Chapter 8 Overview: Techniques of Integration

Differentiation is a relatively straight-forward process, though it is sometimes tedious. That tedium arises from all the algebraic simplifications. The great difficulty with integration is undoing all that algebra. While there are only 20 or so derivative formulas, there are hundreds of integration formulas, each undoing a different simplification. In this chapter, we will concentrate three general techniques:

- Partial Fractions
- Integration by Parts
- Trig Substitution

Integration by Parts is a formula/process which reverses the Product Rule. It works in very specific situations where the U-Substitutions do not. Trig Substitution is a process for dealing with radicals. Partial Fractions undoes "common denominators" so that U-Sub or trig inverse formulas will work.

All three of these techniques are Algebra-intense, with multiple substitutions and algebraic manipulations.

8.1 Rational Integrals

In this section we begin to take a look into integrals of fractions. Most of these integrals require a lot of algebra. Here are some basic ground rules

1. Before beginning any integral with fractions, the degree of the numerator must be strictly less than the degree of the denominator. If it isn't, you must use long division.

2. If the numerator's degree is one less than the denominator's, it is a u-substitution situation. This may involve some algebraic manipulations we have not seen until now.

3. If the denominator is quadratic and not factorable, completing the square may reveal a $\tan^{-1}$ integral.

4. If the denominator is factorable, Partial Fractions is the technique to use.

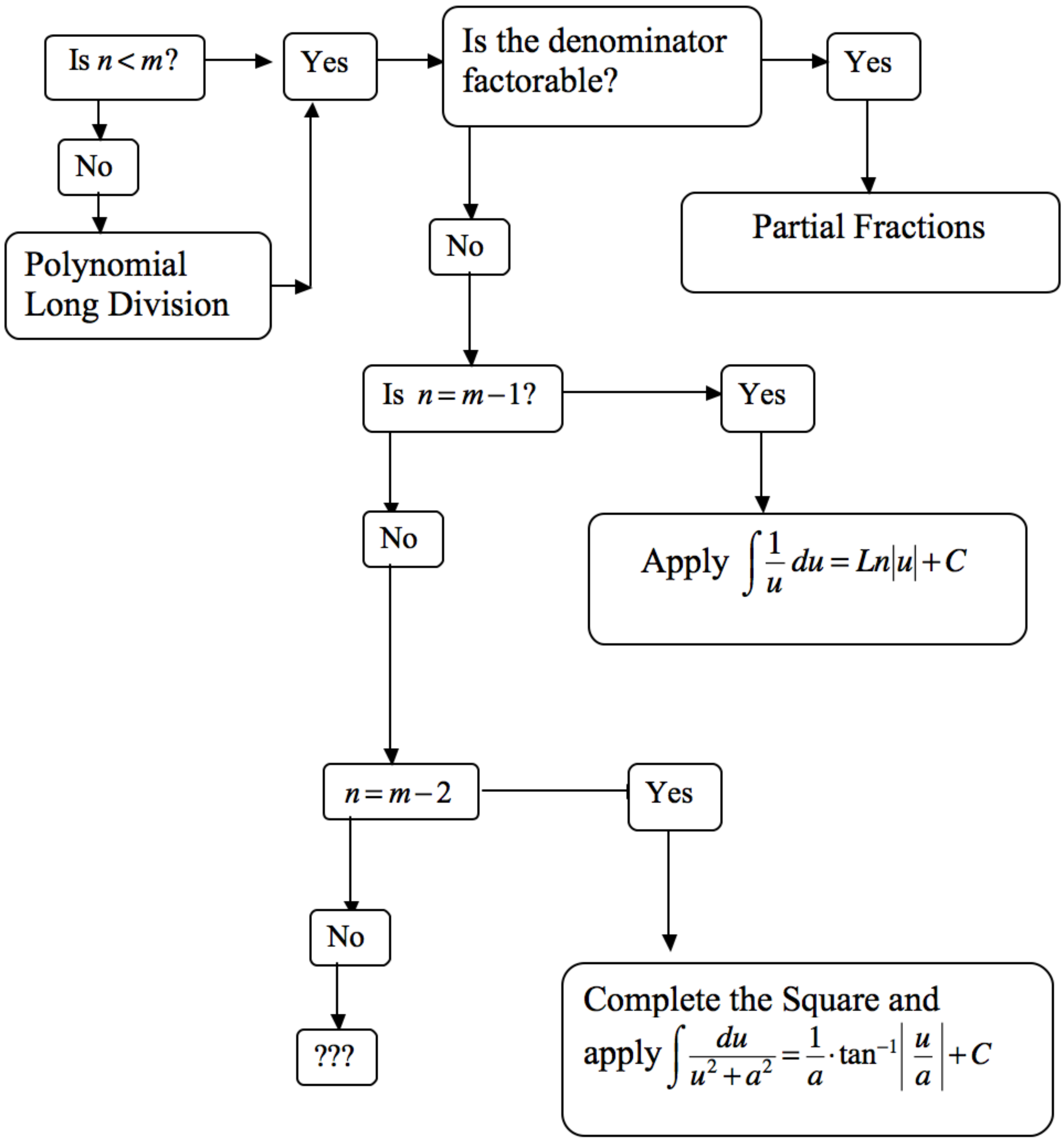
OBJECTIVES
Determine the appropriate technique to apply to a rational integral.

Remember: Based on Completing the Square, there are more widely applicable versions of the Inverse Trig integral rule:

$$\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{1}{|u|\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$$

Decision Chart for $\int \frac{Ax^n + \ldots}{Bx^m + \ldots} dx$



Ex 1 Evaluate $\int \frac{x^3+x}{x-1}\,dx$

Do the Long Division and Integrate.

$$
\begin{array}{r|l}
 & x^2+x+2 \\
\hline
x-1 & x^3+0x^2+x \\
 & -\left(x^3-1x^2\right) \\
\hline
 & x^2+x \\
 & -\left(x^2-x\right) \\
\hline
 & 2x \\
 & -\left(2x-2\right) \\
\hline
 & 2
\end{array}
$$

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2+x+2+\frac{2}{x-1} \right) dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C$$

The integral becomes slightly more complicated when the denominator is quadratic and not factorable, but let's first look at a rational function where $n=m-1$.

Ex 2 $\int \frac{2x-1}{x^2-4x+8}dx$

$$\begin{aligned}\int \frac{2x-1}{x^2-4x+8}dx &= \int \frac{2x-4+3}{x^2-4x+8}dx \\ &= \int\left(\frac{2x-4}{x^2-4x+8}+\frac{3}{x^2-4x+8}\right)dx \\ &= \ln\left|x^2-4x+8\right|+\int \frac{3}{x^2-4x+8}dx+c \\ &= \ln\left|x^2-4x+8\right|+?+c\end{aligned}$$

The new integral has $n=m-2$. We need to use $\int \frac{du}{u^2+a^2}=\frac{1}{a}\cdot\tan^{-1}\left|\frac{u}{a}\right|+C$. Let's just look at this new integral:

Ex 3 $\int \frac{3}{x^2-4x+8}dx$

To apply the Tangent Inverse rule, we need to complete the square:

$$\begin{aligned}\int \frac{3}{x^2-4x+8}dx &= \int\left(\frac{3}{\left(x^2-4x+4\right)-4+8}\right)dx \\ &= 3\int\left(\frac{1}{\left(x-2\right)^2+4}\right)dx \\ &= \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right)+c\end{aligned}$$

So to go back and finish Example 2:

Ex 2 (finished): $\int \frac{2x-1}{x^2-4x+8}dx = \ln\left|x^2-4x+8\right| + ? + c$

$$= \ln\left|x^2-4x+8\right| + \frac{3}{2}\tan^{-1}\left(\frac{x-2}{2}\right) + c + c$$

Ex 4 $\int \frac{4x^2-3x-2}{x^2-x+1}dx$

$$\int \frac{4x^2-3x-2}{x^2-x+1}dx = \int \left(4 + \frac{x-6}{x^2-x+1}\right)dx$$

$$= 4x + \int \frac{x-6}{x^2-x+1}dx$$

This new fraction has a numerator of one degree less than the denominator. Therefore, there must be a u-sub. But is $u = x^2 - x + 1$, then $du = (2x-1)dx$. We must manipulate the numerator to get this *du*.

$$4x + \int \frac{x-6}{x^2-x+1}dx = 4x + \frac{1}{2}\int \frac{2(x-6)}{x^2-x+1}dx$$

$$= 4x + \frac{1}{2}\int \frac{2x-12}{x^2-x+1}dx$$

$$= 4x + \frac{1}{2}\int \frac{2x-1-11}{x^2-x+1}dx$$

$$= 4x + \frac{1}{2}\int \frac{2x-1}{x^2-x+1} + \frac{-11}{x^2-x+1}dx$$

$$= 4x + \frac{1}{2}\int \frac{2x-1}{x^2-x+1}dx - \frac{11}{2}\int \frac{1}{x^2-x+1}dx$$

$$= 4x + \frac{1}{2}\ln\left(x^2-x+1\right) - \frac{11}{2}\int \frac{1}{x^2-x+1}dx$$

This new fraction has an un-factorable quadratic denominator, so we will complete the square.

$$4x+\frac{1}{2}\ln\left(x^2-x+1\right)-\frac{11}{2}\int\frac{1}{x^2-x+1}dx$$

$$=4x+\frac{1}{2}\ln\left(x^2-x+1\right)-\frac{11}{2}\int\frac{1}{\left(x^2-x+\frac{1}{4}\right)-\frac{1}{4}+1}dx$$

$$=4x+\frac{1}{2}\ln\left(x^2-x+1\right)-\frac{11}{2}\int\frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}dx$$

Note that this integral fits the Tan Inverse rule, with $u=x-\frac{1}{2}$ and $a=\frac{\sqrt{3}}{2}$:

$$=4x+\frac{1}{2}\ln\left(x^2-x+1\right)-\frac{11}{2}\left(\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right)+C$$

$$=4x+\frac{1}{2}\ln\left(x^2-x+1\right)-\frac{11}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)+C$$

8.1 Free Response Homework

1. $\int \frac{x-1}{x^2-4x+5}dx$

2. $\int \frac{dx}{x^2+25}$

3. $\int \frac{dx}{x^2-x+2}$

4. $\int \frac{e^x}{e^{2x}+7}dx$

5. $\int \frac{x}{x^2+x+1}dx$

6. $\int \frac{2x^3}{2x^2-4x+3}dx$

7. $\int \frac{x^2+1}{x-2}dx$

8. $\int \frac{x^3+x}{x^2+x+1}dx$

9. $\int \frac{x^3-2x^2-x-15}{x^2+2x+5}dx$

10. $\int \frac{3}{x^2+6x+13}dx$

11. $\int \frac{x}{x-7}dx$

12. $\int \frac{x+5}{x^2+2x+5}dx$

13. $\int \frac{x}{x^2+6x+13}dx$

14. $\int \frac{9x^2+18x}{x^3+3x^2+5}dx$

15. $\int \frac{2x+2}{\left(x^2+2x+4\right)^2}dx$

16. $\int \frac{3x^3+13x^2+19x+6}{x^2+4x+5}dx$

8.1 Multiple Choice Homework

1. What is the best method to evaluate $\int \frac{dx}{x\left(4x^2-9\right)}$?

a) Integration by Parts b) Substitution c) Partial Fractions

d) Completing the Square e) Formula

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2. What is the best method to integrate $\int \frac{5}{x^2+x+5}dx$?

a) Integration by Parts

b) Partial Fractions

c) U-Substitution

d) Complete the Square

e) Long Division

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3. What is the best method to evaluate $\int \frac{1}{x^2+4x+7}\,dx =$?

a) Integration by Parts b) Substitution c) Partial Fractions

d) Completing the Square e) None of these

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4. $\int \frac{e^{x^2}-2x}{e^{x^2}}dx =$

a) $-e^{-x^2}+c$
b) $-e^{x^2}+c$
c) $x-e^{x^2}+c$
d) $x+e^{-x^2}+c$
e) $x-e^{-x^2}+c$

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5. $\displaystyle\int \frac{x^2-4}{x^2+4}\,dx=$

a) $\frac{1}{2}\tan^{-1}\frac{x}{2}+c$
b) $\ln\left|x^2+4\right|+c$
c) $\ln\left|x^2+4\right|+\frac{1}{2}\tan^{-1}\frac{x}{2}+c$
d) $x-4\tan^{-1}\frac{x}{2}+c$
e) $x-8\tan^{-1}\frac{x}{2}+c$

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6. $\displaystyle\int \frac{t}{t+5}\,dt=$

a) $\ln\left|t+5\right|+c$ b) $\frac{t^2}{2}\ln\left|t+5\right|+c$ c) $t-5\ln\left|t+5\right|+c$
d) $-5\ln\left|t+5\right|+c$ e) $t+5\ln\left|t+5\right|+c$

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7. $\displaystyle\int_0^4 \left(\frac{\theta}{\sqrt{\theta^2+9}}\right)d\theta=$

a) 2 b) $\frac{2}{15}$ c) 1 d) 5

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8. $$\int_0^{\ln 3} \frac{e^x}{\left(1-e^x\right)^2+4}\,dx =$$

a) $\frac{\pi}{8}$

b) $\frac{\pi}{4}$

c) $\ln 3$

d) $-\frac{1}{2}\tan^{-1}(\ln 3)$

e) $-\frac{1}{2}\tan^{-1}\left(\frac{3}{2}\right)$

8.2 Rational Integrals – Partial Fractions

So what about when the denominator IS factorable? Consider these two integrals:

$$\int\left[\frac{1}{x-1}+\frac{3}{x+2}\right]dx \text{ and } \int\frac{4x-1}{x^2+x-2}dx$$

The first integral is easy to do, but the second seems not to be. Actually, though, they are the same integral. If we make common denominators between $\frac{1}{x-1}$ and $\frac{3}{x+2}$, we see that

$$\begin{aligned}\frac{1}{x-1}+\frac{3}{x+2} &= \frac{1(x+2)}{(x-1)(x+2)}+\frac{3(x-1)}{(x+2)(x-1)}\\ &= \frac{(x+2)+3(x-1)}{(x-1)(x+2)}\\ &= \frac{x+2+3x-3}{(x-1)(x+2)}\\ &= \frac{4x-1}{(x-1)(x+2)}\end{aligned}$$

So the question becomes "How can we reverse this process?" The reverse of common denominators is called Partial Fractions.

