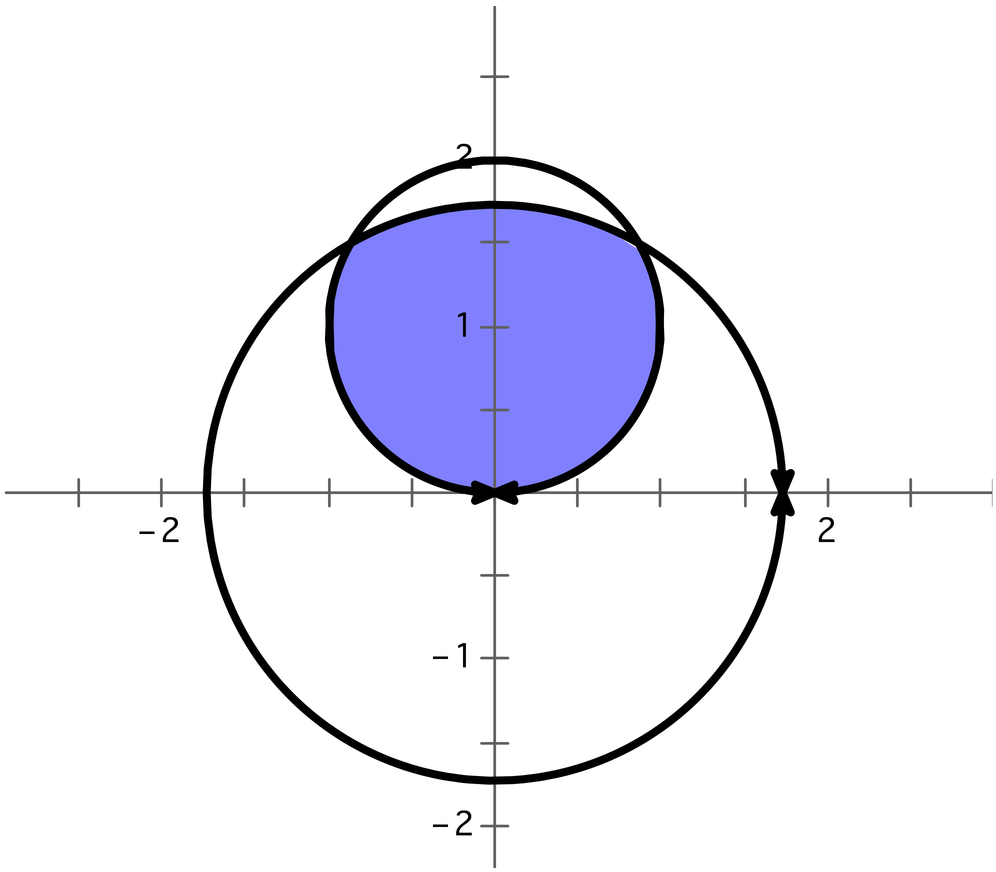
NB The two integrals do not have the same boundaries, therefore they must be solved separately.

$$
\begin{aligned}
A &= \frac{1}{2}\int_{\pi/2}^{\pi} (4 + 2\sin 2\theta)^2 d\theta \\
&= \frac{1}{2}\int_{\pi/2}^{\pi} \left(16 + 16\sin 2\theta + 4\sin^2 2\theta\right) d\theta \\
&= \int_{\pi/2}^{\pi} \left(8 + 8\sin 2\theta + 2\sin^2 2\theta\right) d\theta \\
&= \int_{\pi/2}^{\pi} (8) d\theta + \int_{\pi/2}^{\pi} (4\sin 2\theta) 2 d\theta + \int_{\pi/2}^{\pi} \left(2\sin^2 2\theta\right) d\theta \\
&= \left[8\theta - 4\cos 2\theta + \theta - \frac{1}{4}\sin 4\theta \right]_{\pi/2}^{\pi} \\
&= (9\pi - 4 + 2\pi - 0) - \left(\frac{9\pi}{2} + 4 - 0 \right) \\
&= \frac{9\pi}{2} - 8
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{1}{2}\int_{0}^{\pi/2} (4)^2 d\theta \\
&= \int_{0}^{\pi/2} 8 d\theta \\
&= 8\theta\Big]_{0}^{\pi/2} \\
&= 4\pi
\end{aligned}
$$

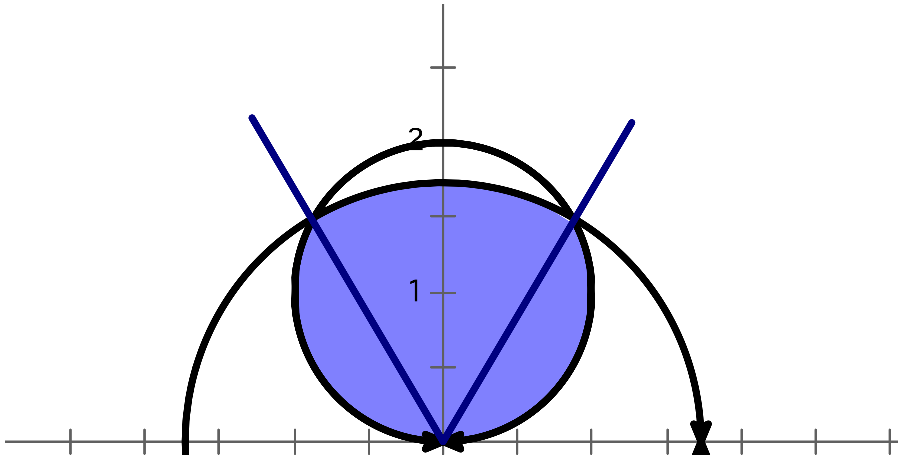
Therefore, the area of region R is $\frac{17\pi}{2} - 8 + 4\pi = \frac{17\pi}{2} - 8$.

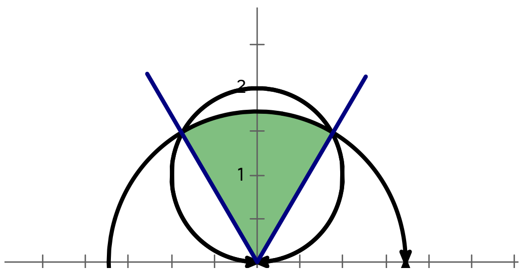
Ex 5 Let R be the region inside the circles $r=\sqrt{3}$ and $r=2\sin\theta$.

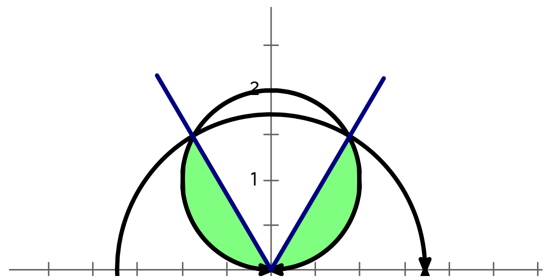


To find the boundaries, set the equations equal to each other:

$$2\sin\theta=\sqrt{3}$$
$$\sin\theta=\frac{\sqrt{3}}{2}$$
$$\theta=\begin{cases}\pi/3\pm 2\pi n\\ 2\pi/3\pm 2\pi n\end{cases}$$
$$\theta=\pi/3,\ 2\pi/3$$







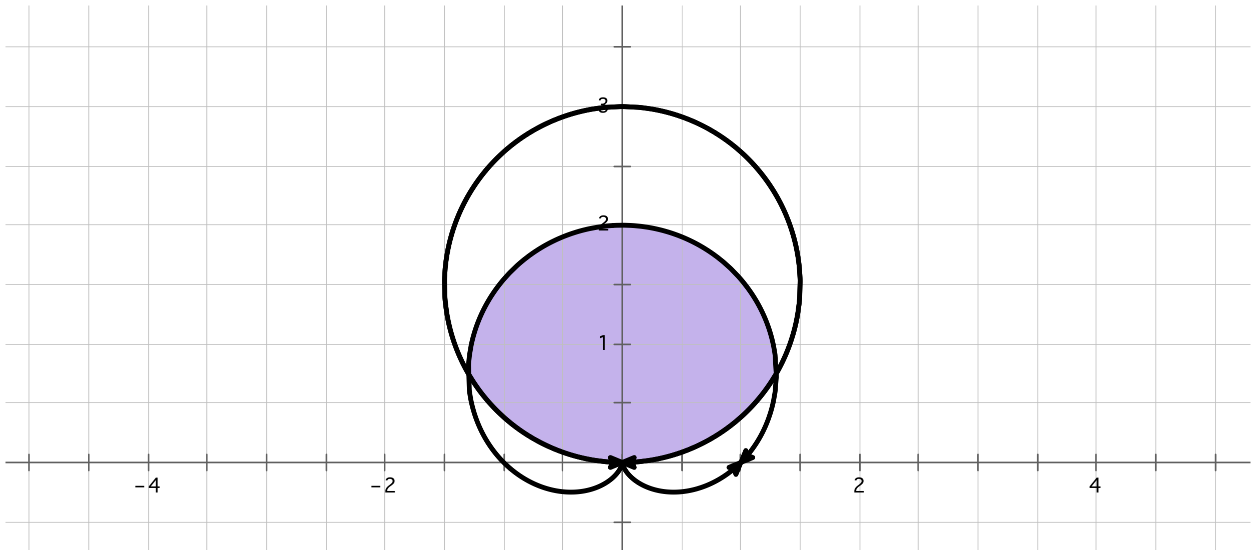
The areas of the two sections of region *R* are:

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/3}^{2\pi/3}\left(\sqrt{3}\right)^2 d\theta \\ &= \frac{1}{2}\int_{\pi/3}^{2\pi/3} 3\,d\theta \\ &= \left[\frac{3}{2}\theta\right]_{\pi/3}^{2\pi/3} \\ &= \frac{3}{2}\left[2\pi/3 - \pi/3\right] \\ &= \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} A &= 2\int_0^{\pi/3}\frac{1}{2}\left(2\sin\theta\right)^2 d\theta \\ &= 4\int_0^{\pi/3}\sin^2\theta\,d\theta \\ &= 4\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/3} \\ &= 4\left[\frac{\pi}{6} + \frac{\sqrt{3}}{8}\right] \\ &= \frac{2\pi}{3} + \frac{\sqrt{3}}{2} \end{aligned}$$

$$\text{Total area} = \frac{7\pi}{6} + \frac{\sqrt{3}}{2}$$

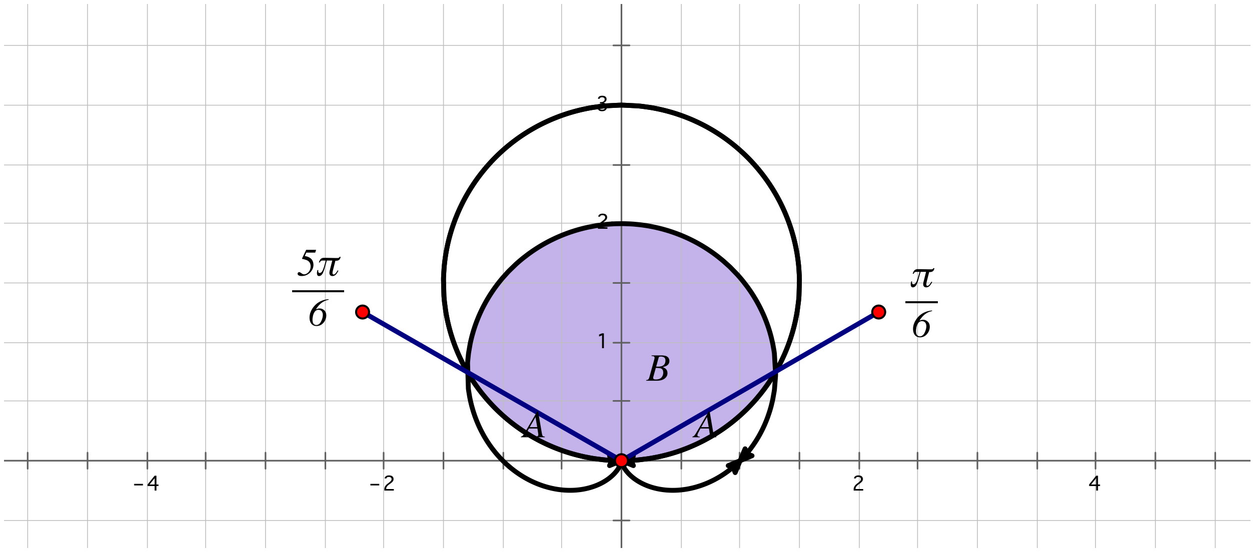
Ex 6 Find the area inside of both $r=3\sin\theta$ and $r=1+\sin\theta$.