Partial Fractions

1. Create separate fractions each with a factor of the original denominator and dummy variables in the numerators.
2. Recreate the common denominator process.
3. Equate the created numerator with the original numerator.
4. Solve for the dummy variables.

Ex 1 $\int \frac{4x-1}{x^2+x-2}dx$

1. $\frac{4x-1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

2. $\frac{A}{x-1} + \frac{B}{x+2} = \frac{A(x+2)+B(x-1)}{(x-1)(x+2)}$

3. $A(x+2)+B(x-1) = 4x-1$

4. There are several ways to solve for *A* and *B*. Linear combination, substitution, and Cramer's Rule come to mind. The simplest way though, is to plug in numbers for x that would eliminate one of the dummy variables:.

$$x=-2 \Rightarrow A(-2+2)+B(-2-1)=4(-2)-1 \Rightarrow -3B=-9 \Rightarrow B=3$$

$$x=1 \Rightarrow A(1+2)+B(1-1)=4(1)-1 \Rightarrow 3A=3 \Rightarrow A=1$$

So,

$$\int \frac{4x-1}{x^2+x-2}dx = \int \left[\frac{1}{x-1}+\frac{3}{x+2}\right]dx$$
$$= Ln(x-1)+3Ln(x+2)+C=$$
$$= Ln\left[(x-1)(x+2)^3\right]+C$$

OBJECTIVES

Determine the appropriate technique to apply to a rational integral.
Apply the Partial Fractions technique.

Ex 2 $\int \frac{1}{x^2 - x - 2} dx$

$$\int \frac{1}{x^2 - x - 2} dx = \int \frac{1}{(x+1)(x-2)} dx$$

<u>Scratch Work</u>

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$\frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

$$1 = A(x-2) + B(x+1)$$
$$x = 2 \Rightarrow 3B = 1 \Rightarrow B = 1/3$$
$$x = -1 \Rightarrow -3A = 1 \Rightarrow A = -1/3$$

$$\int \frac{1}{x^2 - x - 2} dx = \int \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2} \right] dx$$
$$= -\frac{1}{3} \int \frac{1}{x+1} dx + \frac{1}{3} \int \frac{1}{x-2} dx$$
$$= -1/3 \ln|x+1| + 1/3 \ln|x-2| + C$$
$$= 1/3 \ln\left|\frac{x-2}{x+1}\right| + C$$

Simple Linear Partial Fractions lead us to our last formula to memorize:

$$\int \frac{1}{u^2 - a^2} du = \frac{1}{2a} \ln\left|\frac{u-a}{u+a}\right| + C$$

Not to be confused with $\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$

Ex 3 $\int \frac{x^3-4x-10}{x^2-x-6}dx$

$$\int \frac{x^3-4x-10}{x^2-x-6}dx$$

Scratch Work

$$\begin{array}{r} x+1+\dfrac{3x-4}{x^2-x-6} \\ x^2-x-6\overline{)\quad x^3-4x-10} \\ \underline{-\left(x^3-x^2-6x\right)} \\ x^2+2x-10 \\ \underline{-\left(x^2-x-6\right)} \\ 3x-4 \end{array}$$

$$\begin{aligned} \int \frac{x^3-4x-10}{x^2-x-6}dx &= \int\left[x+1+\frac{3x-4}{x^2-x-6}\right]dx \\ &= \int\left[x+1+\frac{3x-4}{(x-3)(x+2)}\right]dx \end{aligned}$$

$$\frac{3x-4}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

$$\frac{3x-4}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

$$3x-4=A(x+2)+B(x-3)$$

Just for variety, let's consider another way to solve for A and B. (This method may be more successful in future sections.) Since like term add to like terms, we can say that

$$\left\{\begin{matrix} 3x=Ax+Bx \\ -4=2A-3B \end{matrix}\right\} \text{and } \left\{\begin{matrix} A+B=3 \\ 2A-3B=-4 \end{matrix}\right\}$$

We can use substitution, linear combinations, Cramer's Rule – we can even graph the system of equations to find the solution.

$$\left.\begin{array}{l} A+B=3 \\ 2A-3B=-4 \end{array}\right\} \Rightarrow A=1, B=2$$

So,

$$\begin{aligned}\int \frac{x^3-4x-10}{x^2-x-6}\,dx &= \int\left[x+1+\frac{3x-4}{(x-3)(x+2)}\right]dx \\ &= \int\left[x+1+\frac{1}{x-3}+\frac{2}{x+2}\right]dx \\ &= \frac{x^2}{2}+x+\ln|x-3|+2\ln|x+2|+C\end{aligned}$$

Ex 4 $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$

$$\begin{aligned}\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx &= \int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx \\ &= \int\left(\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}\right)dx\end{aligned}$$

Scratch Work

$$\frac{A}{x}+\frac{B}{2x-1}+\frac{C}{x+2}$$

$$=\frac{A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)}{x(2x-1)(x+2)}$$

$$A(2x-1)(x+2)+Bx(x+2)+Cx(2x-1)=x^2+2x-1$$

$$x = 0 \Rightarrow A = \frac{1}{2}$$
$$x = -2 \Rightarrow C = \frac{1}{10}$$
$$x = \frac{1}{2} \Rightarrow B = \frac{1}{5}$$

$$\begin{aligned}
\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} dx &= \int \left(\frac{1}{2}\frac{1}{x} + \frac{1}{5}\frac{1}{2x-1} + \frac{1}{10}\frac{1}{x+2} \right) dx \\
&= \frac{1}{2}\int \frac{1}{x} dx + \frac{1}{5}\int \frac{1}{2x-1} dx + \frac{1}{10}\int \frac{1}{x+2} dx \\
&= \frac{1}{2}\int \frac{1}{x} dx + \frac{1}{5}\left(\frac{1}{2}\int \frac{1}{2x-1} 2dx \right) + \frac{1}{10}\int \frac{1}{x+2} dx \\
&= \frac{1}{2} Ln|x| + \frac{1}{10} Ln|2x-1| + \frac{1}{10} Ln|x+2| + C
\end{aligned}$$

Ex 5 $\displaystyle\int \frac{e^x}{e^{2x} + 3e^x + 2} dx$

This integral is a bit trickier. It looks like the numerator is one degree less than the denominator, but this is an exponential integrand and exponentials have a variable degree. The u – sub $\left\{ \begin{matrix} u = e^x \\ du = e^x dx \end{matrix} \right\}$ reveals the true problem:

$$
\begin{aligned}
\int \frac{e^x}{e^{2x}+3e^x+2}\,dx &= \int \frac{1}{u^2+3u+2}\,du \\
&= \int \frac{1}{(u+1)(u+2)}\,du \\
&= \int \frac{-1}{(u+1)}+\frac{1}{(u+2)}\,du \\
&= -\ln|u+1|+\ln|u+2|+C \\
&= \ln\left|\frac{u+2}{u+1}\right|+C \\
&= \ln\left(\frac{e^x+2}{e^x+1}\right)+C
\end{aligned}
$$

Ex 6 Determine if the Telescoping Series $\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$ converges or diverges.

$\lim_{n\to\infty}\frac{1}{n(n-1)}=0$; so, it passed the Divergence Test.

Passing the Divergence Test means the function is decreasing, so we can apply the Integral Test.

$\int_2^{\infty}\frac{1}{x(x-1)}dx$ requires partial fractions.

$$\int_2^{\infty}\frac{1}{x(x-1)}dx=\int_2^{\infty}\frac{A}{x}+\frac{B}{x-1}dx$$

$$
\begin{aligned}
&A(x-1)+Bx=1 \\
&x=1\rightarrow B=1 \\
&x=0\rightarrow -A=1\rightarrow A=-1
\end{aligned}
$$

$$
\begin{aligned}
&= \int_2^\infty \frac{-1}{x} + \frac{1}{x-1}\,dx \\
&= \lim_{b\to\infty}\left[-\ln x + \ln(x-1)\right]_2^b \\
&= \lim_{b\to\infty}\left[\ln\frac{(x-1)}{x}\right]_2^b \\
&= \ln\left[\lim_{b\to\infty}\frac{(b-1)}{b}\right] - \ln 2 \\
&= \ln 1 - \ln 2 \\
&= -\ln 2
\end{aligned}
$$

Therefore, $\sum_{n=2}^{\infty}\frac{1}{n(n-1)}$ converges by the Integral Test.

8.2 Free Response Homework

1. $\int \frac{x-9}{(x-2)(x+5)} dx$

2. $\int \frac{1}{(t+4)(t-1)} dt$

3. $\int \frac{x^2}{x+5} dx$

4. $\int \frac{x^2+1}{x^2-x} dx$

5. $\int \frac{x-1}{x^2-4x-5} dx$

6. $\int \frac{x-1}{x^2-4x+5} dx$

7. $\int \frac{x^3+x^2-12x+1}{x^2+x-12} dx$

8. $\int \frac{e^x}{e^{2x}+3e^x+2} dx$

9. $\int \frac{4x^2-7x-12}{x(x+2)(x-3)} dx$

10. $\int \frac{1}{x^2+x-6} dx$

11. $\int \frac{x^4+x^3-x^2-x+1}{x^3-x} dx$

12. $\int \frac{x^3+x^2}{x^3-5x^2-4x+20} dx$

13. $\int \frac{5}{3x^2-10x+8} dx$

14. $\int \frac{x+14}{2x^2-7x-4} dx$

15. Find the area of the region bounded by $y=\frac{x-1}{x^2-5x+6}$ and the x-axis from $x=4$ to $x=6$.

16. Find the area of the region bounded by $y=\frac{3x-4}{x^2-2x-8}$ and the x-axis from $x=-1$ to $x=3$.

17. Find the volume if the region bounded by $y=\frac{1}{\sqrt{4-x^2}}$ and the x-axis on $x\in[-1, 1]$ is revolved about the x-axis.

18. Find the volume of a solid where the region bounded by $y = \frac{x}{\sqrt{3+4x-x^2}}$, and the x-axis on $x \in [0, 4]$ is revolved about the x-axis.

8.2 Multiple Choice Homework

1. $\int \frac{x}{x^2+2x-8}\, dx =$

a) $\frac{1}{2}\ln(x-2)(x+4)+c$

b) $\frac{1}{3}\ln|x-2|+\frac{2}{3}\ln|x+4|+c$

c) $-\frac{2}{3}\ln|x-2|-\frac{1}{3}\ln|x+4|+c$

d) $-\frac{1}{3}\ln|x-2|-\frac{2}{3}\ln|x+4|+c$

e) $\frac{1}{2}\ln|(x-2)(x+4)|+c$

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2. $\int \frac{2x-3}{x^2+9x+18}\, dx =$

a) $\ln\left|(x+9)^3(x+2)\right|+c$

b) $\ln\left|\frac{(x+6)^5}{(x+3)^3}\right|+c$

c) $3\ln|x+9|-\ln|x+2|+c$

d) $\ln\left|x^2+9x+18\right|+c$

e) $5\ln|x+6|+3\ln|x+3|+c$

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3. $\int \frac{1}{x^2-5x+4}\,dx=$

a) $\frac{1}{3}\ln\left|\frac{x-1}{x-4}\right|+c$ b) $\frac{1}{3}\ln\left|\frac{x-4}{x-1}\right|+c$

c) $\frac{1}{3}\ln\left|(x-4)(x-1)\right|+C$ d) $\frac{1}{3}\ln\left|(x-4)(x+1)\right|+C$

e) $\frac{1}{3}\ln\left|(x+4)(x-1)\right|+C$

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4. $\int \frac{1}{e^{-x}(e^x-1)(e^x+1)}\,dx=$

a) $e^{-x}+\frac{1}{2}\ln\left|\frac{e^x+1}{e^x-1}\right|+c$ b) $-e^{-x}-\frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right|+c$

c) $\frac{1}{2}\ln\left|\frac{e^x+1}{e^x-1}\right|+c$ d) $\frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right|+c$

e) $e^{-2x}+\ln\left|\frac{e^x-1}{e^x+1}\right|+c$

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5. $\int \frac{1}{x^2+x-6}\,dx=$

a) $\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|+c$ b) $\frac{1}{5}\ln\left|\frac{x+3}{x-2}\right|+c$ c) $\frac{1}{5}\ln\left|(x-2)(x+3)\right|+c$

d) $\left(\ln|x+3|\right)\left(\ln|x-2|\right)+c$ e) $\left(\ln|x-3|\right)\left(\ln|x+2|\right)+c$

8.2 Free Response Homework Set B

1. $\int \frac{3}{x^2+3x-4}dx$

2. $\int \frac{2x^3}{x^2+4x+3}dx$

3. $\int \frac{x-8}{x^2+3x-10}dx$

4. $\int \frac{1}{x^2+x}dx$

5. $\int \frac{x}{x+2}dx$

6. $\int \frac{8x-4}{x^2+2x-3}dx$

7. $\int \frac{8}{(x+2)(x-1)}dx$

8. $\int \frac{nx}{x^2-mx}dx$

9. $\int \frac{x^3}{2x^2-5x+2}dx$

10. Does $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ converge

11. $\int \frac{x^3-3x-10}{x^2-x-6}dx$

8.3: Models of Exponential Growth and Decay

There are three models of growth and decay which we will consider:

Vocabulary:

1. ***Unbounded Exponential Growth*** **–** the rate is directly proportional to the amount of material present.

2. ***Simple Bounded Growth*** **–** the rate is directly proportional to the amount of material which has not changed yet.

3. ***Logistic Growth*** **–** the rate of change of y is jointly proportional to the amount which has changed and to the amount which has not yet been changed.

OBJECTIVES

Understand growth and decay in terms of differential equations.
Solve differential equations in a growth and decay context.
Recognize the carrying capacity in a growth setting.
Determine when the maximum growth rate in a logistic growth setting.
Know the solution to a logistic differential equation.

Unbounded Growth

In previous math and science classes, exponential growth had been explored, and it was generally considered any growth that followed an exponential equation, like $y = y_0 e^{kt}$. But this equation actually arises from solving the differential equation $\frac{dy}{dt} = ky$. This equation states that the rate of change of y, $\frac{dy}{dt}$, is directly proportional to y itself. In other words, the rate is determined by the amount of material present.

Unbounded Growth and Decay are described by the two equations:

Differential Equation: $\frac{dy}{dt} = ky$

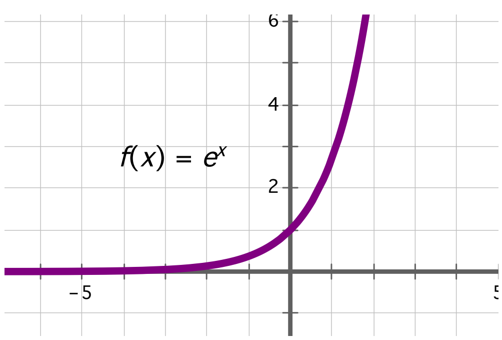
General Solution: $y = Ae^{kt}$ *or* $y_0 e^{kt}$

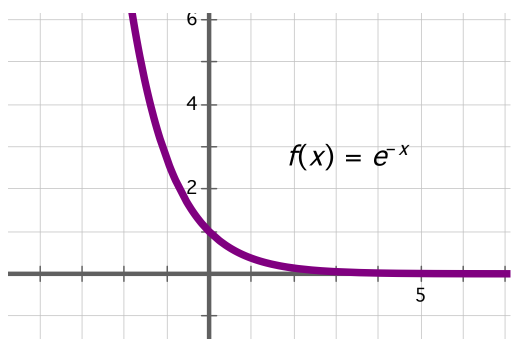
The solution comes from separating the variable to solve the differential equation:

$$\frac{dy}{dt} = ky \rightarrow \frac{1}{y}dy = kt \rightarrow \int \frac{1}{y}dy = \int kdt \rightarrow \ln|y| = kt + c \rightarrow |y| = e^{kt+c} \rightarrow y = Ae^{kt}.$$

$y = y_0 e^{kt}$ is how the solution is given in many science and Algebra 2 courses.

Hopefully, we recall the graphs from PreCalculus:





The difference between growth and decay is the sign of the *k*-value.

EX 1 If there are 100 bacteria in a petri dish and the number doubles every 10 minutes, how long will it take for there to be a million bacteria?

$$t = 0 \rightarrow A = 100, \text{ so } y = 100e^{kt}$$

In ten minutes, there will be 200 bacteria, so

$$\begin{aligned} 200 &= 100e^{k(10)} \\ 2 &= e^{k(10)} \\ \ln 2 &= 10k \\ .06931 &= k \end{aligned}$$

$$\text{Therefore, } y = 100e^{.06931t}$$

$$1000000 = 100e^{.06931t}$$
$$10000 = e^{.06931t}$$
$$\ln 10000 = .06931t$$

$$t = 132.878 \text{ min}$$

EX 2 Radium has a half-life of 1690 years. How much of a 75gm. sample will be left in 400 years?

We will skip the solving of the differential equation here as we know it will be $y = Ae^{kt}$

$$.5A = Ae^{k(1690)}$$
$$.5 = e^{k(1690)}$$
$$\ln .5 = k(1690)$$
$$k = \frac{\ln .5}{1690}$$

$$y = Ae^{kt} \rightarrow y = 75e^{\left(\frac{\ln .5}{1690}\right)400} = 63.652gm$$

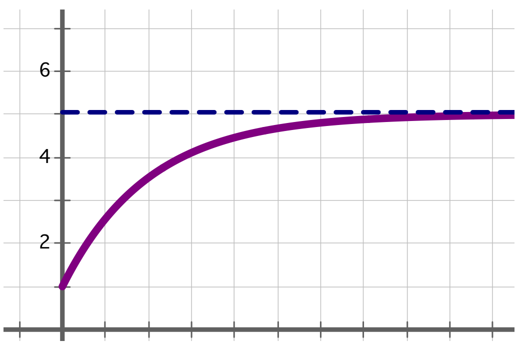
Simple Bounded Growth

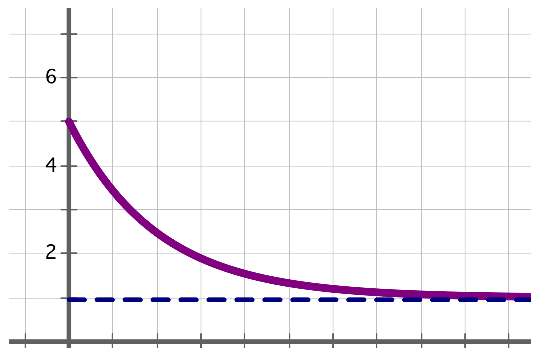
Differential Equation: $\frac{dy}{dt} = k(A - y)$

General Solution: $y = A - Be^{kt}$

We will demonstrate the solving of this differential equation in the examples, as this separation of variables is likely to be the FRQ on the AB Exam.

The Simple Bounded Growth and Decay equations look like this:





The decay curve usually has its boundary at $y = 0$.

NB. These problems are often the Separation of Variables Problems on the AP Calculus FRQ Exam.

EX 3 Suppose that a population of wolves follows the simple bounded growth model. If these wolves have a population limit of 5000, the differential equation would be $\frac{dw}{dt} = k(5000 - w)$.

(a) Find the general solution to the differential equation. Show the steps.

(b) If there are 1000 wolves at time $t = 0$ and 1100 at $t = 1$, find the particular solution to the differential equation.

(c) How many wolves are there at time $t = 20$?

(a)
$$\frac{dw}{dt} = k(5000 - w)$$
$$\frac{1}{5000 - w} dw = kdt$$
$$\int \frac{1}{5000 - w} dw = \int k\, dt$$
$$-\int \frac{1}{5000 - w} (-dw) = \int k\, dt$$
$$-\ln|5000 - w| = kt + c$$
$$\ln|5000 - w| = -kt + c$$
$$|5000 - w| = e^{-kt+c}$$
$$5000 - w = Be^{-kt}$$
$$w = 5000 - Be^{-kt}$$

(b)
$$(0, 1000) \to 1000 = 5000 - Be^{0} \to \mathrm{B} = 4000$$
$$(1, 1100) \to 1100 = 5000 - 4000e^{-k}$$
$$e^{-k} = 0.975$$
$$-k = \ln 0.975 \to k = 0.025317808$$
$$w = 5000 - 4000e^{0.025t}$$

(c) $w(20) = 5000 - 4000e^{0.025(20)} \approx 2589$ wolves.

Ex 4: AP Calculus 2012 #5

Note the use the first and second derivatives to determine the shape of the curve.

A common instance of Simple Bounded Exponential is Newton's Law of Cooling and Heating:

> Newtons' Law of Cooling and Heating states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the surrounding atmosphere. For cooling (i.e., the object is warmer than the environment), the differential equation can be written as:
>
> $$\frac{dy}{dt} = k(y - A)$$
>
> where y is the temperature of the object and A is the temperature of the surrounding environment.
>
> If the object is colder than the environment and is warming, the equation is
>
> $$\frac{dy}{dt} = k(A - y)$$

NB. These two equations are actually the same equation, but, in one case, k is a negative number and in the other k is positive.

EX 5 Suppose that some muffins are taken out of the oven and are at a temperature of 250 $F°$. The ambient room is at a pleasant 70 $F°$. In 15 minutes, they have fallen to 125 $F°$.

(a) Find the particular solution to this cooling situation.
(b) When do the muffins reach 75 $F°$?
(c) When do the muffins reach 65 $F°$?

(a) $\frac{dy}{dt} = k(y-70) \rightarrow \frac{1}{y-70}dy = kdt$

$$\int \frac{1}{y-70}dy = \int k\,dt$$
$$\ln|y-70| = kt + c$$
$$|y-70| = e^{kt+c} \rightarrow y - 70 = Be^{kt}$$

$$(0, 250) \rightarrow 250 - 70 = Be^{0} \rightarrow B = 180$$
$$(15, 125) \rightarrow 125 - 70 = 180e^{k(15)} \rightarrow k = -0.079$$

$$y = 70 + 180e^{-0.079t}$$

(b) $75 = 70 + 180e^{-0.079t} \rightarrow 5 = 180e^{-0.079t} \rightarrow t = \frac{1}{-0.079}\ln\frac{5}{180} = 4.536 \text{ min}$

(c) $65 = 70 + 180e^{-0.079t} \rightarrow -5 = 180e^{-0.079t} \rightarrow t = error$
The biscuits never reach 65°

Logistic Growth

Logistic growth situations are also bounded, but, unlike Simple Bounded Growth problems, the growth relies on both the amount changed and the amount unchanged. The two most common situations modeled by logistic growth equations are the spread of a disease and the spread of a rumor. Consider a rumor. The rate at which the rumor spreads is much greater in the beginning. But, as more people hear the rumor, fewer new people are available hear about it. So, the rate at which it spreads slows down. A rumor spreading is dependent on how many people have heard the rumor and how many people have not heard the rumor. There is a limit to how many people will hear this rumor. This limit is the carrying capacity.

Three Facts You Need to Know About Logistic Growth:

Given a Logistic growth equation in the forms $\frac{dy}{dt} = Ky(A - y)$ or $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$

NB. K and k will be different numbers, with $K = \frac{k}{A}$

1. $\lim_{t \to \infty} y = A$-- there is a horizontal asymptote on y of $y = A$ (i.e. there is a limit to how many people will hear the rumor, etc.).

2. The maximum growth rate happens at $y = \frac{A}{2}$.

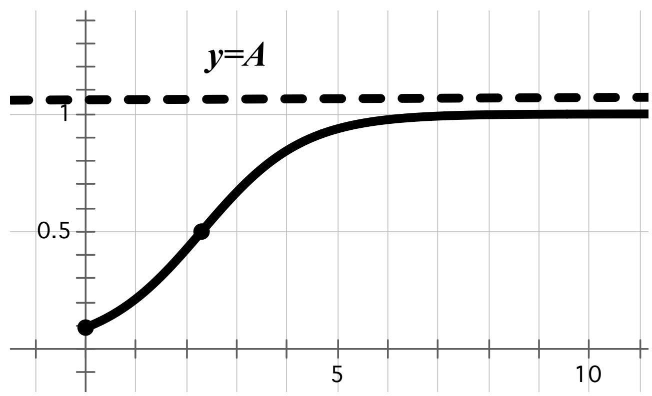
3. The solution to the logistic growth differential equation $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$ is $y = \frac{A}{1 + Be^{-kt}}$.

Note: The AP Biology Exam uses $\frac{dN}{dt} = r_{\max} N\left(\frac{K - N}{K}\right)$ where $r_{\max}$ is the maximum growth rate and K is the carrying capacity (what is usually denoted as A in Calculus).

Differential Equation: $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$

General Solution: $y = \frac{A}{1 + Be^{-kt}}$

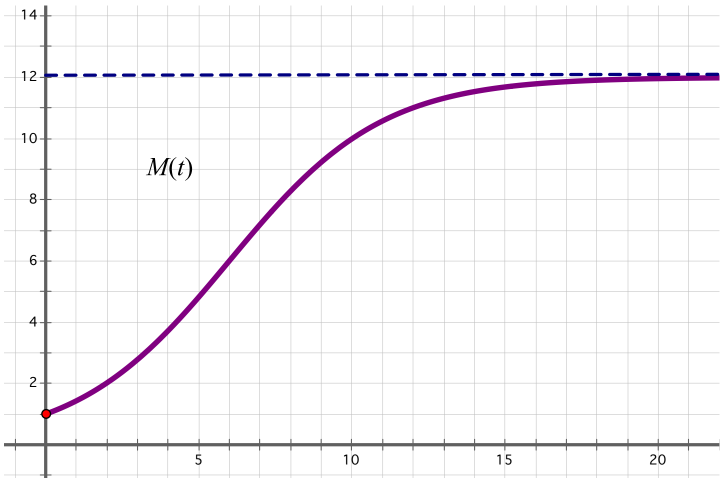
Logistic growth curves look like this:



Note that it looks like an exponential growth curve at the beginning (concave up) and a simple bounded growth curve at the end (concave down).