The boundaries are often the biggest problem. We can find them by setting the equations equal to each other because the points of intersection will be where the *r*–values are equal.

$$3\sin\theta=1+\sin\theta$$
$$\sin\theta=\frac{1}{2}$$
$$\theta=\begin{cases}\dfrac{\pi}{6}\pm 2\pi n\\ \dfrac{5\pi}{6}\pm 2\pi n\end{cases}\rightarrow\theta=\frac{\pi}{6},\frac{5\pi}{6}$$

The areas of the two sections of region *R* are:



$$A = 2\int_0^{\pi/6} \frac{1}{2}(3\sin\theta)^2 d\theta$$
$$= \int_0^{\pi/6} 9\sin^2\theta d\theta$$
$$= 9\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{\pi/6}$$
$$= 9\left[\pi/12 - \sqrt{3}/8\right]$$
$$= 2.764$$

$$B = 2\int_{\pi/3}^{\pi/2} \frac{1}{2}(1+\sin\theta)^2 d\theta$$
$$= \int_{\pi/3}^{\pi/2} (1+\sin 2\theta + \sin^2\theta) d\theta$$
$$= \left[\theta - \frac{1}{2}\cos 2\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_{\pi/3}^{\pi/2}$$
$$= \left[\frac{\pi}{2} + \frac{1}{2} + \frac{\pi}{4} - 0\right] - \left[\frac{\pi}{3} - \frac{1}{4} + \frac{\pi}{6} - \frac{\sqrt{3}}{8}\right]$$
$$= 2.002$$

Total area = 4.766

Polar Area Summary

0. Always draw the region first (not just graph on the calculator).

1. Area inside one curve

- Find the boundaries
- Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

2. Area Inside Two Curves

- **Break the picture into two regions and make two problems**
- Find the boundaries for each region
  - There are likely to be different boundaries for the two regions
- Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ to each problem

3. Area Inside One Curve and Outside another

- Find the boundaries
  - These are usually the points of intersection, but they could involve $r = 0$ or $\theta = 0$
- Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} (Outer\ r)^2\, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} (Inner\ r)^2\, d\theta$ to each problem

9.6: Free Response Homework

Find the area described.

1. Inside $r^2 = 6\cos 2\theta$ and outside $r = \sqrt{3}$

2. Inside $r = 1 + \cos\theta$ and outside $r = 3\cos\theta$

3. Inside $r = -\cos\theta$ and outside $r = \sec\theta - 2\cos\theta$

4. Inside $r = 4\sin\theta$ and outside $r = 2$

5. Inside $r = 3\cos\theta$ and outside $r = 2 - \cos\theta$

6. Inside $r = 3\cos\theta$ and outside $r = 1 + \cos\theta$

7. The area of the region inside the polar curve $r = 4\cos\theta$ and outside the polar curve $r = 2$ is given by

a) $\frac{1}{2}\int_0^{\pi}\left(4\cos\theta - 2\right)^2 d\theta$ b) $\frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(4\cos\theta - 2\right)^2 d\theta$

c) $\frac{1}{2}\int_0^{\pi}\left(16\cos^2\theta - 4\right)d\theta$ d) $\frac{1}{2}\int_0^{\pi/3}\left(16\cos^2\theta - 4\right)d\theta$

e) $\int_0^{\pi/3}\left(16\cos^2\theta - 4\right)d\theta$

8. Enclosed by the inner loop of $r = 2 + 3\cos\theta$

9. Inside both $r = 4\sin\theta$ and $r = 2$

10. Inside both $r = 1 + \cos\theta$ and $r = 3\cos\theta$

11. Inside both $r = \sec\theta - 2\cos\theta$ and $r = \frac{1}{2}$

12. Inside both $r = 4\cos 4\theta$ and $r = 2$

13. Inside both $r^2 = 6\cos 2\theta$ and $r = \sqrt{3}$

9.6: Multiple Choice Homework

1. Which of these integrals represents the total area shared by the graph of both $r = 2\cos\theta$ and $r = 2\sin\theta$?

a) $2\int_0^{\frac{\pi}{4}} (\sin^2\theta) d\theta$

b) $4\int_0^{\frac{\pi}{4}} (\sin^2\theta) d\theta$

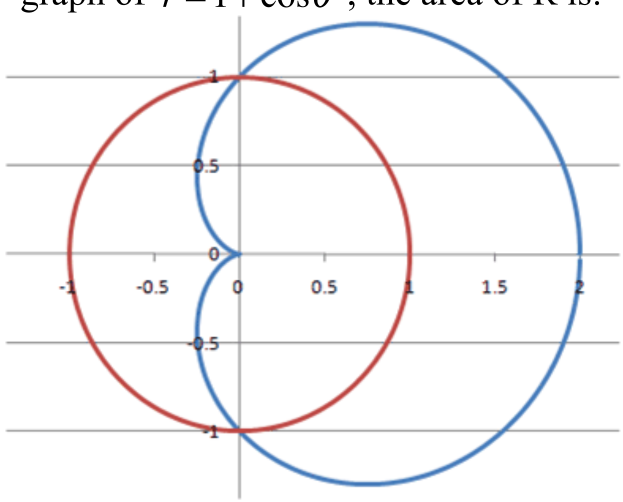
c) $2\int_0^{\frac{\pi}{2}} (\sin^2\theta) d\theta$

d) $4\int_0^{\frac{\pi}{4}} (\cos^2\theta) d\theta$

e) $2\int_0^{\frac{\pi}{4}} (\cos^2\theta - \sin^2\theta) d\theta$

---

2. The graphs of the polar curves $r = 1$ and $r = 1 + \cos\theta$ are shown in the figure below. If R is the region that is inside the graph of $r = 1$ and outside of the graph of $r = 1 + \cos\theta$, the area of R is:



a) 1.127 b) 1.215 c) 1.275 d) 1.235 e) 1.375

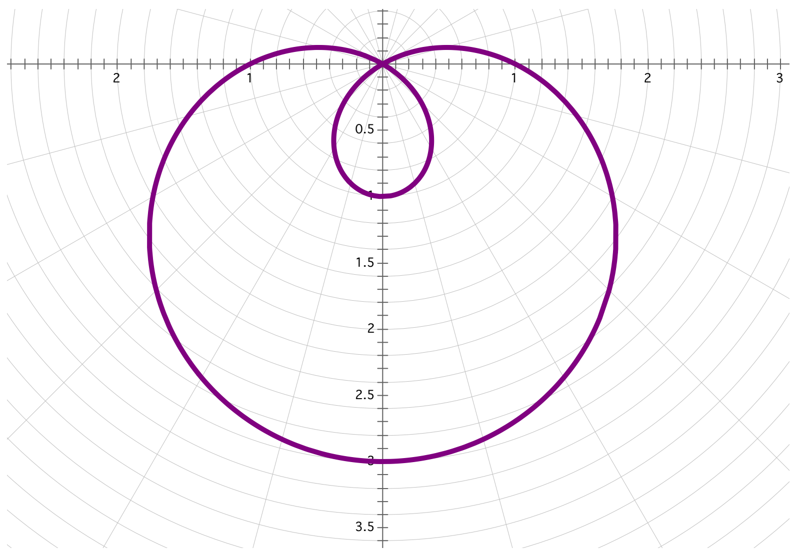
---

3. What is the total area between the polar curves $r = 4\cos(5\theta)$ and $r = 7\cos(5\theta)$?

a) 14.137 b) 7.069 c) 25.918 d) 51.836

---

4. Below is the graph of the polar curve $r = 1 - 2\sin\theta$ on $\theta \in [0, 2\pi]$.



Which of the following integrals represents the area inside the outer loop but outside the inner loop of the curve?

a) $\int_{0}^{2\pi} \frac{1}{2} r^2 \, d\theta$ b) $\int_{\pi/6}^{-11\pi/6} \frac{1}{2} r^2 \, d\theta$

c) $\int_{\pi/6}^{-11\pi/6} \frac{1}{2} r^2 \, d\theta - \int_{5\pi/6}^{\pi/6} \frac{1}{2} r^2 \, d\theta$ d) $\int_{5\pi/6}^{2\pi} \frac{1}{2} r^2 \, d\theta - \int_{0}^{5\pi/6} \frac{1}{2} r^2 \, d\theta$

e) $\int_{5\pi/6}^{\pi/6} \frac{1}{2} r^2 \, d\theta - \int_{\pi/6}^{-11\pi/6} \frac{1}{2} r^2 \, d\theta$

---

5. What is the area inside the polar curve $r=\sqrt{3}\sin\theta$ and outside $r=1+\cos\theta$?

a) $\frac{\sqrt{3}}{4}$ b) $\frac{3\sqrt{3}}{4}$ c) $\frac{3\sqrt{3}}{2}$ d) $\frac{3\sqrt{3}}{8}$

---

6. What is the area inside the polar curve $r=-6\cos\theta$ and outside $r=3$?

a) $\frac{9\pi}{2}$ b) 3π c) $\frac{3}{2}\left(4\pi-3\sqrt{3}\right)$ d) $\frac{3}{2}\left(4\pi-3\sqrt{3}\right)$

---

7. What is the area inside the outer loop and inside the outer loop of the polar curve $r=-3+6\cos\theta$?

a) $3\left(2-\sqrt{2}\right)$ b) $\frac{9}{2}\left(4\pi+3\sqrt{3}\right)$ c) $9\left(\pi+3\sqrt{3}\right)$ d) 9π