The solution is a little complicated and involves a technique beyond the scope of this class which is called Partial Fractions. For further information and examples, see the Readable Calculus (BC version) 8.2 and 8.3. **Logistic Growth is not a part of the AP Calculus AB curriculum.** On the BC Calculus Exam, logistic growth is usually a multiple-choice question.

Ex 5 Consider the logistic growth curve below.



Which of the following statements is false?

a) $\lim_{t\to\infty} M(t) = 12$

b) The fastest rate of growth occurs when $M(t) = 6.5$.

c) The solution equation is $M(t) = \dfrac{12}{1+11e^{-.04t}}$.

d) At $t = 10$, $M'(t) > 0$ and $M''(t) < 0$.

$\lim_{t\to\infty} M(t) = 12$ is true. The end behavior is the horizontal asymptote $y = A = 12$.

"The fastest rate of growth occurs when $M(t) = 6.5$" is false. The fastest rate of growth occurs when $M(t) = \dfrac{A}{2} = 6$.

$M(t) = \dfrac{12}{1+11e^{-.04t}}$ is the solution equation.

$M'(t) > 0$ and $M''(t) < 0$ at $t = 10$, because the curve is increasing and concave up.

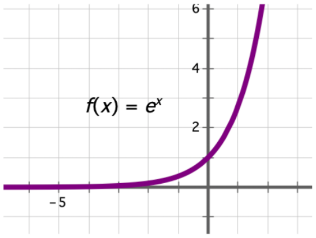
So the answer is B.

In summary,

Exponential Growth

Differential Equation: $\frac{dy}{dt} = ky$

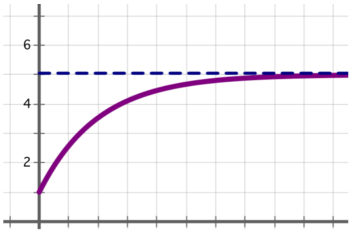
General Solution: $y = Ae^{kt}$ *or* $y_0 e^{kt}$



Simple Bounded Growth

Differential Equation: $\frac{dy}{dt} = k(A - y)$

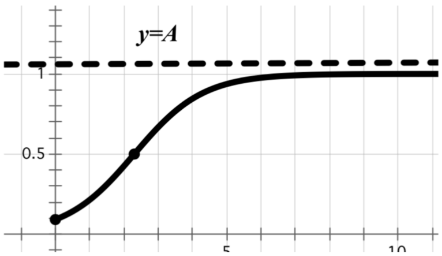
General Solution: $y = A - Be^{kt}$



Logistic Growth

Differential Equation: $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$

General Solution: $y = \frac{A}{1 + Be^{-kt}}$



8.3 Homework

1. The Body Weight Problem

According to a simple physiological model, an athletic adult needs 20 calories per day per pound of body weight to maintain his weight. If he consumes more or fewer calories than those required to maintain his weight, his weight *W* changes at a rate proportional to the difference between the number of calories consumed and the number of needed to maintain his current weight. Let $\frac{dW}{dt} = \frac{1}{3500}(3600 - 20W)$, where *W* is the person's weight at time *t* (measured in days).

a) What is the limit to this person's weight $W(t)$?
b) Assume that the 3600 in the equation is the person's daily caloric intake. If that is reduced to 3000, what would be the limit to this person's weight?
c) Find the particular solution to $\frac{dW}{dt} = \frac{1}{3500}(3600 - 20W)$ if the person weights 200 pounds when $t = 0$.
d) Use your solution in c) to determine how long it would take this person's weight to reach 190 pounds. [Show the set up, but solve by graphing.]

---

2. Alien Population Problem

The rate at which the population of a certain alien creature on another planet grows according to the differential equation $\frac{dA}{dt} = .005(100 - A)$, where *A* in creatures at time *t* days. The equation $y = A(t)$ is the particular solution to the differential equation wherein there are 10 creatures at time $t = 0$.

(a) Find the equation of the line tangent to $y = A(t)$ at $(0, 10)$.

(b) Use the line tangent found in (a) to approximate the number of creatures at time $t = 12$ days.

(c) Find the particular solution to $\frac{dA}{dt} = .005(100 - A)$ with the initial condition $A(0) = 10$.

(d) Determine whether the alien creature population is changing at an increasing or a decreasing rate at time $t = 12$ days. Explain your reasoning.

---

3. The Ebbinghaus Model Problem

As you know, when a course ends, students start to forget the material they have learned. One model (called the Ebbinghaus model) assumes that the rate at which a student forgets material is proportional to the difference between the amount *y* (material which is currently forgotten) and the total amount of material learned. Based on this, the rate of loss would be determined by

$$\frac{dy}{dt} = k(100 - y),$$

where *y* is the percentage of material forgotten and t is measured in weeks since the end of class. At the end of the class $(t = 0)$, the students have not forgotten anything $(y = 0)$.

(a) Find the general solution to the differential equation.

(b) Find the particular solution to the differential equation if the students have forgotten half the material $(y = 50)$ after four weeks.

(c) How much material do they **still remember** after the Summer Break, 10 weeks later?

---

4. The Hot Coffee Problem

A cup of coffee is made with boiling water at a temperature of 100 C°, in a room at temperature 20 C°. After two minutes, it has cooled to 80 C°. According to Newton's Law of Cooling, the temperature of the coffee follows the differential equation

$$\frac{dy}{dt} = k(y - 20),$$

where *y* is the temperature of the coffee at time *t* minutes.

a) Find the particular solution to the differential equation.

b) What is its temperature after five minutes?

c) The coffee will be perceived as "cold" when the temperature drops below 40 C°. At what time *t* will this occur?

---

5. The Ice Cream Tempering Problem

Dr. and Mrs. Quattrin enjoy the occasional pint of Ben & Jerry's Chocolate Fudge Brownie, but they prefer to let it soften before digging in. Research shows that ice cream tempers (softens) more evenly by putting it in the refrigerator rather than on the kitchen counter. When the ice cream comes out of the freezer, its temperature is -1 F°. According to Newton's Law of Cooling, the temperature of the ice cream follows the differential equation

$$\frac{dy}{dt} = k(37 - y),$$

where *y* is the temperature of the ice cream at time *t* minutes.

a) If the temperature of the ice cream is 2.5 F° after 20 minutes, find the particular solution to the differential equation.

b) The ice cream cannot get any warmer than the temperature inside the refrigerator. What is that temperature. Indicate units.

c) When will the temperature of the ice cream reach 4 F° (the ideal tempered temperature)?

---

6. The Boiled Beet Problem

At time $t = 0$, boiled beets are taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the beets is 90 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the beets is greater than 25°C for all times $t > 0$. The internal temperature of the beets at time t minutes can be modeled by the function B that satisfies the differential equation $\frac{dB}{dt} = -\frac{1}{4}(B - 25)$, where $B(t)$ is measured in degrees Celsius and $B(0) = 89$.

a) Write an equation for the line tangent to the graph of $B(t)$ at $t = 0$. Use this equation to approximate the internal temperature of the beets at time $t = 3$.

b) Use $\frac{d^2B}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the beets at time $t = 3$.

c) For $0 \le t \le 10$, an alternate model for the internal temperature of the beets at time t minutes is the function $b(t)$ that satisfies the differential equation $\frac{db}{dt} = -(b - 25)^{2/3}$, where $b(t)$ is measured in degrees Celsius and $b(0) = 89$. Find an expression for $b(t)$. Based on this model, what is the internal temperature of the beets at time $t = 3$?

---

7. The Chinook Salmon Problem

The fishing industry is a major part of California's economy. A catch-and-release study of Chinook salmon on the Sacramento Delta near Rio Vista was undertaken in 2008. Over 60 days, the rate at which new fish were caught and released followed the equation $\frac{dF}{dt} = .004F(100 - F)$, where $\frac{dF}{dt}$ was measured in number of smolt (young salmon) caught per day.

a) If $F(0) = 10$, what is $\lim_{t\to\infty} F(t)$?

b) Using the correct units, explain $\lim_{t\to\infty} F(t)$.

c) If $F(0) = 25$, how many smelt are captured and release when $\frac{dF}{dt}$ is at its greatest?

d) Data from a different study showed $\frac{dF}{dt} = .004(100 - F)$, where $F(0) = 10$. Use separation of variables to solve the differential equation.

---

8. The Deer Population Proplem

The population of deer in a forest is modeled by $\frac{dP}{dt} = .5P - .0005P^2$.

a) If $P(0) = 10$, what is $\lim_{t\to\infty} P$.

b) If $P(1) = 30$, find $\lim_{t\to\infty} P$.

c) Find the particular solution to $\frac{dP}{dt} = .05P - .0005P^2$ where $P(0) = 100$.

d) What would the population be when it is growing the fastest?

---

9. The Philosopher's Stone Problem

Medieval alchemist Pol Maychrowitz believed that the Philosopher's Stone would help them to convert lead into gold. The Stone was never found, but, if it had been found and worked, Pol assumed he could convert 12 pounds of lead over a 72-hour period and that the conversion rate would follow a logistic growth curve $\frac{dG}{dt}=1.2G\left(3-\frac{G}{4}\right)$. (By the way, if he had succeeded, Pol would probably have been burned at the stake.)

a) If $G(0)=1$, what is $\lim_{t\to\infty} G$.

b) How much gold would have been transmuted when the transformation way occurring the fastest?

c) If $G(0)=1$, state the particular solution to the logistic differential equation.

d) Suppose Pol was incorrect and the actual growth rate followed the separable differential equation $\frac{dG}{dt}=1.2\left(3-\frac{G}{4}\right)$, instead of the logistic equation above. If $G(0)=1$, state the particular solution to the separable differential equation.

---

10. The Body Farm Problem

Research at the University of Tennessee Anthropological Research Facility, (aka The Body Farm) shows that a 233 lb. male body buried six feet underground without a coffin will decompose to a 33 lb. skeleton in 12 days. The table below shows $W(t)$, the rate of decomposition of the flesh in pounds per day, between $t=0$ and $t=12$ days.

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$W(t)$	0	2.4	5.7	13.2	22.0	36.5	44.1	36.5	22.5	10.9	4.9	2.1	0.1

a) Approximate $W'(7)$ and explain thr result using the appropriate units.

b) Use a trapezoidal approximation to find the total weight of the body which decomposed between $t=0$ and $t=8$ days.

c) Body decomposition rate depends on both the amount of material that has decomposed and the amount not yet decomposed. Thus, body decomposition is modeled by a logistic growth equation. In this case, the differential equation is $\frac{dN}{dt}=.2N(200-N)$. At what weight of deposed flesh is the rate the highest?

d) The data on the table can be modeled by $R(t)=50e^{-.2(t-6)^2}$. Write an equation for $0 \le t \le 12$ which would determine the weight of the body at any time *t*.

---

11. The Black Hole Accretion Problem II

Like stars, black holes accrete (gain) mass from solar gasses. It was always assumed that this accretion was never-ending, ultimately, black holes would swallow the universe. But the research into supermassive black holes (SMBH) by Columbia Inayoshi and Haiman seems to indicate that there is a limit to how large these black holes can get. SMBH sizes are on the scale of $10^9 M_{suns}$ (solar masses). For simplicity, we will refer to these units as Kellar-masses (not a real thing). The research seems to indicate that the size limit for SMBHs is 1000 Kellar-masses. If the growth were logistic, one model might be

$$\frac{dM}{dt}=.256M(1000-M).$$

a) If $M(0)=10$, how many Kellar-masses would a SMBH attain when the accretion rate was the highest?

b) If $M(0)=10$, state the particular solution to the logistic differential equation.

c) Further research might show the growth to be exponential rather than logistic. Assuming a new model of $\frac{dM}{dt} = .256(1100 - M)$, what would be $\lim_{t \to \infty} M$?

d) If $M(0) = 10$ and $\frac{dM}{dt} = .256(1100 - M)$, state the particular solution to the logistic differential equation.

---

12. The Monkey Island Problem

A research team is studying a group of monkeys living on Monkey Island in Cambodia. When they begin observing the monkeys$(t = 0)$, there are 20 monkeys on the island. The researchers determine that the population P grows logistically at a rate of $\frac{dP}{dt} = 3P - \frac{P^2}{40}$ monkeys per year.

---

a) According to this logistic model, what is the maximum population of monkeys on the island?

b) Further research shows the growth to be a bounded exponential function rather than a logistic growth function. Assuming a new model of $\frac{dM}{dt} = \frac{1}{40}(130 - M)$, find the particular solution to the differential equation, given that $M(0) = 20$.

c) Find $\lim_{t \to \infty} M$. Explain the meaning of this result, using the correct units.