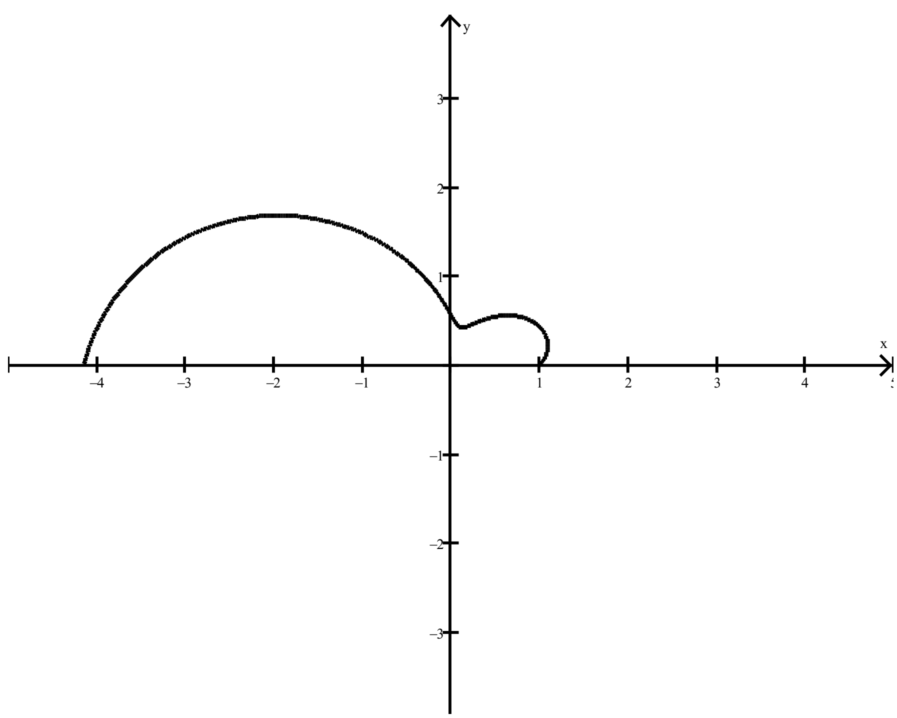
9.7: Derivatives and Polar Graphs

Polar coordinates can be considered a form of parametric mode. θ is the parameter (like *t*) which not only determines *r*, but , due to the conversions, can determine (x, y). When considered this way, there are two derivatives of particular interest with polar coordinates: $\frac{dr}{d\theta}$ and $\frac{dy}{dx}$.

> OBJECTIVES
>
> Determine and interpret intervals of increasing or decreasing of a polar curve.
> Find slopes of lines tangent to polar curves.

The derivative $\frac{dr}{d\theta}$ is the rate of change of *r* in terms of θ. We consider maximums and minimums, as well as intervals of increasing and decreasing, by looking at $\frac{dr}{d\theta}$. But the interpretation is not of high and low points. Rather, the extremes would be points furthest and nearest the origin/pole.

Ex 1 $r = \theta + \cos 2\theta$ on $\theta \in [0, \pi]$. What are the points closest and furthest from the origin?



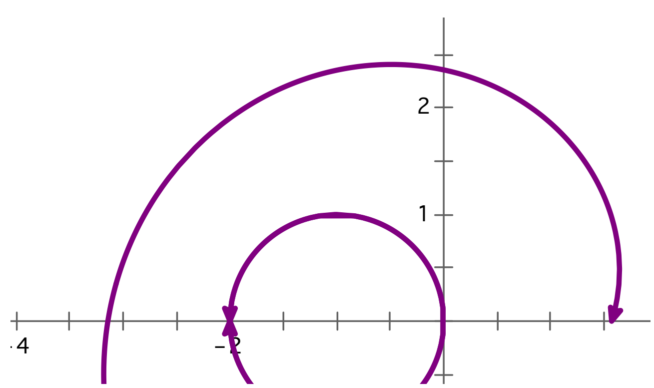
$$\frac{dr}{d\theta}=1-2\sin 2\theta=0$$
$$\sin 2\theta=\frac{1}{2}$$
$$2\theta=\begin{cases}\frac{\pi}{6}\pm 2\pi n\\ \frac{5\pi}{6}\pm 2\pi n\end{cases}$$
$$\theta=\frac{\pi}{12}\text{ and }\frac{5\pi}{12}$$

Don't forget the endpoints: $\theta=0$ and π. Now we need to find the *r*-values.

$$r(0)=1$$
$$r\left(\frac{\pi}{12}\right)=1.128$$
$$r\left(\frac{5\pi}{12}\right)=.443$$
$$r(\pi)=3$$

So the point nearest the pole is $\left(.443,\frac{5\pi}{12}\right)$ and the point furthest from the pole is $(3,\pi)$.

Ex 2 Consider the graphs of $r_1=\frac{1}{2}(\theta+\pi)$ and $r_2=-2\cos\theta$ on $x\in[0,\pi]$. For any specific value of θ, what is the closest the two graphs are to one another?



$$\begin{aligned} \text{distance} &= r_1 - r_2 \\ &= \frac{1}{2}(\theta + \pi) - (-2\cos\theta) \\ &= \frac{1}{2}\theta + \frac{\pi}{2} + 2\cos\theta \end{aligned}$$

$$\frac{d}{d\theta}\left(\frac{1}{2}\theta + \frac{\pi}{2} + 2\cos\theta\right) = \frac{1}{2} - 2\sin\theta = 0$$

$$\sin\theta = \frac{1}{4}$$

$$\theta = \sin^{-1}\frac{1}{4} = \begin{cases} .253 \pm 2\pi n \\ 2.889 \pm 2\pi n \end{cases}$$

θ	$r_1 - r_2$
0	3.571
0.253	3.512
2.889	1.045
π	1.142

The nearest they come is 1.045 units.

Tangent lines

The derivative $\frac{dy}{dx}$ is still the slope of the tangent line. As we saw in the first section, $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$. With θ serving as *t*, $\frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)}$. This leads to this formula:

$$\frac{dy}{dx}=\frac{r\cos\theta+\frac{dr}{d\theta}\sin\theta}{-r\sin\theta+\cos\theta\frac{dr}{d\theta}}$$

Often, it is easier to convert the polar equation to Cartesian in order to find $\frac{dy}{dx}$.

Ex 3 Find the Cartesian equation of the line tangent to the curve in $r=\theta+\cos 2\theta$ at $\theta=\frac{\pi}{3}$.

First we need the point $(x,\,y)$:

$$x=r\cos\theta=(\theta+\cos 2\theta)\cos\theta\Rightarrow x\left(\frac{\pi}{3}\right)=.274$$
$$y=r\sin\theta=(\theta+\cos 2\theta)\sin\theta\Rightarrow y\left(\frac{\pi}{3}\right)=.474$$

Then we need the slope of the tangent line:

$$\frac{dy}{dx}=\frac{\frac{dr}{d\theta}\sin\theta+r\cos\theta}{\cos\theta\frac{dr}{d\theta}-r\sin\theta}$$
$$=\frac{(1-2\sin 2\theta)\sin\theta+(\theta+\cos 2\theta)\cos\theta}{\cos\theta(1-2\sin 2\theta)-(\theta+\cos 2\theta)\sin\theta}$$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}}=\frac{\left(1-2\sin 2\left(\frac{\pi}{3}\right)\right)\sin\left(\frac{\pi}{3}\right)+\cos\left(\frac{\pi}{3}\right)\left(\frac{\pi}{3}+\cos 2\left(\frac{\pi}{3}\right)\right)}{\cos\left(\frac{\pi}{3}\right)\left(1-2\sin 2\left(\frac{\pi}{3}\right)\right)-\left(\frac{\pi}{3}+\cos 2\left(\frac{\pi}{3}\right)\right)\sin\left(\frac{\pi}{3}\right)}$$
$$=.429$$

And the tangent line is:

$$y-.474=.429(x-.274)$$

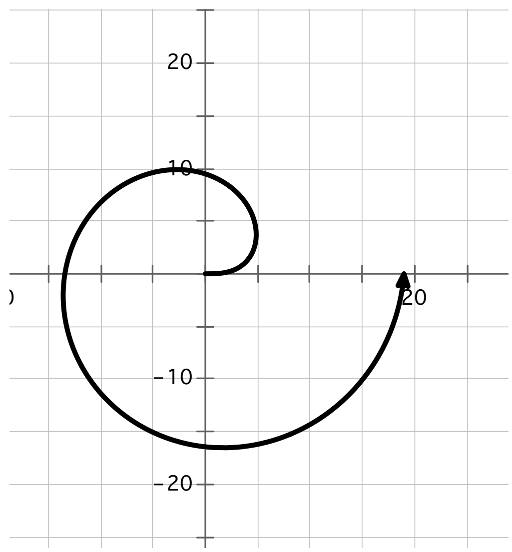
Ex 2 might have been easier to do in either Polar coordinates, rather than mixing the two together. A line in Polar form is $r=a\sec(\theta-\alpha)$. $r\left(\frac{\pi}{3}\right)=.547$ represents the a and α. So the answer in polar form is

$$r=.547\sec\left(\theta-\frac{\pi}{3}\right)$$

Ex 4 Let R be the region bounded by $r=\sqrt{\theta}$ on $\theta\in[0,2\pi]$.

a. Sketch and shade the region R.
b. Find the area of R.
c. Find the equation of the tangent line at $\theta=\frac{\pi}{2}$.

a.



b. $$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \; d\theta$$
$$= \frac{1}{2}\int_{0}^{2\pi} \left(\sqrt{\theta}\right)^2 \; d\theta$$
$$= \frac{1}{2}\int_{0}^{2\pi} \theta \; d\theta$$
$$= \frac{\theta^2}{4}\Bigg]_0^{2\pi} = 4\pi^2$$

c. $$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}$$
$$= \frac{1}{2\sqrt{\theta}} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}$$

Ex 4 BC2014 #2

9.7 Free Response Homework

Find the points closest to and furthest from the pole (points that are the relative extremes of *r*) for the given domain.

1. $r = 1 + 2\sin\left(\frac{\theta}{2}\right)$ on $\theta \in [0, 2\pi]$

2. $r = \sqrt{1 + .8\sin^2\theta}$ on $\theta \in [0, 2\pi]$

3. $r = 5(1 + \cos\theta)$ on $\theta \in [0, 2\pi]$

4. $r = \frac{1}{2}(\theta + \pi)$ on $\theta \in [0, 2\pi]$

5. $r = \theta + \sin 3\theta$ on $\theta \in [\pi, 2\pi]$

6. $r = \sin\theta + \cos 2\theta$ on $\theta \in [0, 2\pi]$

Find the slope of the line tangent to the given equation at the given value of θ

7. $r = \frac{1}{\theta}$ at $\theta = \pi$

8. $r = Ln\ \theta$ at $\theta = e$

9. $r = \sin 3\theta$ at $\theta = \frac{\pi}{6}$

10. $r = \sin\theta + \cos\theta$ at $\theta = \frac{\pi}{4}$

Find the points where the tangent line is (a) horizontal and (b) vertical for the domain $\theta \in [0, 2\pi]$.