---

13. The Popcorn Problem I

While preparing to watch a horror film, Mr. Maychrowitz decides to make himself some microwave popcorn. As the popcorn begins to pop, he notices that the number of kernels popped could be represented by a logistic growth curve. After repeated trials and careful analysis, he ascertains that there are 300 unpopped kernels of corn in an average bag of popcorn and that the first kernel of popcorn pops after 20 seconds in the microwave. He therefore posits that the differential equation $\frac{dP}{dt} = 5.7P\left(1 - \frac{P}{300}\right)$ could appropriately model the situation, where P = the number of kernels of corn that have popped after 30 seconds in the microwave. For the function $P(t)$, t is minutes past the first 0.5 minutes in the microwave (so when the microwave has been going for 30 seconds, $t = 0$ minutes)

a) How many kernels of popcorn have popped when the rate of popping is the fastest?
b) Given that $P(0) = 1$, what is $\lim_{t \to \infty} P$?
c) Suppose that at 30 seconds ($t = 0$), 6 kernels of popcorn have popped. Given that fact, what is the $\lim_{t \to \infty} P$?
d) What does the quantity, $\left(1 - \frac{P}{300}\right)$, mean?
e) Find the solution curve for this particular differential, given that $P(0) = 1$.

---

14. The South Park Zombie Problem

In a South Park episode, an epidemic of zombie-ism breaks out after Kenny is killed and his blood is replaced by Worchester Sauce. The spread of cases of zombie-ism could be modeled by the logistic equation $\frac{dZ}{dt} = 0.45Z\left(1 - \frac{Z}{666}\right)$, where t is the time in days after they killed Kenny.

a) How many people does this model assume live in South Park?
b) How many zombies will there be when the fastest rate of growth of zombie-ism occurs?
c) What would the solution equation to the problem be logistic growth equation?
d) A different, non-logistic model for the growth rate of Kenny's zombies might be $\frac{dK}{dt} = 0.45\left(1 - \frac{K}{666}\right)$. Assuming $K(0) = 1$ (Kenny is the first zombie), find the particular solution to t his separable differential equation.

---

15. AP 2004 BC#5.

8.3 Multiple Choice Homework

1. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt}=\frac{1}{4000}P(400-P)$, where $P(0) = 100$. What value of $P(t)$ shows the fastest rate of growth?

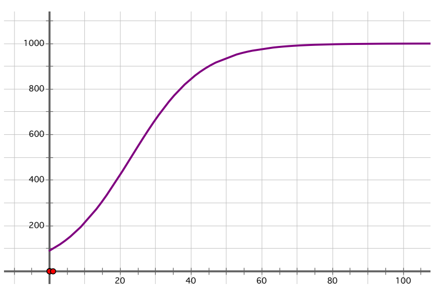
a) 10 b) 100 c) 200

d) 400 e) 4000

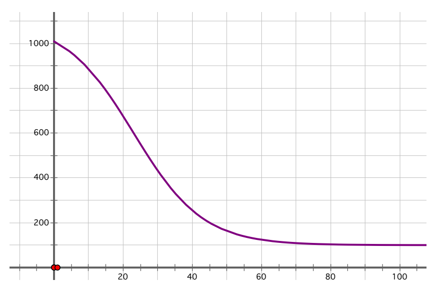
---

2. Which of the following graphs is of the solution $y=f(x)$ to the differential equation $\frac{dy}{dx}=.1y(1000-y)$?

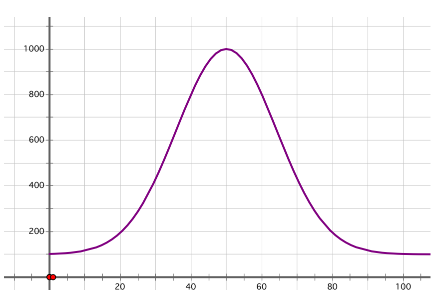
a)



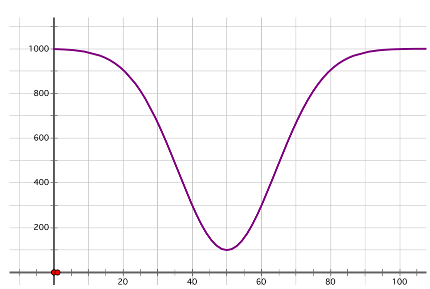
b)



c)



d)



e) None of these

---

3. The population P(t) of a species satisfies the logistic differential equation $\frac{dP}{dt} = \frac{1}{4000}P(400 - P)$, where P(0) = 100. What is the end behavior of $P(t)$?

a) $P = 10$ b) $P = 100$ c) $P = 200$

d) $P = 400$ e) $P = 4000$

---

4. The function F satisfies the logistic growth equation $\frac{dF}{dt} = \frac{F}{30}\left(2 - \frac{F}{650}\right)$, where F(0)=95. Which of the following statements is false?

a) $\lim_{t \to \infty} F(t) = 1300$ b) $\frac{dF}{dt}$ has a minimum value when $F = 95$.

c) $\frac{d^2F}{dt^2} = 0$ when $F = 650$. d) When $F < 650$, $\frac{dF}{dt} > 0$ and $\frac{d^2F}{dt^2} < 0$

---

5. The population of bears grows according to the logistic equation

$$\frac{dB}{dt} = 2B - 0.02B^2$$

where *B* is the number of bears and *t* is measured in years. Which of the following statements is false?

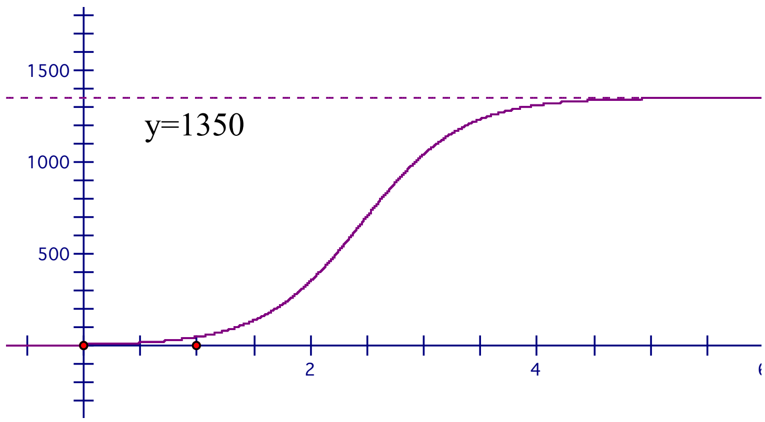
I. The growth rate of bears is greatest at $B = 50$
II. If $B > 100$, the population is decreasing.
III. $\lim_{t \to \infty} B(t) = 50$

a) I only b) II only c) I and II only

d) I and III only e) I, II, and III

---

6. The rate at which a rumor spreads at school of 1350 students can be modeled by the graph below, where R is the number of students that have heard the rumor t hours after 9am.



Which of these equations could not possibly be the logistic growth equation for this model?

a) $\frac{dR}{dt} = 2.272R\left(1 - \frac{R}{1350}\right)$ b) $\frac{dR}{dt} = 2.272(1350 - R)$

c) $\frac{dR}{dt} = -2.272R(1350 - R)$ d) $\frac{dR}{dt} = 2.272R(1350 - R)$

e) $\frac{dR}{dt} = 0.002R\left(1 - \frac{R}{1350}\right)$

---

7. The number of wildcats in a portion of the Sierra Nevada mountain range is modeled by the function W that satisfies the logistic differential equation $\frac{dW}{dt} = 0.4W\left(1 - \frac{W}{300}\right)$, where t is the time in years and $W(0) = 500$. What is $\lim_{t\to\infty} W(t)$?

a) 150 b) 300 c) 500

d) 600 e) 1000

---

9. The growth rate of a population $y(t)$ of dolphins is modeled by the logistic growth equation $\frac{dy}{dt} = \frac{y}{2}(120 - y)$. If $y(0) = 30$, which of these describes the future behavior of the population?

a) The population will increase towards 60 dolphins
b) The population will increase towards 120 dolphins
c) The population will decrease towards 120 dolphins
d) The population will decrease towards 60 dolphins

---

10. What is the solution curve to the logistic growth equation $\frac{dy}{dt} = 10y\left(1 - \frac{y}{100}\right)$ given that $y(0) = 20$?

a) $y = \frac{100}{1+4e^{-0.1t}}$ b) $y = \frac{100}{1+4e^{10t}}$ c) $y = \frac{100}{1+4e^{-10t}}$

d) $y = \frac{100}{1+20e^{-10t}}$ e) $y = \frac{100}{1+20e^{-0.1t}}$

8.4 Integration By Parts

Integration by Substitution is the most common integration method because it reverses the most common differentiation method--the Chain Rule. Integration By Parts reverses the Product Rule.

Integration By Parts: $\int u\,dv = uv - \int v\,du$

Here is the proof. Let's start with the product *uv* and take its derivative.

$(uv)' = uv' + vu'$	Now let's integrate both sides
$\int (uv)' = \int (uv' + vu')$	Apply the FTC to the left side of the equation
$uv = \int (uv' + vu')$	Distribute the integral
$uv = \int u\,dv + \int v\,du$	Isolate one integral
$\int u\,dv = uv - \int v\,du$	

Two questions arise immediately:

1. How do I recognize that I need to use integration by parts?
2. How do I choose my *u* and *dv*?

OBJECTIVES

- Identify integrals where Integration by Parts is appropriate.
- Apply the Integration by Parts method.

The answer to number 1 is, until you gain more experience, you guess and check. Integration by Parts problems often look like Integration by Substitution problems, at first, because they are both integration of a product. But the substitution does not work.

The answer to the second question is that it depends on the functions involved it he product. A simple mnemonic developed by Professor Fawal is **LIPTE**.

LIPTE = Logs, Inverse (trig, that is), Polynomials, Trig, Exponentials.

This is the (descending) order in which you choose u. If the product is a Log times a polynomial, the log is u. If the product is an inverse and an exponential, the inverse is u.

Dr. Quattrin says the key is which of the two functions is more easily integrated and which is easily differentiated. There are really only three cases to deal with.

I. A polynomial times a trig or exponential function: u = the polynomial.
II. A polynomial times a log or trig inverse: dv = the polynomial.
III. An exponential times a sinusoidal: your choice.

Case I: A polynomial times a trig or exponential function.

Ex1 Evaluate $\int xe^{5x}\,dx$.

Let $u = x$ and $dv = e^{5x}dx$. Choose your *u* and *dv*.

$$\begin{aligned} &Let\ \ u = x && dv = e^{5x}dx \\ &\quad du = dx && v = \frac{1}{5}e^{5x} \end{aligned}$$

Find *du* and *v*.

$$\int xe^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{5}\int e^{5x}\,dx$$

Apply the IBP formula. Note that the new integral has a more simpler integrand—one we already know how to integrate. If our integrand had become worse, it means we picked the wrong *u* and *dv*.

$$\int xe^{5x}\,dx = \frac{1}{5}xe^{5x} - \frac{1}{5}e^{5x} + C$$

Integrate and add *C*.

You can check your answer through differentiation.

Steps to Integrating By Parts:

0. **Check the Chain Rule first!!**
1. Determine that your integral *can* be evaluated using integration by parts.
2. Choose your u and dv.
3. Find du and v.
4. Apply $\int u dv = uv - \int v du$.
5. Evaluate your new integral and add C.
6. Repeat steps 1 thru 5 if necessary.

Ex 2 Evaluate $\int x^2 \sin x \, dx$.

Again, we have Case I a polynomial multiplied by a trig/exp function.

$$\int x^2 \sin x \, dx \qquad\qquad \begin{array}{ll} u = x^2 & dv = \sin x dx \\ du = 2x dx & v = -\cos x \end{array}$$

$$\begin{aligned} \int x^2 \sin x \, dx &= x^2(-\cos x) - \int -\cos x \; 2x \, dx \\ &= -x^2 \cos x + 2\int x \cos x \, dx \end{aligned}$$

This new integral is also a Case I Integration by Parts integral. Now,

$$\begin{array}{ll} u = x & dv = \cos x dx \\ du = dx & v = \sin x \end{array}$$

$$\begin{aligned} \int x^2 \sin x dx &= -x^2 \cos x + 2\int x \cos x \, dx \\ &= -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right] \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + C \end{aligned}$$

You can check your answer through differentiation.