11. $r = 3\cos 2\theta$

12. $r = \sin\theta + \cos\theta$

13. $r = e^{\theta}$

14. Explain the difference between $\frac{dy}{dx}$, $\frac{dy}{dt}$, $\frac{dy}{dt}$, and $\frac{dr}{d\theta}$.

15. 2003BC # 3

16. 2005BC # 2

17. 2007BC # 3

18. 2013BC # 2

9.7: Multiple Choice Homework

1. What is the slope of the line tangent to the polar curve $r = -1 + 2\sin\theta$ at the point where $\theta = \frac{\pi}{4}$?

a) $-1+\sqrt{2}$ b) $\sqrt{2}$ c) $\frac{-1+\sqrt{2}}{\sqrt{2}}$ d) $-1+2\sqrt{2}$

---

2. What is the slope of the line tangent to the polar curve $r = 6(1+\cos\theta)$ at the point where $\theta = \frac{\pi}{2}$?

a) $2\sqrt{2}$ b) -2 c) 1 d) $\frac{\sqrt{3}}{3}$

---

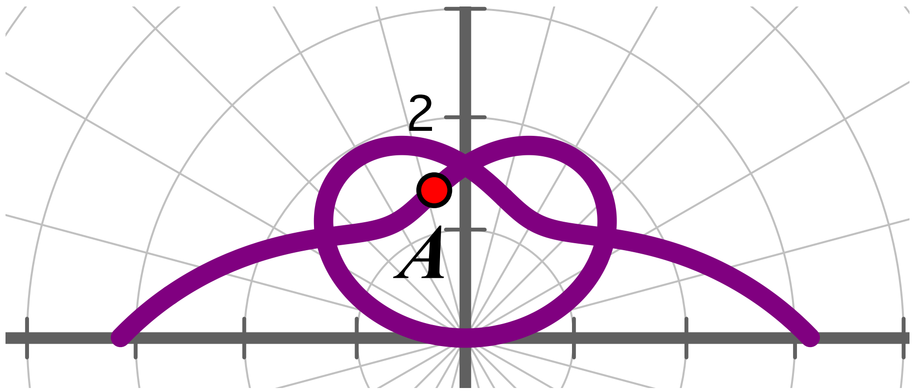
3. The function $g(\theta) = 2\theta^3 - 15\theta^2 + 36\theta$ satisfies $g(\theta) \geq 0$ for $\theta \geq 0$. During the time interval $0 \leq t \leq 2\pi$ seconds, a particle moves along the polar curve $r = g(\theta)$ so that at time *t* seconds, $\theta = t$. On what intervals of time *t* is the distance between the particle and the origin increasing?

a) $0 \leq t \leq 3$ b) $0 \leq t \leq 2\pi$

c) $2 \leq t \leq 3$ d) $0 \leq t \leq 2$ and $3 \leq t \leq 2\pi$

---

4. The graph below is $r = \theta + \sin 2\theta$ on $\theta \in [-\pi, \pi]$.



Which is true at point A of the polar graph shown?

a) $\frac{dy}{dx} = \frac{dr}{d\theta}$ b) $\frac{dy}{dx} > \frac{dr}{d\theta}$ c) $\frac{dy}{dx} < \frac{dr}{d\theta}$ d) not enough information

---

5. A completely artificial coordinate system called Maychrowitzian defines points as (μ, Q), where $x = Q\cot\mu$ and $y = Q\sin\mu$. Which of the following would be the Maychrowitzian form of the slope of the line tangent to $Q = 3 + \csc\mu$?

a) $(3+\csc\mu)\cos\mu$

b) $\dfrac{\cot\mu + (3+\csc\mu)\cos\mu}{(3+\csc\mu)\csc^2\mu + \cot^2\mu\csc\mu}$

c) $\dfrac{(3+\csc\mu)\csc^2\mu + \cot^2\mu\csc\mu}{\cot\mu - (3+\csc\mu)\cos\mu}$

d) $\dfrac{\cot\mu - (3+\csc\mu)\cos\mu}{(3+\csc\mu)\csc^2\mu + \cot^2\mu\csc\mu}$

e) $\dfrac{(3+\csc\mu)\csc^2\mu + \cot^2\mu\csc\mu}{\cot\mu + (3+\csc\mu)\cos\mu}$

---

6. What is the slope of the line tangent to the polar curve $r = 5\cos 3\theta$ at the point where $\theta = \frac{\pi}{3}$?

a) $\sqrt{3}$ b) $\frac{\sqrt{3}}{3}$ c) $-\frac{\sqrt{3}}{3}$ d) $-\sqrt{3}$

9.8: Intro to AP: Polar Coordinate FRQs

OBJECTIVES

Determine and interpret intervals of increasing or decreasing of a polar curve.
Find slopes of lines tangent to polar curves.

Key Ideas:

Know the formulas.
r-values can now be negative.
Know the conversion formulas.
Know the basic shapes and their equations.
Remember the unit circle.
The hardest part of the problem is finding the boundaries.
The boundaries are usually the points of intersections
Think about the derivatives as rates of change and consider what is changing in terms of what.

Common Sub-Topics:

- Area
- Conversions between polar, cartesian, and parametric
- Change in distance
- Interpretation of derivatives in one or more coordinate systems
- Average Value
- Use of Calculator

Formulas:

Conversions:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

$$r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \pm\cos^{-1}\frac{x}{r}$$

Arc Length: $L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$

Slope: $\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \cos\theta\frac{dr}{d\theta}}$

Polar Area Summary

0. Always draw the region first (not just graph on the calculator).

1. Area inside one curve
   - Find the boundaries
   - Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$

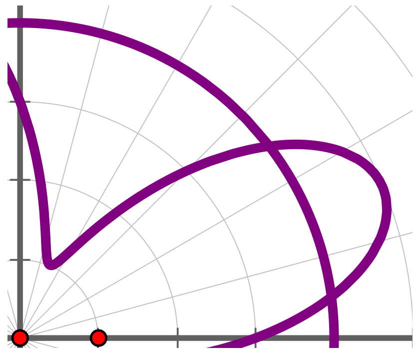
2. Area Inside Two Curves
   - **Break the picture into two regions and make two problems**
   - Find the boundaries for each region
     - There are likely to be different boundaries for the two regions
   - Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$ to each problem

3. Area Inside One Curve and Outside another

   - Find the boundaries
     - These are usually the points of intersection, but they could involve $r = 0$ or $\theta = 0$
   - Apply the formula $A = \int_{\alpha}^{\beta} \frac{1}{2}(\mathit{Outer}\ r)^2\ d\theta - \int_{\alpha}^{\beta} \frac{1}{2}(\mathit{Inner}\ r)^2\ d\theta$ to each problem

The derivative $\frac{dr}{d\theta}$ is the rate of change of r in terms of θ. We consider maximums and minimums, as well as intervals of increasing and decreasing, by looking at $\frac{dr}{d\theta}$. But the interpretation is not of high and low points. Rather, the extremes would be points furthest and nearest the origin/pole.

EX 1 The graph below shows the curves $r_1 = 3 + 2\sin 4\theta$ and $r_2 = 4$ in Quadrant I.



a. Find the points of intersection of the two curves. Show the algebra.

b. Set up, but do not evaluate, an expression involving one or more integrals that would find the area inside **both** $r_1 = 3 + 2\sin 4\theta$ and $r_2 = 4$ in Quadrant 2.

c. Find the value of θ that corresponds to the point on $r_1 = 3 + 2\sin 4\theta$ that is closest to the pole. Justify your answer.

d. For the curve r_1, write an expression for $\frac{dy}{dx}$ in terms of θ.

---

a. Find the points of intersection of the two curves. Show the algebra.

$$3+2\sin 4\theta = 4$$
$$\sin 4\theta = \frac{1}{2}$$
$$4\theta = \begin{cases} \pi/6 \pm 2\pi n \\ 5\pi/6 \pm 2\pi n \end{cases}$$
$$\theta = \begin{cases} \pi/24 \pm \frac{\pi}{2}n \\ 5\pi/24 \pm \frac{\pi}{2}n \end{cases}$$
$$\theta = \pi/24,\ 5\pi/24$$

---

b. Set up, but do not evaluate, an expression involving one or more integrals that would find the area inside **both** $r_1 = 3+2\sin 4\theta$ and $r_2 = 4$ in Quadrant 2.

$$A = \int_0^{\pi/24} \frac{1}{2}(3+2\sin 4\theta)^2 d\theta + \int_{\pi/24}^{5\pi/24} \frac{1}{2}(4)^2 d\theta + \int_{5\pi/24}^{\pi/2} \frac{1}{2}(3+2\sin 4\theta)^2 d\theta$$

---

c. Find the value of θ that corresponds to the point on $r_1 = 3+2\sin 4\theta$ that is closest to the pole. Justify your answer.

$$\frac{dr}{d\theta} = 8\cos 4\theta = 0 \rightarrow \cos 4\theta = 0$$
$$4\theta = \cos^{-1} 0 = \pm\frac{\pi}{2} \pm 2\pi n \rightarrow \theta = \pm\frac{\pi}{8} \pm \frac{\pi}{2}n \rightarrow \theta = \frac{\pi}{8} \text{ and } \frac{3\pi}{8}$$

θ	r
0	3
$\frac{\pi}{8}$	5
$\frac{3\pi}{8}$	1
$\frac{\pi}{2}$	3

$\left(\frac{3\pi}{8}, 1\right)$ is closest to the pole.

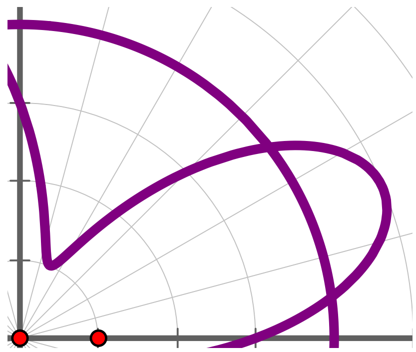
---

d. For the curve r_1, write an expression for $\frac{dy}{dx}$ in terms of θ.

$$\frac{dy}{dx} = \frac{r\cos\theta + \frac{dr}{d\theta}\sin\theta}{-r\sin\theta + \frac{dr}{d\theta}\cos\theta} = \frac{(3+2\sin 4\theta)\cos\theta + (8\cos 4\theta)\sin\theta}{-(3+2\sin 4\theta)\sin\theta + (8\cos 4\theta)\cos\theta}$$

9.8 Free Response Homework

1. The graph below shows the curves $r_1 = 3 + 2\sin 4\theta$ and $r_2 = 4$in Quadrant I.

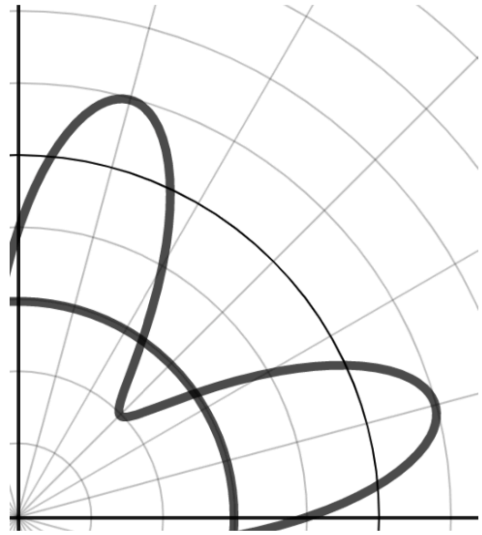


(a) Find the points of intersection of the two curves. Show the algebra.

(b) Set up, but do not evaluate, an expression involving one or more integrals that would find the area inside **both** $r_1 = 3 + 2\sin 4\theta$ and $r_2 = 4$ in Quadrant 2.

(c) Find the value of θ that corresponds to the point on $r_1 = 3 + 2\sin 4\theta$ that is closest to the pole. Justify your answer.

(d) For the curve r_1, write an expression for $\dfrac{dy}{dx}$ in terms of θ.