Tabular Integration (as short-hand for Case I with higher powers)

Ex 3: $\int x^4 e^{2x}\,dx$

$$
\begin{array}{lllll}
u_1 = x^4 & & dv_1 = e^{2x} & & \\
u_2 = 4x^3 & \searrow & v_1 = \frac{1}{2}e^{2x} & \longrightarrow & u_1v_1 = x^4\left(\frac{1}{2}e^{2x}\right) \\
u_3 = 12x^2 & \searrow & v_2 = \frac{1}{4}e^{2x} & \longrightarrow & u_2v_2 = 4x^3\left(\frac{1}{4}e^{2x}\right) \\
u_4 = 24x & \searrow & v_3 = \frac{1}{8}e^{2x} & \longrightarrow & u_3v_3 = 12x^2\left(\frac{1}{8}e^{2x}\right) \\
u_5 = 24 & \searrow & v_4 = \frac{1}{16}e^{2x} & \longrightarrow & u_4v_4 = 24x\left(\frac{1}{16}e^{2x}\right) \\
 & \searrow & v_5 = \frac{1}{32}e^{2x} & \longrightarrow & u_5v_5 = 24\left(\frac{1}{32}e^{2x}\right)
\end{array}
$$

$$\int x^4 e^{2x}\,dx = \frac{1}{2}x^4e^{2x} - x^3e^{2x} + \frac{3}{2}x^2e^{2x} - \frac{3}{2}xe^{2x} + \frac{3}{4}e^{2x} + c$$

Case II: A polynomial times a log or trig inverse

Ex 4 $\int x\ln x\;dx$

Now we have a polynomial multiplied by a logarithm—that is, Case II. The reason

For this example I will tell you to let $u = \ln x$ and $dv = xdx$.

$$\int x\ln x\;dx \qquad \begin{array}{ll} u = \ln x & dv = xdx \\ du = \frac{1}{x}dx & v = \frac{x^2}{2} \end{array}$$

$$\int x \ln x \, dx = \frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$
$$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$$
$$= \frac{x^2}{2} \ln x - \frac{1}{2} \frac{x^2}{2} + C$$
$$= \frac{x^2}{2} \ln x - \frac{1}{4} x^2 + C$$

Ex 5 $\int \tan^{-1} x \; dx$

At first, this does not appear to be a product. But dx is actually 1 dx, and 1 is a polynomial. So,

$$u = \tan^{-1} x \qquad dv = dx$$
$$du = \frac{1}{1+x^2} dx \qquad v = x$$

$$\int \tan^{-1} x \; dx$$

$$\int \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{1+x^2} dx$$
$$= x \tan^{-1} x - \frac{1}{2} \int \frac{2x}{1+x^2} dx$$
$$= x \tan^{-1} x - \frac{1}{2} \ln\left(1+x^2\right) + C$$

Note that this second integral requires substitution.

Case III: An exponential times a sinusoidal. (This is the most interesting case.)

Ex 6 $\int e^x \cos x \; dx$

$\int e^x \cos x \; dx$

$$u = \cos x \qquad dv = e^x dx$$
$$du = -\sin x dx \qquad v = e^x$$

$$\begin{aligned}\int e^x \cos x \; dx &= e^x \cos x - \int (-\sin x) \; e^x \, dx \\ &= e^x \cos x + \int e^x \sin x \; dx\end{aligned}$$

This new integral is also a Case III integration by parts integral. Since we chose the trig function to be u the first time, we must choose the trig function to be u again. Otherwise, we will just undo the first step and return to the start.

$$u = \sin x \qquad dv = e^x dx$$
$$du = \cos dx \qquad v = e^x$$

$$\begin{aligned}\int e^x \cos x \; dx &= e^x \cos x - \int (-\sin x) \; e^x \, dx \\ &= e^x \cos x + \int e^x \sin x \; dx \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x dx\end{aligned}$$

This new integral is the same as the original, so we can consider it a "like term"" with the left side of the equation and add it over.

$$\int e^x \cos x \; dx = e^x \cos x + e^x \sin x - \int e^x \cos x dx$$
$$\int e^x \cos x dx + \int e^x \cos x dx = e^x \cos x + e^x \sin x + C$$
$$2\int e^x \cos x dx = e^x \cos x + e^x \sin x + C$$

$$\int e^x \cos x dx = \frac{1}{2} e^x \cos x + \frac{1}{2} e^x \sin x + C$$

Case III also gives rise to the last case of Trig Integrals in Chapter 2, which we could not solve then.

Review:

$$\int \sin^n x \ \cos^m x \ dxa$$

The odd powered term is du.
If both powers are even, use the double angle rules.

$$\int \sec^n x \tan^m x \ dx \text{ or } \int \csc^n x \cot^m x \ dx$$

If the sec power is even or the tan power is odd, that term is du.
If the sec power is odd AND the tan power is even, we need integration by parts.

This last case, which we could not do before, can finally be addressed.

Ex 7 Evaluate $\int \sec^3 x \ dx$.

Let's take a look at this integrand before we get going. Notice the power on secant is odd – it's 3. And the power on tangent is even (0 is considered even, for our purposes).

$$\int \sec^3 x \ dx = \int \sec x \cdot \sec^2 x \ dx \qquad \begin{matrix} u = \sec x & dv = \sec^2 x dx \\ du = \sec x \tan x dx & v = \tan x \end{matrix}.$$

$$\begin{aligned} \int \sec x \cdot \sec^2 x dx &= \sec x \tan x - \int \tan x \sec x \tan x dx \\ &= \sec x \tan x - \int \sec x \tan^2 x dx \\ &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1\right) dx \\ &= \sec x \tan x - \int \left(\sec^3 x - \sec x\right) dx \\ &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\ \int \sec^3 x dx &= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \end{aligned}$$

$$
\begin{aligned}
2\int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \\
2\int \sec^3 x dx &= \sec x \tan x + \ln\left|\sec x + \tan x\right| + C \\
\int \sec^3 x dx &= \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln\left|\sec x + \tan x\right| + C
\end{aligned}
$$

Ex 8 $\int e^{\sqrt{x}} dx$

As with $\int \tan^{-1} x \; dx$, this integral does not involve product, so it does not look like in integration by parts problem. But we also cannot directly integrate it. A u-substitution reveals the true nature of this integral.

$$
\begin{aligned}
&u = \sqrt{x} \\
&du = \frac{1}{2\sqrt{x}} dx \rightarrow dx = 2\sqrt{x} \; du
\end{aligned}
$$

Therefore, $\int e^{\sqrt{x}} dx = \int 2ue^{u} du$ and it is a case I. In fact, it is double Ex 1.

$$
\begin{aligned}
\int e^{\sqrt{x}} dx &= \int 2ue^{u} du \\
&= 2\int ue^{u} dx \\
&= 2ue^{u} - 2e^{u} + C \\
&= 2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C
\end{aligned}
$$

Ex 9 Does $\sum_{n=0}^{\infty} ne^{-n^2}$ converge?

$\lim_{n\to\infty} ne^{-n^2} = \lim_{n\to\infty} \frac{n}{e^{n^2}} = 0$; so it might converge.

$\int_1^{\infty} xe^{-x^2}\,dx$ requires a u-substitution.

$$\begin{aligned}
\int_1^{\infty} xe^{-x^2}\,dx &= -\frac{1}{2}\int_1^{\infty} -2xe^{-x^2}\,dx \\
&= -\frac{1}{2}\int_{-1}^{+\infty} e^{u}\,dx \\
&= -\frac{1}{2}\lim_{b\to-\infty} e^{u}\Big|_{-1}^{b} \\
&= \left(-\frac{1}{2}\lim_{b\to-\infty} e^{b}\Big|\right) - \left(-\frac{1}{2}e^{-1}\right) \\
&= 0 - \left(\frac{-1}{2e}\right) \\
&= \frac{1}{2e}
\end{aligned}$$

Since the integral equals a real number, the series must converge to a real number.

8.4 Free Response Homework

Evaluate the integrals.

1. $\int xe^{2x}dx$

2. $\int t^3e^{-t}dt$

3. $\int x^2 \cos 4x dx$

4. $\int m^2e^{2m}dm$

5. $\int (x^3+1)e^{-x}dx$

6. $\int x^7e^x dx$

7. $\int (\ln x)^2 dx$

8. $\int \cos^{-1} x \; dx$

9. $\int e^{-\theta} \cos 2\theta \; d\theta$

10. $\int x^2 \cos 3x \; dx$

11. $\int_1^4 \ln\sqrt{y} \; dy$

12. $\int e^{2\theta} \sin 3\theta \; d\theta$

13. $\int x \tan^{-1} x \; dx$

14. $\int \cos(\ln x) \; dx$

15. $\int \ln(3x+2) dx$

16. $\int x \ln(3x^2+2) dx$

17. $\int \cos\sqrt{x} dx$

18. $\int \sin^{-1} x dx$

19. If $\int f(x)e^{2x}dx = \frac{1}{2}f(x)e^{2x} - \int 12x^3e^{2x}dx$, then what is $f(x)$?

20. Use the Integral Test to determine if $\sum_{n=1}^{\infty} ne^{-n^2}$ converges or diverges.

21. Use the Integral Test to determine if $\sum_{n=1}^{\infty} ne^{-n}$ converges or diverges.

22. $\int \sec^2 x \; \sqrt{\tan x} \; dx$

23. $\int_1^\infty \frac{\ln x}{x^2}\, dx$

24. Find the average value of $x^2 \ln x$ on $x \in [1, 3]$.

25. Find the area bounded by $y = x\sin x$ and $y = (x-2)^2$.

8.4 Multiple Choice Homework

x	$h(x)$	$h'(x)$
0	2	5
3	-3	11

1. Let h be a function defined and continuous on the closed interval [0, 4]. If $\int_0^3 h(x)dx = 8$, then $\int_0^3 xh'(x)dx =$

a) -23 b) -18 c) -17 d) -12 e) 36

---

2. $\int x^2 \sin x\ dx =$

a) $-x^2\cos x - 2x\sin x - 2\cos x + c$

b) $-x^2\cos x + 2x\sin x + 2\cos x + c$

c) $-x^2\cos x + 2x\sin x + 2\cos x + c$

d) $-\frac{x^3}{3}\cos x + c$

e) $2x\cos x + c$

---

3. $\int x \sin 3x \, dx =$

a) $-\frac{1}{3}x\cos(3x)+\frac{1}{9}\sin(3x)+c$

b) $\frac{1}{3}x\cos(3x)+\frac{1}{9}\sin(3x)+c$

c) $\frac{1}{3}x\cos(3x)-\frac{1}{6}\sin(3x)+c$

d) $-\frac{1}{3}\cos(3x)-\frac{1}{6}\sin(3x)+c$

e) $3\sin(3x)+c$

---

4. $\int xe^{2x} \, dx =$

a) $\frac{1}{4}e^{2x}(2x-1)+c$ b) $\frac{1}{2}e^{2x}(2x-1)+c$ c) $\frac{1}{4}e^{2x}(4x-1)+c$

d) $\frac{1}{2}e^{2x}(x-1)+c$ e) $\frac{1}{4}e^{2x}(x-1)+c$

---

5. $\int_0^{\pi} x \sin x dx =$

a) $-\pi$ b) $-\frac{\pi^2}{2}$ c) 0 d) $\frac{\pi^2}{2}$ e) π

8.5 Integration of Radicals and Trig Substitutions

Some radical integrals can be easily dealt with by u – substitution and some cannot. As with Integration by Parts, only experience will lead you to quick and correct decisions about which technique to use. Trig substitution is the technique we will use to deal with a variety of radical integrals. Instead of letting u be some function of x, we will let x be some function of θ.

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $dx = a\cos\theta\ d\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta\ d\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\ d\theta$	$\sec^2\theta - 1 = \tan^2\theta$

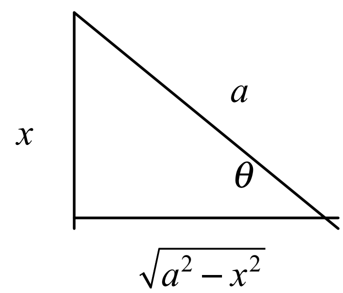
These substitutions are based on the fact that the radicands will convert to perfect squares.

Steps to Integrating With a Trig Sub:

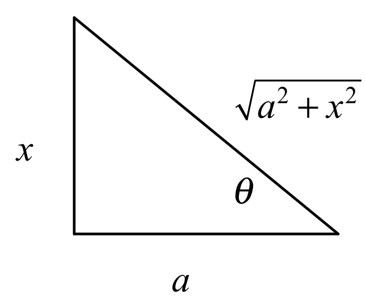
1. Determine that your integral ***should*** be evaluated using trig substitution.
   a) Don't forget the trig inverse rules and Back Substitution.
2. Make the appropriate trig sub, making sure to sub out for dx.
3. Use a trig identity to simplify your integrand.
4. Integrate and add C.
5. Back substitutes to x.

This last step requires looking at right triangles and SOHCAHTOA. Once the sides of the triangle are identified, any trig function can be replaces with its SOHCAHTOA referenced sides and an angle can be replaced by the trig inverse of the original substitution.

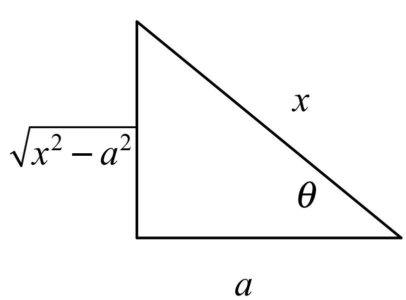
Case I. $\sqrt{a^2 - x^2}$ and $x = a\sin\theta$. In this case, the opposite side is x and the hypotenuse is a, making $\sqrt{a^2 - x^2}$ the adjacent side.



Case II. $\sqrt{a^2 + x^2}$ and $x = a\tan\theta$. In this case, the opposite side is x and the adjacent side is a, making $\sqrt{a^2 + x^2}$ the hypotenuse.



Case III. $\sqrt{x^2 - a^2}$ and $x = a\sec\theta$. In this case, the hypotenuse is x and the adjacent side is a, making $\sqrt{x^2 - a^2}$ the opposite side.



OBJECTIVE
Integrate radical integrands using trig substitution.

Ex1 Evaluate $\int x^3\sqrt{4-x^2}\,dx$.