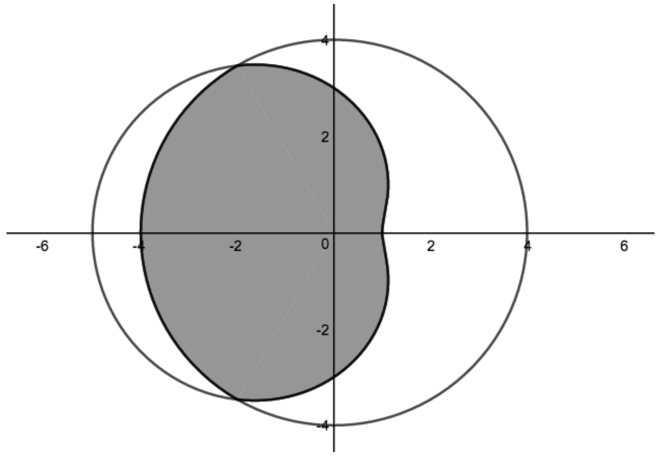
2. The graph below shows the curves $r_1 = 4 + 2\sin(6\theta)$ and $r_2 = 3$ in Quadrant I.

(a) Find the points of intersection of the two curves in Quadrant 1. Show the algebra/trig.

(b) Set up, but do not evaluate, an expression involving one or more integrals that would find the area inside **both** $r_1 = 4 + 2\sin(6\theta)$ and $r_2 = 3$ in Quadrant 1.

(c) Find the value of θ that corresponds to the point on $r_1 = 4 + 2\sin(6\theta)$ that is closest to the pole in Quadrant 1. Justify your answer.

(d) For the curve $r_1 = 4 + 2\sin(6\theta)$, find all points in Quadrant 1 where the tangent line would be horizontal.

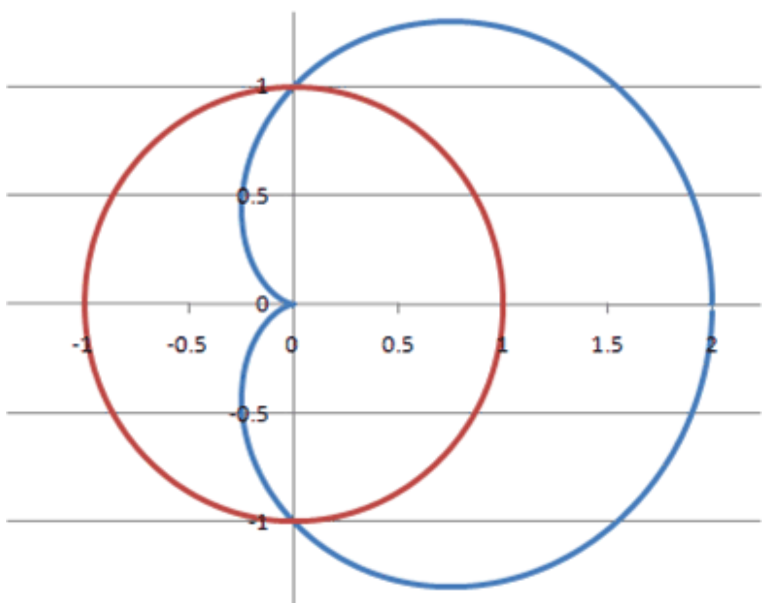


3. The polar graphs $r = 3 - 2\cos\theta$ and $r = 4$ are shown below.

(a) Find the exact values of points of intersection (r, θ) of the two curves.

(b) Find the area of the shaded region.

(c) A particle travels along the curve $r = 3 - 2\cos\theta$ with time $t = \theta$. Write the particle's position vector $\langle x(t), y(t) \rangle$ and use it to find the speed of the particle when $t = 1$.



4. The graphs of the polar curves $r = 1$ and $r = 1 + \cos\theta$ are shown in the figure above.

(a) Find the area of the region inside the graph of $r = 1$ and outside of the graph of $r = 1 + \cos\theta$.

(b) Find the area of the region inside the graph of both $r = 1$ and $r = 1 + \cos\theta$.

(c) Find the coordinate points (r, θ) on $r = 1 + \cos\theta$ where the tangent line is horizontal.

---

5. Let R be the region inside the graph of the $r_1 = 2$ and outside the graph of $r_2 = 2\cos(3\theta)$

(a) Sketch and shade the region R.

(b) Find the area of R.

(c) There are two points on the graph of $r_2 = 2\cos(3\theta)$where the *x*-coordinate is 1. Find the y-coordinates of those two points. Is *r* increasing or decreasing at those points?

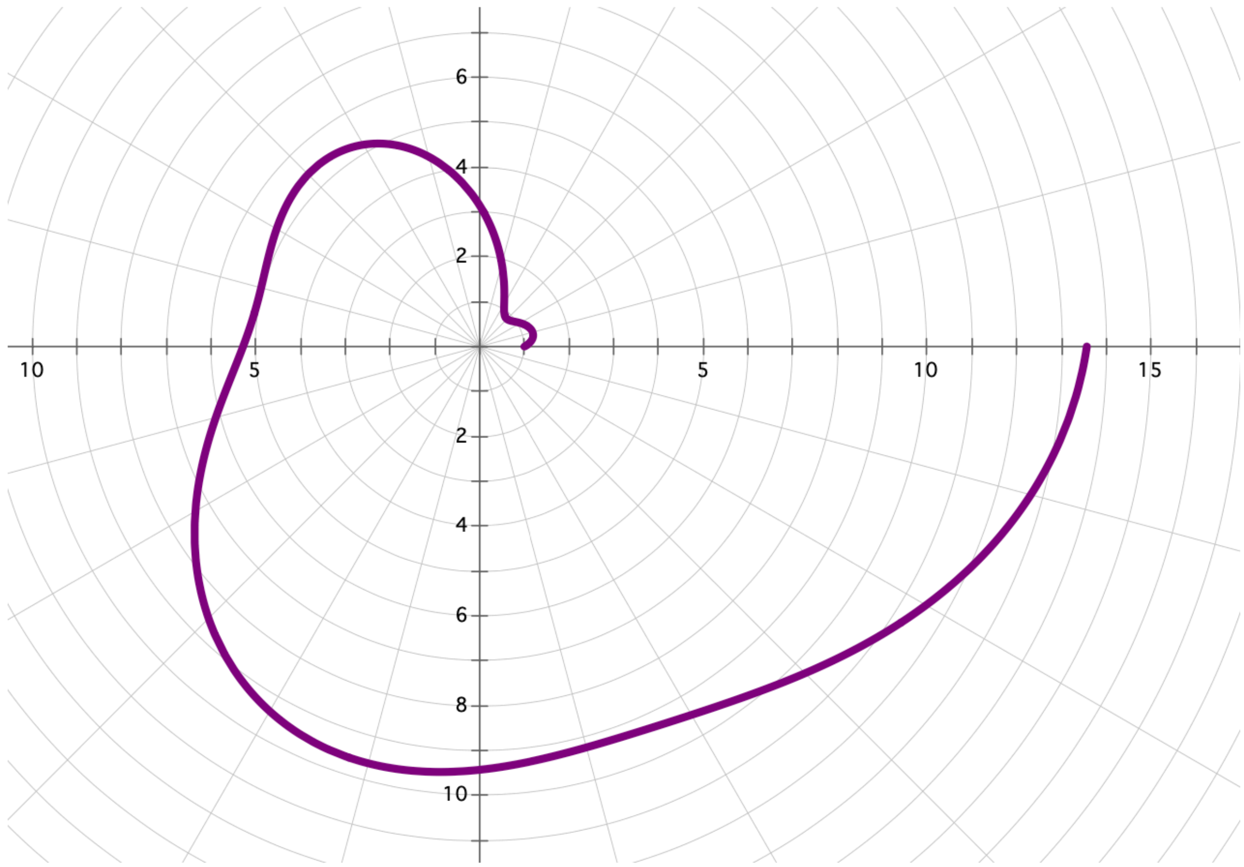
---

6. Let R be the region inside $r_1 = 2$ and outside $r_2 = 2(1 - \sin\theta)$.

(a) Sketch and shade the region R.

(b) Find the area of R.

(c) There are two points on the graph of r = 2(1- s i n) where the x-coordinate is 1. Find the *y*-coordinates of those two points. Is *r* increasing or decreasing at those points?



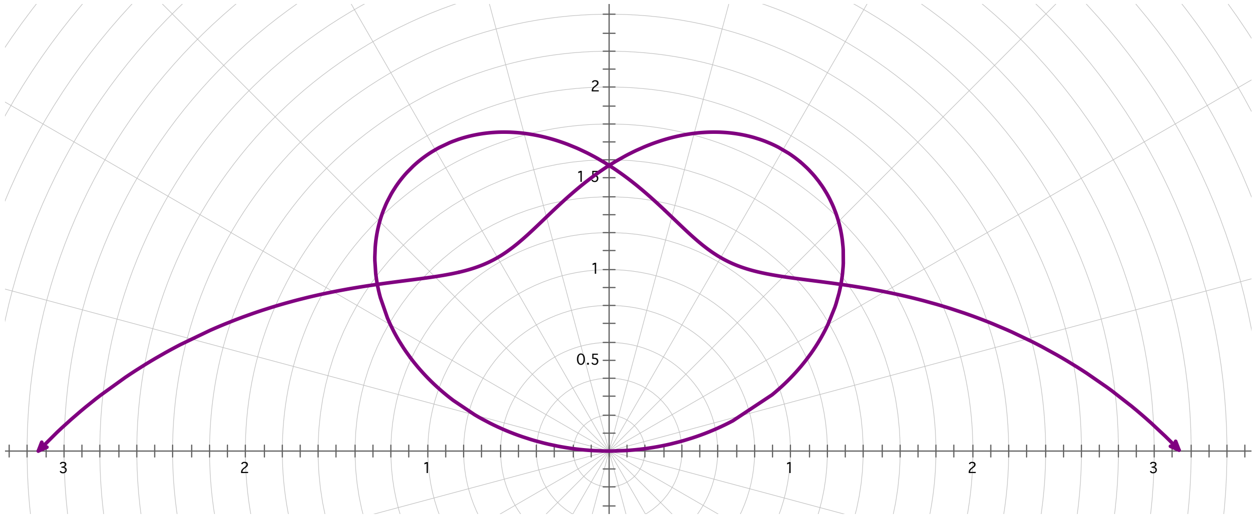
7. The graph of the polar curve $r(\theta) = 2\theta + \cos(3\theta)$ is shown above.

(a) Find the area in the third quadrant enclosed by the coordinate axes and the graph of *r*.

(b) Find the point *P* on the curve *r* with x-coordinate 3. Find the angle θ that corresponds to the point *P*. Show the work that leads to your answers.

(c) A particle travels along *r* so that its position at time *t* is $x(t), y(t)$ and the angle θ increases at π radians per second. Find $\frac{dy}{dt}$. Interpret the meaning of your answer in the context of this problem.

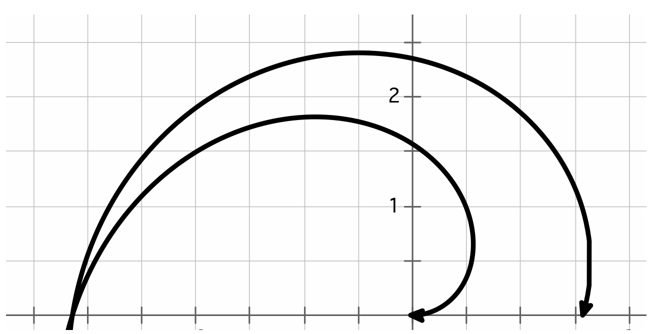
8. The graph below is $r = \theta + \sin 2\theta$ on $\theta \in [-\pi, \pi]$.



(a) Find the length of the curve.

(b) Assume the point where the curve intersects itself in QI has $\theta = -2.487$. Find the area of the region enclosed by $r = \theta + \sin 2\theta$ from $\theta = -\pi$ to the point of intersection.

(c) Find the formula for the slope of the line tangent to $r = \theta + \sin 2\theta$ in terms of θ.



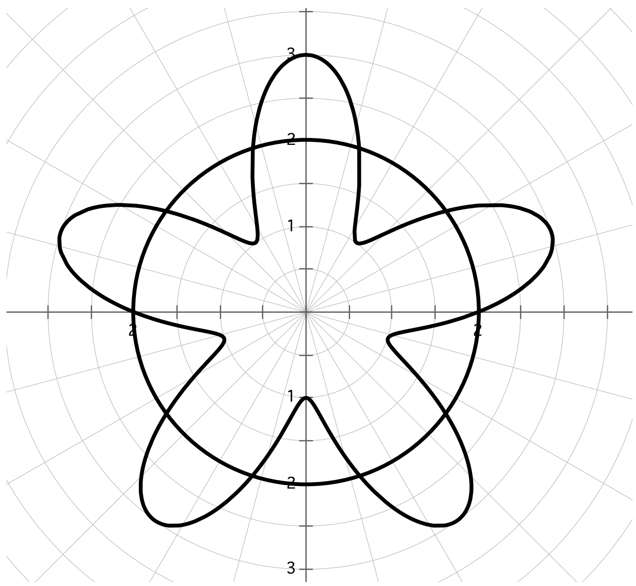
9. Consider the region R bounded by $r_1 = \theta$ and $r_2 = \frac{1}{2}(\theta + \pi)$ on $\theta \in [0, \pi]$.

(a) Find the area of region R.

(b) Let $\theta = k$ be the radial line that splits region R into two equal areas. Set up, but do not solve, an integral expression that would solve for k.

(c) For all $\theta \in [0, \pi]$, let $f(\theta)$ be the distance between points on the two curves. Find the average value of $f(\theta)$ on $\theta \in [0, \pi]$.

(d) Find the perimeter of region R.



10. The graphs of $r_1 = 2 + \sin(5\theta)$ and $r_2 = 2$ are shown above.

(a) What is the total area outside r_1 and inside r_2?

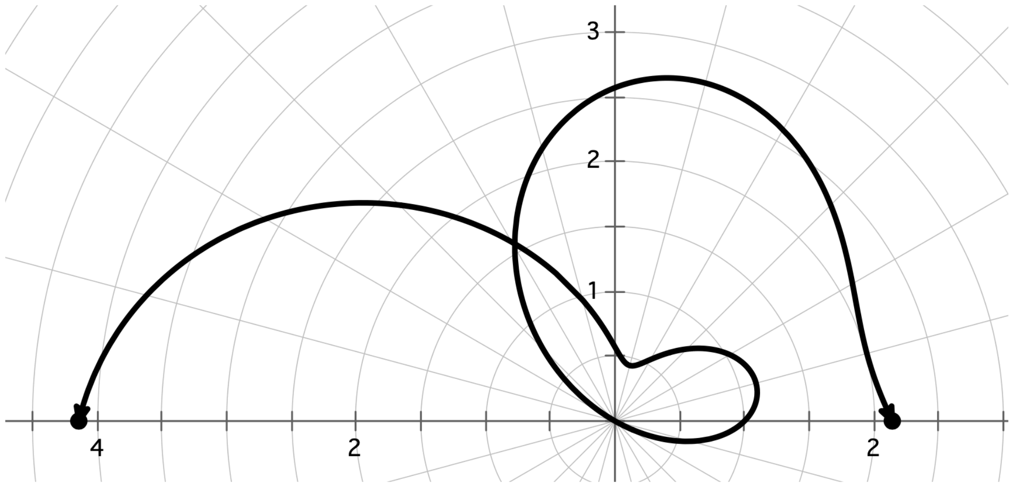
(b) Assume the point where the curve intersects itself in QI has $\theta = -2.487$. Find the area of the region enclosed by $r = \theta + \sin 2\theta$ from $\theta = -\pi$ to the point of intersection.

(c) Find the formula for the slope of the line tangent to $r = \theta + \sin 2\theta$ in terms of θ.

---

11. The graph below is $r = \theta + \cos 2\theta$ on $\theta \in [-\pi, \pi]$

$$\theta + \cos 2\theta = -1.547 \to \theta = -0.671$$



(a) What are the points in the interval $\theta \in (-\pi, 0)$ (beside the pole) that are furthest and closest to the origin?

(b) If the curve crosses itself at $r = -1.547$ and 1.574, find the area of the region enclosed by the loop.

(c) Find the y-coordinate of the point(s) on where $x=-3$. Is r increasing or decreasing at that/those points?

---

12. BC2003 form B # 3

13. BC2005 # 2

14. BC2007 # 3

15. BC2013 # 2

16. BC2019 # 2

Parametric and Polar Practice Test

1. Give the length of the curve determined by $\left\{ \begin{array}{c} x = 3t^2 \\ y = t^3 + 4t \end{array} \right\}$ on $t \in [0, 2]$.

a) 20.145 b) 20.119 c) 20.127

d) 20.135 e) 20.154

---

2. The area inside one petal of the polar graph $r = 4\sin 2\theta$ is

a) 2π b) 4π c) 8π d) π e) 2

---

3. The equation of the line tangent to $\left\{ \begin{array}{c} x = e^t \\ y = t^2 + 5t \end{array} \right\}$ at $t = 0$ is

a) $y - 1 = 5x$ b) $y = \dfrac{5(x-1)}{e}$ c) $y - 1 = 5(x-1)$

d) $y - 1 = 5x - 5$ e) $y = 5x - 5$

---

4. A particle moves according to $\left\{ \begin{array}{c} x'(t) = t^2 \\ y'(t) = \sin \pi t \end{array} \right\}$. If the particle is at (1, 0) when $t = 0$, then the position vector at t = 3 is

a) $\left(9, \dfrac{1}{\pi}\right)$ b) $\left(10, \dfrac{2}{\pi}\right)$ c) $(6, -2\pi)$

d) $(10, 2\pi)$ e) $(10, 2)$

---

5. A particle moves according to $\left\{\begin{array}{l} x = 1 - e^t + t \\ y = t^{5/2} - 2t \end{array}\right\}$. Which of the following statements is true?

I. The particle is moving to the right when t=1

II. The particle is moving up when t=0

III. The particle is at rest when t=1

a) None of these b) I only c) I and II only

d) II and III only e) I, II, and III

---

6. Which of the following integrals gives the length of the graph $y = \sin\sqrt{x}$ between $x = a$ and $x = b$, where $0 < a < b$?

a) $\int_a^b \sqrt{x + \cos^2\sqrt{x}}\,dx$ b) $\int_a^b \sqrt{1 + \cos^2\sqrt{x}}\,dx$ c) $\int_a^b \sqrt{\sin^2\sqrt{x} + \frac{1}{4x}\cos^2\sqrt{x}}\,dx$

d) $\int_a^b \sqrt{1 + \frac{1}{4x}\cos^2\sqrt{x}}\,dx$ e) $\int_a^b \sqrt{\frac{1 + \cos^2\sqrt{x}}{4x}}\,dx$

---

7. Which of these integrals represents the area enclosed by the graph of $r = 1 + \cos\theta$?

a) $\int_0^{\pi} \left(1 + \cos^2\theta\right) d\theta$ b) $\int_0^{\pi} \left(1 + \cos\theta\right)^2 d\theta$

c) $\int_0^{2\pi} \left(1 + \cos\theta\right) d\theta$ d) $\int_0^{2\pi} \left(1 + \cos\theta\right)^2 d\theta$

e) $\frac{1}{2}\int_0^{2\pi} \left(1 + \cos^2\theta\right) d\theta$

---

8. A particle moves in the xy-plane so that at time $t \geq 0$ such that its acceleration vector is $\langle -\pi \sin \pi t,\ 2t+1 \rangle$. If, at $t=0$, it velocity is $\langle 0, 1 \rangle$, then how fast is it moving at t=2?

a) 5 b) 7 c) $\sqrt{37}$ d) $\sqrt{40}$ e) $\sqrt{\pi^4 + 4}$

---

9. If $x(t) = \sin^2 t$ and $y(t) = \cos t$, then $\dfrac{d^2 y}{dx^2}$ at $t = \dfrac{\pi}{2}$ is

a) $\sqrt{2}$ b) $\dfrac{\sqrt{2}}{2}$ c) $-\dfrac{\sqrt{2}}{2}$ d) $-\sqrt{2}$ e) DNE

Parametric and Polar Practice Test Part 2

10. Let R be the region outside $r = 2$ and inside $r = 3\sin 2\theta$ on $\theta \in \left[0, \frac{\pi}{2}\right]$.

a. Sketch and shade the region R.

b. Find the area of R.

c. Find the equation of the tangent line to $r = 2(1 - \sin\theta)$ at $\theta = \frac{\pi}{4}$.

11. The velocity vector of a particle moving in the *xy*-plane has components given by $\frac{dx}{dt} = 14\cos\left(t^2\right)\sin\left(e^t\right)$ and $\frac{dy}{dt} = 1 + 2\sin\left(t^2\right)$ for $0 \le t \le 1.5$.
At time $t = 0$, the position of the particle is (-2, 3).

a. For $0 < t < 1.5$, find all values of t at which the line tangent to the path of the particle is vertical.

b. Write an equation for the line tangent to the path of the particle at $t = 1$.

c. Find the speed of the particle at $t = 1$.

d. Find the acceleration vector of the particle at $t = 1$.

Chapter 9 Answer Key

9.1 Free Response Answer Key

1. (a) $\langle \frac{1}{2}t^{-1/2},\ -\sin t\rangle$, (b) $s(t)=\sqrt{\frac{1}{4t}+\sin^2 t}$, (c) $\langle -\frac{1}{4}t^{-3/2},\ -\cos t\rangle$

2. (a) $\langle 5\cos t,\ 2t\rangle$, (b) $s(t)=\sqrt{25\sin^2 t+4}$, (c) $\langle -5\sin t,\ 2\rangle$

3. (a) $\langle 2t-5,\ -8t\rangle$, (b) $s(t)=\sqrt{68t^2-20t+25}$, (c) $\langle 2,\ -8\rangle$

4. (a) $\langle \sec\theta\tan\theta,\ \sec^2\theta\rangle$, (b) $s(\theta)=\sec\theta\sqrt{\tan^2\theta+\sec^2\theta}$,
(c) $\langle \sec^3\theta+\tan^2\theta\sec\theta,\ 2\sec^2\theta\tan\theta\rangle$

5. $x(5)=-31.333,\ y(5)=16.620$

6. $x(5)=-28.667,\ y(5)=5.981$

7. $x(5)=7.382,\ y(5)=3269016.39$

8. $x(5)=-2.487,\ y(5)=3.770$

9. $x(5)=10.877,\ y(5)=dne$

10. $x(5)=-.051,\ y(5)=2.300$

11. 3.168

12. 11.305

13. 40.431

14. 31.312

15. L=2.958, distance=11.832

16-19. See AP website

9.1: Multiple Choice Answer Key

1.	E	2.	B	3.	B	4.	B	5.	C	6.	C
7.	E	8.	A	9.	E	10.	E				

9.2 Free Response Answer Key

1. $y-1=-2(x-1)$

2. $y-6=\frac{3}{4}(x-19)$

3. $y-1=\frac{-2}{e}(x-e)$

4. $y+\pi=\frac{1}{\pi}x$

5. $\frac{dy}{dx}=\frac{2-3t}{4t^2}$ and $\frac{d^2y}{dx^2}=\frac{3t-4}{16t^6}$.