$\int x^3\sqrt{4-x^2}\,dx$ Make this trig sub $\begin{cases} x = 2\sin\theta \\ dx = 2\cos\theta\ d\theta \end{cases}$.

$$\int x^3\sqrt{4-x^2}\,dx = \int\left(8\sin^3\theta\right)\sqrt{4-4\sin^2\theta}\left(2\cos\theta\right)d\theta$$

$$= 16\int \sin^3\theta\cos\theta\sqrt{4\left(1-\sin^2\theta\right)}\ d\theta$$ Use the trig identity

$$= 16\int \sin^3\theta\cos\theta\sqrt{4\cos^2\theta}\ d\theta$$

$$= 32\int \sin^3\theta\cos^2\theta\ d\theta$$ This is a trig integral you have learned about already.

$$= -32\frac{\cos^3\theta}{3} + 32\frac{\cos^5\theta}{5} + C$$

Now keep in mind, we started the problem in *x*'s, so we must end in *x*'s.

From the triangles above, we know the third side of the triangle must be $\sqrt{4-x^2}$. Since $\cos\theta = \frac{Adjacent}{Hypotenuse}$, then $\cos\theta = \frac{\sqrt{4-x^2}}{2}$ and this is what we wanted to use to substitute u back out of (and x back into) the answer.

$$\int x^3\sqrt{4-x^2}\ dx = -32\frac{\cos^3\theta}{3} + 32\frac{\cos^5\theta}{5} + C$$

$$= -32\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^3}{3} + 32\frac{\left(\frac{\sqrt{4-x^2}}{2}\right)^5}{5} + C$$

$$= -32\frac{\frac{\left(\sqrt{4-x^2}\right)^3}{2^3}}{3} + 32\frac{\frac{\left(\sqrt{4-x^2}\right)^5}{2^5}}{5} + C$$

$$= -\frac{4}{3}\left(4-x^2\right)^{3/2} + \frac{1}{5}\left(4-x^2\right)^{5/2} + C$$

There are two methods for evaluating this integral. One is a trig substitution and the other is a rather crafty u – sub. Not every trig sub integral will have two methods.

Ex 1 (again) $\int x^3\sqrt{4-x^2}\,dx$

This problem could also have been done by a u-substitution.

Let $u = 4-x^2$
$du = -2x\,dx$

Here is your u – sub.

$$\int x^3\sqrt{4-x^2}\,dx = \int x\cdot x^2\sqrt{4-x^2}\,dx$$

Split the integral up, so we can get to our u and du.

$$= -\frac{1}{2}\int -2x\cdot x^2\sqrt{4-x^2}\,dx$$

Multiply by a constant to get your du.

$$= -\frac{1}{2}\int \left(4-u\right) u^{1/2}\ du$$

Sub out for u.

$$= -\frac{1}{2}\int \left(4u^{1/2} - u^{3/2}\right)\ du$$

$$= -\frac{1}{2}\left[4\cdot\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right] + C$$

Sub back for x.

$$= -\frac{4}{3}\left(4-x^2\right)^{3/2} + \frac{1}{5}\left(4-x^2\right)^{5/2} + C$$

Same answer as in before.

Ex 2 $\int \sqrt{x^2+1}\ dx$

$\int \sqrt{x^2+1}\ dx$ According to the chart, use $\begin{cases} x = \tan\theta \\ dx = \sec^2\theta\ d\theta \end{cases}$

$\int \sqrt{x^2+1}\ dx = \int \sqrt{\tan^2\theta+1}\left(\sec^2\theta\right)\ d\theta$ Make the trig sub

$= \int \sqrt{\sec^2\theta}\sec^2\theta\ d\theta$ Use your trig identity

$= \int \sec^3\theta d\theta$ We remember this integral from the previous section.

$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C$

According to the triangles $x = \tan\theta$ and $\sec\theta = \sqrt{x^2+1}$. Now sub back out for $\tan\theta$ and $\sec\theta$.

$$\begin{aligned}\int \sqrt{x^2+1}\ dx &= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right| + C \\ &= \frac{1}{2}\sqrt{x^2+1}\cdot x + \frac{1}{2}\ln\left|\sqrt{x^2+1} + x\right| + C\end{aligned}$$

Ex 3 $\int \sqrt{x^2-9}\ dx$

$\int \sqrt{x^2-9}\ dx$ According to the chart, use $\begin{cases} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta d\theta \end{cases}$

$\int \sqrt{x^2-9}\ dx = \int \sqrt{9\sec^2\theta-9}\left(3\sec\theta\tan\theta\right)\ d\theta$ Make the trig sub

$= \int \sqrt{9\tan^2\theta}\left(3\sec\theta\tan\theta\right)d\theta$ Use your trig identity

$= 9\int \sec\theta\tan^2\theta\, d\theta = 9\int u\, du$

U-substitution $\begin{cases} u = \sec\theta \\ du = \sec\theta\tan\theta d\theta \end{cases}$

$= \frac{9}{2}u^2 + c = \frac{9}{2}\sec^2\theta + c$

$$=\frac{1}{2}\left(\frac{3}{x}\right)^2+c=\frac{9}{2x^2}+c$$

Ex 4 $\int \frac{x\ dx}{\sqrt{3-2x-x^2}}$

Before we start the trig subs, we need to complete the square to make the radical conform to one of our three cases.

$$\int \frac{x\ dx}{\sqrt{3-2x-x^2}}=\int \frac{x\ dx}{\sqrt{3+\quad-\left(\quad+2x+x^2\right)}}$$

$$=\int \frac{x\ dx}{\sqrt{3+1-\left(1+2x+x^2\right)}}$$

$$=\int \frac{x\ dx}{\sqrt{4-\left(1+x\right)^2}}$$

Now make the u – sub $\begin{cases} u=x+1 \\ x=u-1. \\ du=dx \end{cases}$

$$=\int \frac{(u-1)du}{\sqrt{4-u^2}}$$

Use Case III substitutions

$$=\int \frac{2\sin\theta-1}{2\cos\theta}\left(2\cos\theta\right)d\theta$$

Simplify

$$=\int\left(2\sin\theta-1\right)d\theta$$

and integrate

$$=-2\cos\theta-\theta+C$$

Substitute back to u

$$=-\sqrt{4-u^2}-\sin^{-1}\left(\frac{u}{2}\right)+C$$

and then substitute back to x

$$=-\sqrt{3-2x-x^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+C$$

8.5 Homework Set A

1. $\int \frac{x^3 \, dx}{\sqrt{x^2+9}}$

2. $\int \frac{dx}{(x^2)\sqrt{x^2-9}}$

3. $\int y^3\sqrt{9-y^2}\,dy$

4. $\int_{\sqrt{2}}^{2} \frac{x^3}{\sqrt{16-x^2}}\,dx$

5. $\int \frac{r \, dr}{(r^2+4)^{5/2}}$

6. $\int \frac{dx}{x\sqrt{x^2+3}}$

7. $\int \sqrt{2x-x^2}\;dx$

8. $\int e^t\sqrt{9-e^{2t}}\,dt$

9. $\int \frac{dx}{\sqrt{x^2-6x+14}}$

8.5 Homework Set B

1. $\int \sqrt{1-9x^2}\,dx$

2. $\int \frac{x}{\sqrt{4x^2-7}}\,dx$

3. $\int \frac{x}{\sqrt{8-2x-x^2}}\,dx$

4. $\int \frac{x}{x^2\sqrt{36-x^2}}\,dx$

5. $\int \frac{x^7}{\sqrt{x^2+3}}\,dx$

6. $\int \frac{du}{u^2\sqrt{25u^2-4}}$

7. $\int \frac{x^2}{\sqrt{9x-x^2}}\,dx$

8. $\int \frac{\sin\theta}{\sqrt{1+\cos^2\theta}}\,d\theta$

9. $\int x\sqrt{1-16x^4}\,dx$

8.6 Partial Fractions – Linear Repeated Fractions

In a previous section, we introduced the integration method of partial fractions. Now we will look closer into that method. Specifically into what happens when there are powers of our factors on the denominator. There is still a bunch of algebra to do, so don't worry.

Steps to Integrating By Partial Fractions:

1. Determine that your integral *can* be evaluated using partial fractions.
2. Set up the appropriate fractions. Linear factors have constants for their numerators. Factors with powers greater than 1 must have a partial fraction for each degree from the highest down to one.
3. Use algebra to determine the coefficients of your fractions.
4. Use the appropriate integration technique to evaluate your integrals.
5. Add C.

OBJECTIVE
Apply Partial Fractions to the proper type of integral.

Ex 1 $\int \frac{4x}{x^3-x^2-x+1}dx$

$\int \frac{4x}{(x-1)^2(x+1)}dx$ Factor the denominator.

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}$$ Set up your partial fractions.

$$\frac{4x}{(x-1)^2(x+1)}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1}=\frac{A(x-1)(x+1)+B(x+1)+C(x-1)^2}{(x-1)^2(x+1)}$$

$$4x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$$

Our fractions are equal, the denominators are the same, so the numerators must also be the same.

Note: We will now use the second method to find the coefficients for my partial fractions.

$$4x = Ax^2 - A + Bx + B + Cx^2 - 2Cx + C$$
$$4x = (Ax^2 + Cx^2) + (Bx - 2Cx) + (-A + B + C)$$

Gather and compare like terms

$$\left.\begin{array}{l} A + C = 0 \\ B - 2C = 4 \\ -A + B + C = 0 \end{array}\right\} \Rightarrow A = 1, B = 2, C = -1$$

Use substitution, linear combination, Cramer's Rule – we can even graph the system of equations to find the solution.

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \left[\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1} \right] dx$$

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$
$$= \ln\left|\frac{x-1}{x+1}\right| - \frac{2}{x-1} + C$$

Ex 2 $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

$$\int \frac{x^2}{(x+1)^3}\,dx$$

The degree in the numerator is less than the degree in the denominator, so we can begin.

$$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Set up your partial fractions. You must make a different fraction for each power of each factor in your denominator. In this case we have one linear factor, repeated three times … so we have three fractions. **Note:** Our factors are linear so the fractions have constants in the numerator, exactly one degree less.

$$\frac{x^2}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

Rationalize the numerator.

$$x^2 = A(x+1)^2 + B(x+1) + C$$

Our fractions are equal, the denominators are the same, so the numerators must also be the same.

$$x = -1 \Rightarrow C = 1$$

Choose a value for x that makes the binomials zero.

$$x = 0 \Rightarrow A + B = 0$$
$$x = 1 \Rightarrow 4A + 2B + 1$$

Choose any other value for x.
Choose another value for x.

$$\begin{Bmatrix} A + B = 0 \\ 4A + 2B + 1 \end{Bmatrix}$$

Solve the system of equations.

$$\Rightarrow A = 1, B = -2$$

$$\int \frac{x^2}{(x+1)^3}dx = \int\left[\frac{1}{x+1} + \frac{-2}{(x+1)^2} + \frac{1}{(x+1)^3}\right]dx$$

Plug your coefficients back into your partial fractions.

$$= \ln|x+1| + 2(x+1)^{-1} - \frac{1}{2}(x+1)^{-2} + C$$

Integrate and add *C*.

Ex 3 $\displaystyle\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx$$

The degree in the numerator is greater than the degree in the denominator so we must long divide.

$$\begin{array}{r|l} & x+1+\dfrac{4x}{x^3-x^2-x+1} \\ x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & -(x^4-x^3-x^2+x) \\ \hline & x^3-x^2+3x+1 \\ & -(x^3-x^2-x+1) \\ \hline & 4x \end{array}$$

$$\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx = \int\left[x+1+\frac{4x}{x^3-x^2-x+1}\right]dx$$

$$\int\left[x+1+\frac{4x}{(x-1)^2(x+1)}\right]dx$$

Note that this fraction is EX 1.

$$\int\left[x+1+\frac{4x}{(x-1)^2(x+1)}\right]dx = \int\left[x+1+\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\right]dx$$

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\,dx = \frac{x^2}{2} + x + \ln|x-1| + \frac{2}{x-1} - \ln|x+1| + C$$

$$= \frac{x^2}{2} + x + \ln\left|\frac{x-1}{x+1}\right| + \frac{2}{x-1} + C$$

Ex 4 $\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2}\,dx$

$$\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2}\,dx = \int \frac{5x^2 + 3x - 2}{(x+2)x^2}\,dx$$

$$= \int \left(\frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+2} \right) dx$$

<u>Scratch Work</u>

$$A(x+2) + Bx(x+2) + Cx^2 = 5x^2 + 3x - 2$$

$$x = 0 \Rightarrow A = -1$$

$$x = -2 \Rightarrow C = 3$$

$$x = 1 \Rightarrow B = 2$$

$$\int \frac{5x^2 + 3x - 2}{x^3 + 2x^2}\,dx = \int \left(\frac{-1}{x^2} + \frac{2}{x} + \frac{3}{x+2} \right) dx$$

$$= \frac{1}{x} + 2Ln|x| + 3Ln|x+2| + C$$

$$= \frac{1}{x} + Ln\left|x^2(x+2)^3\right| + C$$

8.6 Homework Set A

1. $\displaystyle\int \frac{x}{(x+1)^3}\,dx$

2. $\displaystyle\int \frac{1}{(t+5)^2(t-1)}\,dt$

3. $\displaystyle\int \frac{1}{(y-3)(y+2)^2}\,dy$

4. $\displaystyle\int \frac{1}{x^4-x^2}\,dx$

5. $\displaystyle\int \frac{x^2+9x-12}{(3x-1)(x+6)^2}\,dx$

6. $\displaystyle\int \frac{z^2-4z}{(3z+5)^3(z+2)}\,dz$

7. $\displaystyle\int \frac{x^3}{(x+1)^3}\,dx$

8. Find the volume of a solid where the region bounded by $y=\dfrac{x-1}{x^2-5x+6}$, and the x-axis from x=4 to x=6 is revolved about the x-axis.