6. $\frac{dy}{dx}=-2\sin 2t\cos^2 t$ and $\frac{d^2y}{dx^2}=2\cos^2 t\left(\sin^2 2t-4\cos 2t\cos^2 t\right)$.

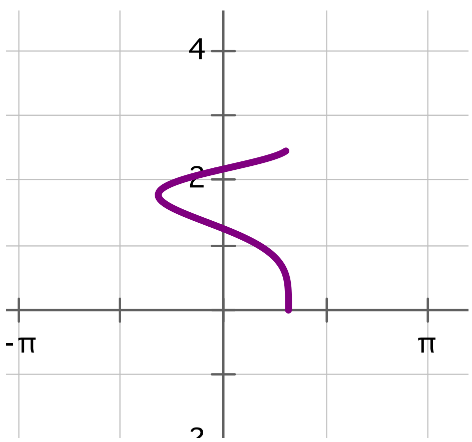
7. $\frac{dy}{dx}=-e^{3t}(2t+1)$ and $\frac{d^2y}{dx^2}=-e^{2t}(6t+5)$.

8. $\frac{dy}{dx}=\frac{1+\ln t}{2t}$ and $\frac{d^2y}{dx^2}=\frac{-\ln t}{4t^3}$.

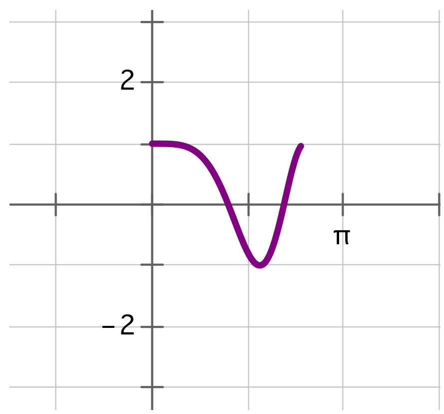
9. Horizontal: $(0,\ -9)$ Vert: $(\pm 2,-6)$

10. Horizontal: $(0,0), \left(\sqrt[3]{2},\ 1.587\right)$ Vert: $\left(\frac{1}{\sqrt[3]{2}},\ 1.260\right)$

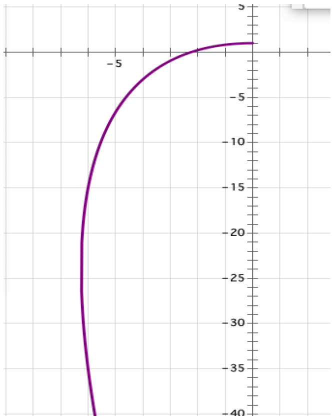
11.



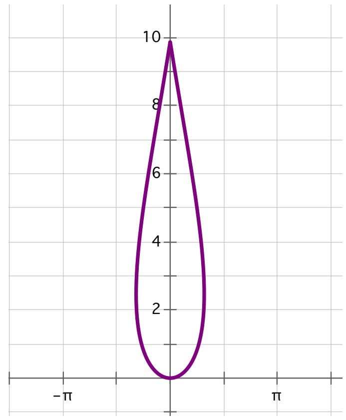
12.



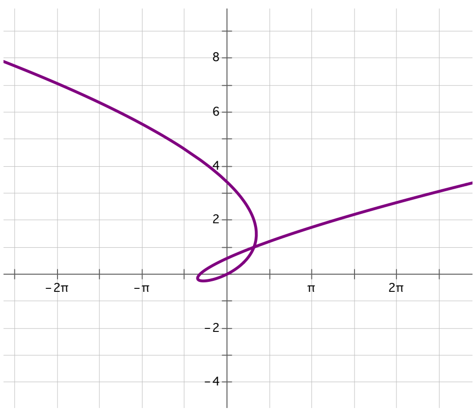
13.



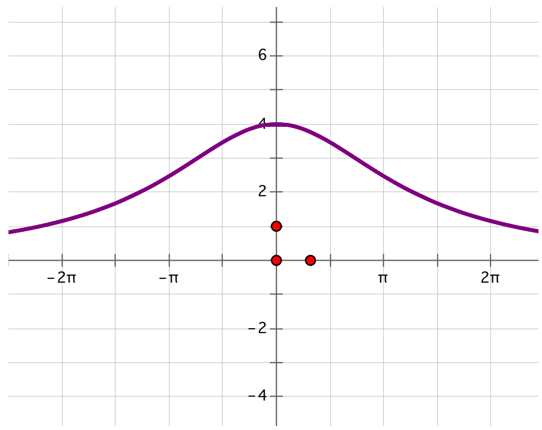
14.



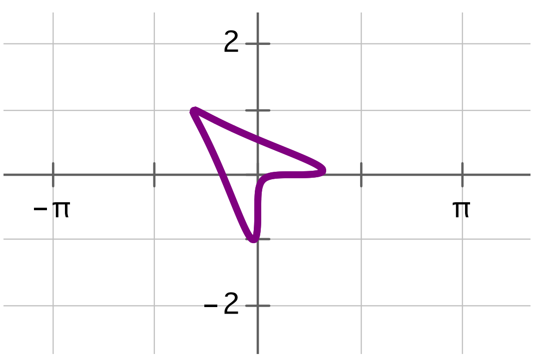
15.



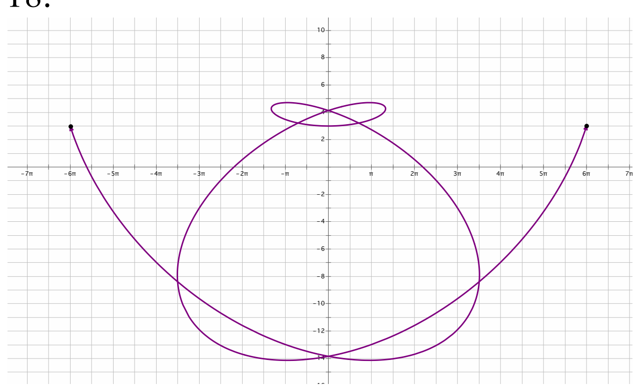
16.



17.



18.



19. $y=\cos x^2$ 20. $y=\dfrac{1}{x}$ 21. $y=e^{1/2x}$ 22. $y=\tan\left(\sec^{-1}x\right)$

23-27. See AP Central

9.2: Multiple Choice Answer Key

1. B 2. C 3. D 4. D 5. B 6. D

7. D 8. B

9.3 Free Response Answer Key

1a) $t=4$ b) 15.133 c) increasing d) 33.866

2a) never b) 0.862 c) increasing d) 8.479

3a) $a(t)=\langle 0.866, 1.890\rangle$ b) $t=2.835$

c) 10.338 d) $\langle -8.238, 3.616\rangle$

4a) 2.647 b) $a(1)=\langle -0.324, 0.132\rangle$

c) 5.641 d) 2.206

5a) $a(1)=\langle -0.250, 0\rangle$ b) $y-3=-8x+2$

c) $t=1.828$ d) 27.018

6a) $a(4)=\langle 0.001, 0.070\rangle$ b) -0.141

c) decreasing d) 16.004

7a) $\langle 73.106, 5.875\rangle$ b) 6.770

c) $(x, y)=(0.887, 6)$ d) 48.297

8a) $\langle 42, 7 \rangle$

b) $(y-1) = \frac{5}{18}(x+2)$

c) $t = -3.661$

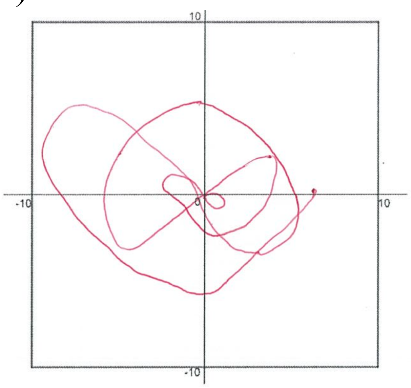
d) 68.938

9a) 13.798

b) 11.693

d) 0.627 seconds

c)



10a) (-3.108, 4.828)

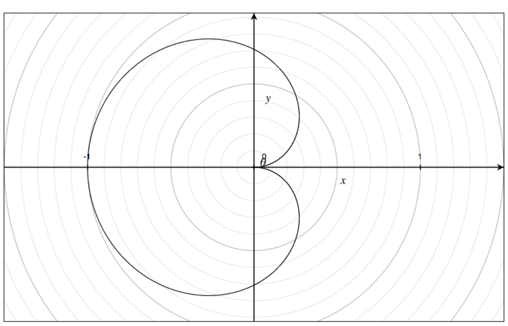
b) 13.785

c) $\left\langle 0.370\ {}^{ft}/_{min^2},\ -77.252\ {}^{ft}/_{min^2} \right\rangle$

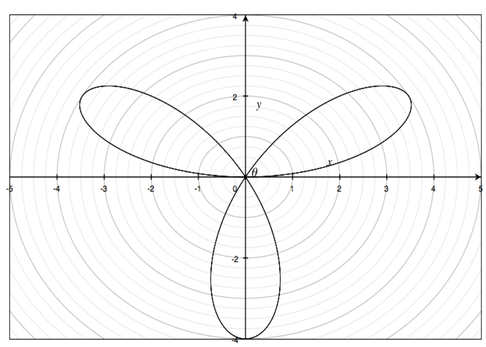
d) $t = \frac{3\pi}{8} = 1.178\ min$

<u>9.4 Free Response Answer Key</u>

1.

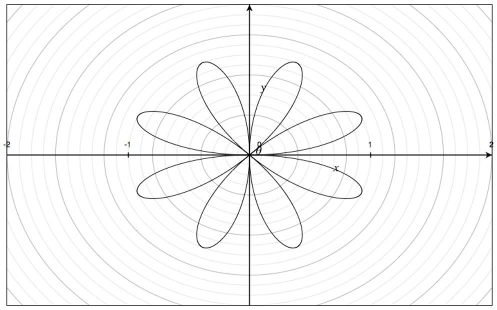


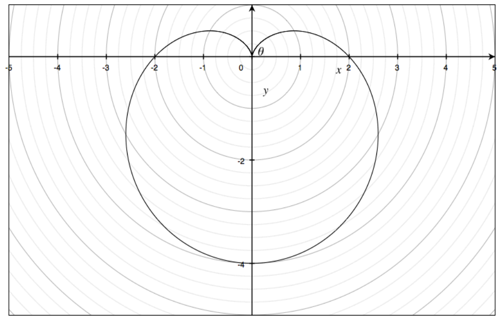
2.



3.

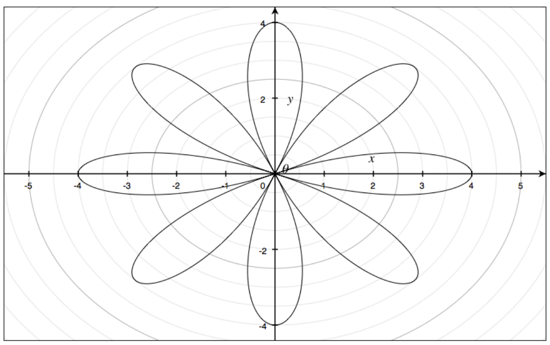
4.

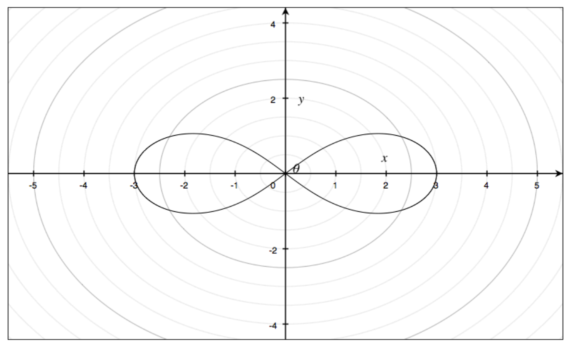




5.

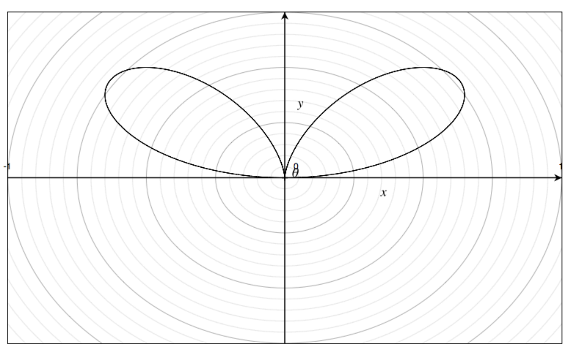
6.

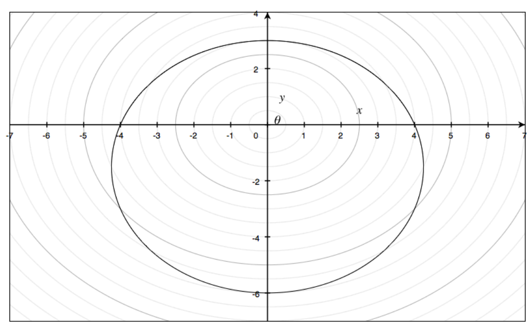




7.

8.





9. $y = 2$

10. $x^2 - 3y^2 + 4y - 1 = 0$

11. $x^4 + 2x^2y^2 + y^4 = 2xy$

12. $r^2 = \sec 2\theta$

13. $r = 4\tan\theta\sec\theta$

14. $r = \dfrac{1}{2\cos\theta - \sin\theta}$

15.	$\theta = \left\{ \frac{7\pi}{12}, \frac{11\pi}{12} \right\}$	16.	$\theta \in \left\{ \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3} \right\}$
17.	$\theta = .301, .643, 3.443, 3.784$	18.	$\theta = .523, .785, 2.356, 2.618$
19.	$\theta = .124, .243, 1.732, 2.440$	20.	$\theta = .735, 1.463, 1.679, 2.446$
21.	$\theta = .355, 2.787, 3.497, 5.928$	22.	$\theta = .169, 2.973, 3.311, 5.526, 6.114$

9.4: Multiple Choice Answer Key

1.	A	2.	C	3.	B	4.	C	5.	D	6.	B
7.	C										

9.5 Free Response Answer Key

1.	$\frac{\pi}{2}$	2.	12.566	3.	$\frac{\pi}{2}$	4.	18.850
5.	8π	6.	9	7.	0.393	8.	35.003
9.	5.168	10.	2.178	11.	0.196	12.	4
13.	11.781	14.	320596.596	15.	134.954		
16.	4.006	17.	8				

9.5: Multiple Choice Answer Key

1.	D	2.	B	3.	D	4.	C	5.	E	6.	E
7.	A	8.	D	9.	C						

9.6 Free Response Answer Key

1. $3\sqrt{3}-\pi$ 2. 7.122 3. $\frac{9\pi}{8}-\frac{5}{4}$ 4. 7.653

5. 5.196 6. 3.142 7. 0.441 8. 0.441

9. 4.913 10. 3.927 11. 0.162 12. 9.827

13. $\pi+3$ 14. B

9.6: Multiple Choice Answer Key

1. B 2. B 3. C 4. C 5. B

9.7 Free Response Answer Key

1. Nearest: $(1, 0), (1, 2\pi)$; Furthest: $(3, \pi)$

2. Nearest: $(1, 0), (1, \pi), (1, 2\pi)$; Furthest: $\left(1.342, \frac{\pi}{2}\right), \left(1.342, \frac{3\pi}{2}\right)$

3. Nearest: $(0, \pi)$; Furthest: $(5, 0), (5, 2\pi)$

4. Nearest: $\left(\frac{\pi}{2}, 0\right)$; Furthest: $\left(\frac{3\pi}{2}, 2\pi\right)$

5. Nearest: $(3.552,\ 2.609)$; Furthest: $(2\pi, 2\pi)$

6. Nearest: $\left(0, \frac{\pi}{2}\right)$; Furthest: $\left(-2, \frac{3\pi}{2}\right)$

7. $\frac{dy}{dx} = -\pi$

8. $\frac{dy}{dx} = 1.019$

9. $\frac{dy}{dx} = -\sqrt{3}$

10. $\frac{dy}{dx} = -1$

11. Vert: $(3, 0), (-1.999, 1.150), (-2.001, 1.991), (3, \pi), (-2.001, 4.292), (-2, 5.133), (3, 2\pi)$
Horiz: $(1.998, .421), \left(-3, \frac{\pi}{2}\right), (1.966, 2.720), (2.001, 3.562), (-3, 4.712), (2.002, 5.863)$

12. Vert: $(1.307, .393), (-.542, 1.963), (-1.306, 3.534), (-.541, 5.105)$
Horiz: $(1.307, 1.178), (-.541, 2.749), (-1.306, 4.320), (.541, 5.890)$

13. Vert: $\left(2.193, \pi/4\right), \left(50.754, 5\pi/4\right)$
Horiz: $\left(10.551, 3\pi/4\right), \left(244.151, 7\pi/4\right)$

14. dy/dx is the rate of change of *y* with respect to *x* - we typically refer to this as the 'slope of the tangent line'
dy/dt is the rate of change of *y* with respect to *t*
dx/dt is the rate of change of *x* with respect to *t*
$dr/d\theta$ is the rate of change of *r* with respect to - refers to how fast the radius of a polar graph is changing as changes.

15-18. See AP Central

9.7: Multiple Choice Answer Key

1. D 2. C 3. D 4. B 5. D 6. B

9.8 Free Response Answer Key

1a) $\theta = \frac{\pi}{24}$ *and* $\frac{5\pi}{24}$

b) $\int_0^{\pi/24} \frac{1}{2}(3+2\sin 4\theta)^2 d\theta + \int_{\pi/24}^{5\pi/24} \frac{1}{2}(4)^2 d\theta + \int_{5\pi/24}^{\pi/2} \frac{1}{2}(3+2\sin 4\theta)^2 d\theta$

c) $\theta = \frac{3\pi}{8}$

(d) $\dfrac{dy}{dx} = \dfrac{(3+2\sin 4\theta)\cos\theta + (8\cos 4\theta)\sin\theta}{-(3+2\sin 4\theta)\sin\theta + (8\cos 4\theta)\cos\theta}$

2a) $\theta = \dfrac{7\pi}{36}$ *and* $\dfrac{11\pi}{36}$

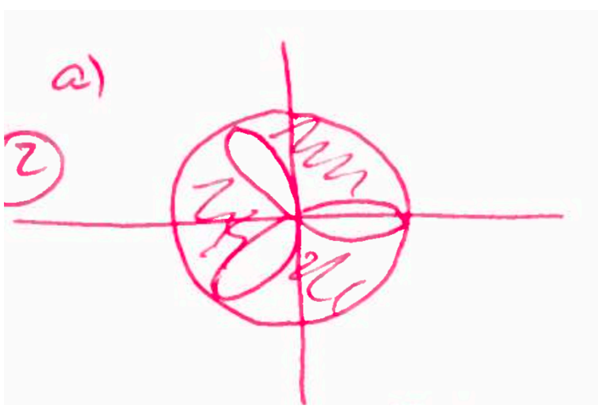
b) $\int_0^{7\pi/36} \frac{1}{2}(3)^2 d\theta + \int_{7\pi/36}^{11\pi/36} \frac{1}{2}(4+2\sin 6\theta)^2 d\theta + \int_{11\pi/36}^{\pi/2} \frac{1}{2}(3)^2 d\theta$

c) $\theta = \dfrac{\pi}{4}$ d) $\theta = 0.440, 0.755, 1.329,$ *or* 1.330

3a) $r\left(\pm\dfrac{2\pi}{3}\right) = 4 \rightarrow \left(\dfrac{2\pi}{3}, 4\right)$ *and* $\left(\dfrac{4\pi}{3}, 4\right)$ b) 28.525

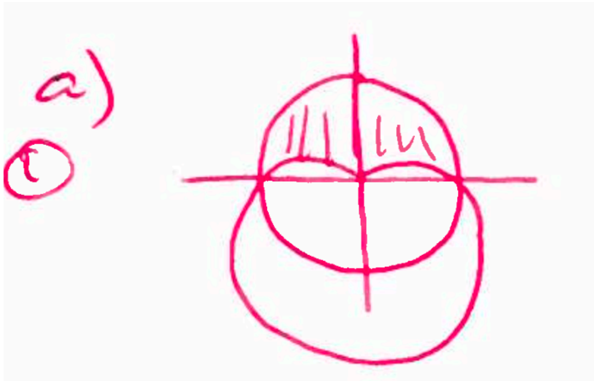
c) 2.553

4a) 1.215 b) 1.927 c) $\left(\pm\dfrac{\pi}{3}, 1.5\right)$

5a) 

b) 6.283

c) decreasing

6a) 

b) 4.858

c) decreasing

7a) 52.850 b) 5.043 c) -16.598

8a) 14.339 b) 1.393 c) $\frac{dy}{dx} = \frac{(\theta + \sin 2\theta)\cos\theta + (1 + 2\cos 2\theta)\sin\theta}{-(\theta + \sin 2\theta)\sin\theta + (1 + 2\cos 2\theta)\cos\theta}$

9a) 14.211 b) 7.105 c) 0.785 d) 15.256

10a) 4.7865 b) 3,215 c) $y - 1.176 = 2.189(x - 1.618)$

11a) $Furthest: \left(-2.880, \frac{-11\pi}{12}\right);\ closest: \left(-2.699, \frac{-7\pi}{12}\right)$

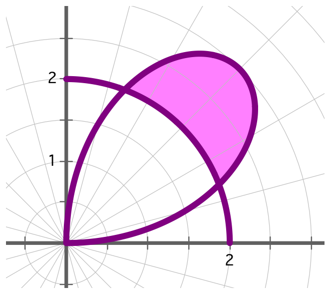
b) $A = \int_{-0.671}^{2.021} \frac{1}{2}\left[(\theta + \cos 2\theta)^2\right] d\theta = 0.857$ c) decreasing

12. – 15. See APCentral.com

Parametric and Polar Practice Test Answer Key

1. D 2. B 3. E 4. B 5. A 6. D

7. B 8. B 9. E

10a. 

10b. 1.328

10c. $y-0.414=0.414(x-0.414)$

11a. $t=-1.144, \quad 1.253$

11b. $y-4.621=0.863(x-9.315)$

11c. 4.105

11d. $\langle -28.425, \quad 2.161 \rangle$