8.6 Homework Set B

1. $\displaystyle\int \frac{5x^2+3x-2}{x^3+3x^2}\,dx$

2. $\displaystyle\int \frac{2y^2-4y+5}{(y-1)(y+2)^2}\,dy$

3. $\displaystyle\int \frac{2x-5}{(x+1)^3}\,dx$

4. $\displaystyle\int \frac{x^3}{(x-3)(x+2)^2}\,dx$

5. $\displaystyle\int \frac{x^2}{(x+2)^3}\,dx$

6. $\displaystyle\int \frac{x^3}{(x+2)^3}\,dx$

7. $\displaystyle\int \frac{1}{(x+6)^2(x-4)^2}\,dx$

8. $\displaystyle\int \frac{2}{y^2(y-1)}\,dy$

8.7 Partial Fractions with Quadratic Factors

In our final section dealing with partial fractions we will now introduce fractions whose denominators contain quadratic factors – think x^2+4.

Steps to Integrating By Partial Fractions:

1. Determine that your integral *can* be evaluated using partial fractions.
2. Set up the appropriate fractions. Linear factors have constants for their numerators. ***Quadratic factors have a linear equation for their numerators.*** Factors with powers greater than 1 must have a fraction for each degree.
3. Use algebra to determine the coefficients of your fractions.
4. Use the appropriate integration technique to evaluate your integrals.
5. Add C.

OBJECTIVE
Apply Partial Fractions to integrals with Quadratic factors.

Ex 1 $\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$

$$\int \frac{3x^2-4x+5}{(x-1)(x^2+1)}\,dx$$

The degree in the numerator is less than the degree in the denominator, so we can begin.

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+1)}$$

Set up your partial fractions. Each factor must have a fraction made for it. And in for each fraction, the numerator must be exactly one degree less than the degree of the denominator. So for our linear factor, the numerator will be a constant, but for our quadratic factor, our numerator will be linear.

$$\frac{3x^2-4x+5}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{(x^2+1)}=\frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}$$ Common denominators.

$$3x^2-4x+5=A(x^2+1)+(Bx+C)(x-1)$$ Our fractions are equal, the denominators are the same, so the numerators must also be the same.

$$3x^2-4x+5=Ax^2+A+Bx^2-Bx+Cx-C$$ Match like terms.

$$\left.\begin{array}{l}A+B=3\\-B+C=-4\\A-C=5\end{array}\right\}\Rightarrow A=2, B=1, C=-3$$ Solve the system of equations.

$$\int\frac{3x^2-4x+5}{(x-1)(x^2+1)}dx=\int\left[\frac{2}{x-1}+\frac{x-3}{(x^2+1)}\right]dx$$ Plug the coefficients back in.

$$\int\left[\frac{2}{x-1}+\frac{x-3}{(x^2+1)}\right]dx=\int\frac{2}{x-1}dx+\int\frac{x}{x^2+1}dx-3\int\frac{1}{x^2+1}dx$$

Split the integral up. In order to do these three integrals we will need to use a natural log, a u – sub, and an inverse tangent, respectively. By this point we should be able to dispense with the actual substitution.

$$=\int\frac{2}{x-1}dx+\frac{1}{2}\int\frac{2x}{x^2+1}dx-3\int\frac{1}{x^2+1}dx$$

$$=2\ln|x-1|+\frac{1}{2}\ln(x^2+1)-3\tan^{-1}x+C$$

Note the absence of Absolute Value signs in the Ln. You should know why.

Ex 2 $\int \frac{-2x^2-10x+12}{(x+2)(x^2+4)}dx$

$$\frac{-2x^2-10x+12}{(x+2)(x^2+4)}=\frac{A}{x+2}+\frac{Bx+C}{(x^2+4)}=\frac{A(x^2+4)+(Bx+C)(x+2)}{(x+2)(x^2+4)}$$

$$\frac{A}{x-1}+\frac{Bx+C}{x^2+2x+2}=\frac{A(x^2+2x+2)}{(x-1)(x^2+2x+2)}+\frac{(Bx+C)(x-1)}{(x-1)(x^2+2x+2)}$$

$$-2x^2-10x+12=A(x^2+4)+(Bx+C)(x+2)$$

$$x=-2\Rightarrow 8A=24\Rightarrow A=3$$

$$\left.\begin{array}{l}A+B=-2\\4A+2C=12\end{array}\right\}\Rightarrow A=3\Rightarrow B=-5,C=0$$

$$\begin{aligned}\int \frac{-2x^2-10x+12}{(x+2)(x^2+4)}dx &= \int\left(\frac{3}{x+2}+\frac{5x}{x^2+4}\right)dx\\ &=3\int\left(\frac{1}{x+2}\right)dx+5\int\left(\frac{x}{x^2+4}\right)dx\\ &=3\int\left(\frac{1}{x+2}\right)dx+5\int\left(\frac{x}{x^2+4}\right)dx\\ &=3\int\left(\frac{1}{x+2}\right)dx+\frac{5}{2}\int\left(\frac{2x}{x^2+4}\right)dx\\ &=3\ln|x+2|+\frac{5}{2}\ln(x^2+4)+c\end{aligned}$$

Ex 3 $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx$

$$\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx = \int\left(\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}\right)dx$$

$$\begin{aligned}1-x+2x^2-x^3 &= A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x \\ &= A(x^4+4x^2+1)+B(x^4+x^2)+C(x^3+x)+Dx^2+Ex \\ &= (A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A\end{aligned}$$

Equating the like-terms' coefficients, we get

$A=1,\ A+B=1,\ C=-1,\ 2A+B+D=2,$ and C+E$=-1$

$A=1,\ B=-1,\ C=-1,\ D=1,$ and E=0

$$\begin{aligned}\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}dx &= \int\left(\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}\right)dx \\ &= \int\left(\frac{1}{x}+\frac{-x-1}{x^2+1}+\frac{x}{(x^2+1)^2}\right)dx \\ &= \int\frac{1}{x}dx-\int\frac{x+1}{x^2+1}dx+\int\frac{x}{(x^2+1)^2}dx \\ &= \int\frac{1}{x}dx-\frac{1}{2}\int\frac{2(x+1)}{x^2+1}dx+\frac{1}{2}\int\frac{2x}{(x^2+1)^2}dx \\ &= \int\frac{1}{x}dx-\frac{1}{2}\int\frac{2x}{x^2+1}dx-\frac{1}{2}\int\frac{2}{x^2+1}dx+\frac{1}{2}\int\frac{2x}{(x^2+1)^2}dx\end{aligned}$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{\left(x^2+1\right)^{-1}}{-1} + C$$

$$= \ln|x| - \frac{1}{2}\ln\left(x^2+1\right) - \frac{1}{2}\tan^{-1}x - \frac{1}{2\left(x^2+1\right)} + C$$

8.7 Homework Set A

1. $\int \frac{x^3}{x^2+1}dx$

2. $\int \frac{x^4+1}{x\left(x^2+1\right)^2}dx$

3. $\int \frac{3y^2-4y+5}{(y-1)\left(y^2+1\right)}dy$

4. $\int \frac{2t^3-t^2+3t-1}{\left(t^2+1\right)\left(t^2+2\right)}dt$

5. $\int \frac{1}{x^3-1}dx$

6. $\int \frac{x^3}{x^3-1}dx$

7. $\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}dx$

8.7 Homework Set B

1. $\int \frac{x^3+x^2+x-1}{\left(x^2+1\right)\left(x^2-1\right)}dx$

2. $\int \frac{x^3+3x^2+2x+1}{x^4+13x^2+36}dx$

3. $\int \frac{\cos x}{\sin^4 x-1}dx$

4. $\int \frac{e^{2x}}{e^{4x}+3e^{2x}+2}dx$

5. $\int \frac{x-1}{\left(x^2+4\right)(x-3)}dx$

6. $\int \frac{x^2-x+5}{x^3+4x}dx$

7. $\int \frac{8}{\left(x^2+16\right)(x-2)}dx$

8. $\int \frac{1}{x^3-1}dx$

9. $\int \frac{x^3}{x^3-1}dx$

8.8 General Integration Techniques and Strategies

You now have several formulas and techniques with which to approach integration. The real problem now is what to use when. First, of course, you need to have these formulas memorized.

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ if\ \ n \neq -1$$

$$\int \frac{1}{u}\,du = Ln|u| + C$$

$$\int e^u\,du = e^u + C$$

$$\int a^u\,du = \frac{a^u}{\text{Ln}a} + C\ \ (a > 0\ \ and\ \ a \neq 1)$$

$$\int \sin u\,du = -\cos u + C$$

$$\int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C$$

$$\int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \tan u\,du = \ln|\sec u| + C$$

$$\int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C$$

$$\int \sin^2 u\,du = \frac{1}{2}u - \frac{1}{4}\sin 2u + C$$

$$\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left|\frac{u}{a}\right| + C^*$$

$$\int \frac{du}{u^2 + a^2} = \frac{1}{a}\cdot\tan^{-1}\left|\frac{u}{a}\right| + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\cdot\sec^{-1}\left|\frac{u}{a}\right| + C^*$$

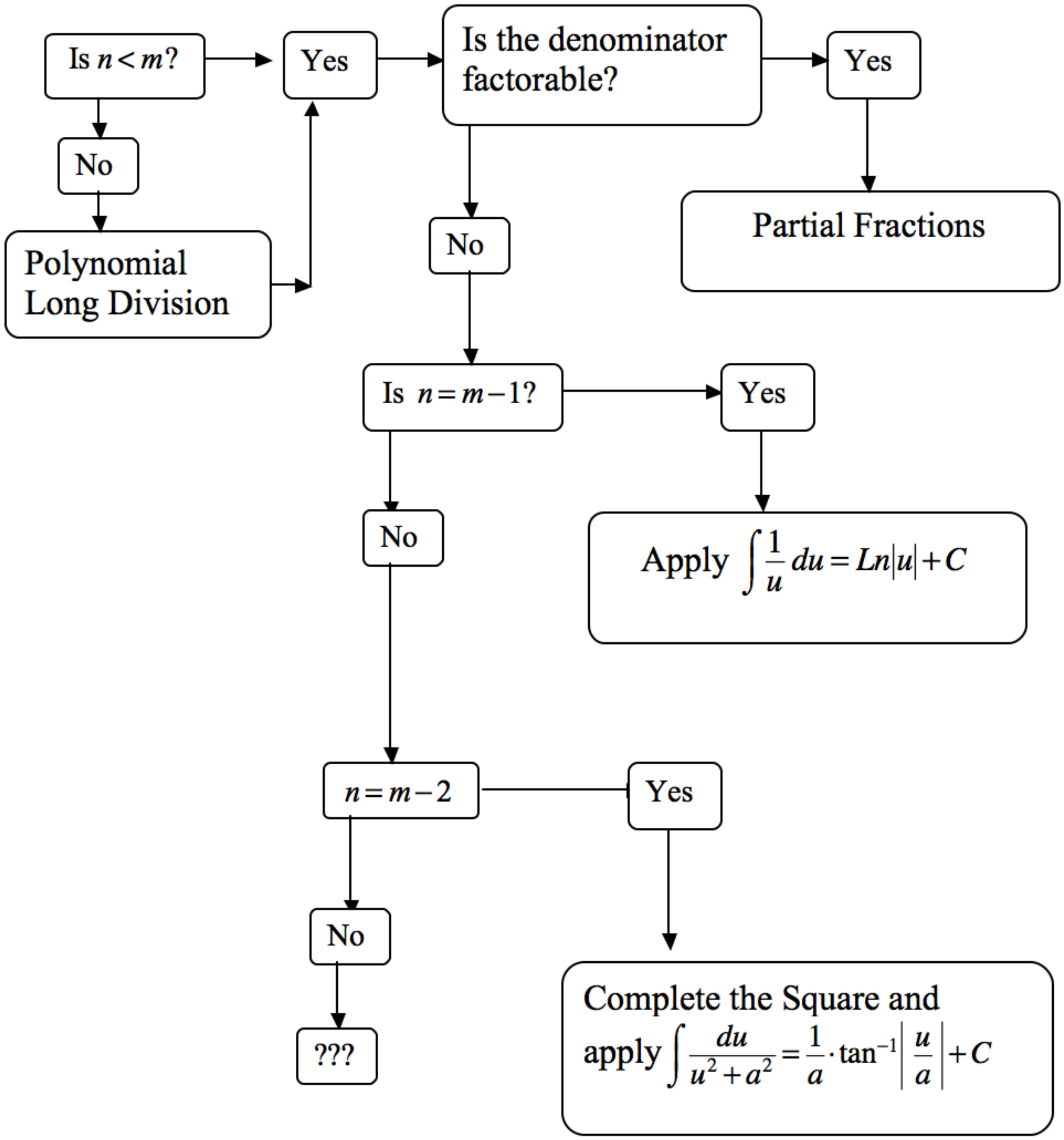
$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) + C$$

$$\int u\,dv = uv - \int v\,du$$

*Note the variations on the inverse trig rules

What To Do When Confronted with an Integral:

1. Apply a memorized formulae.
2. Product of two functions
   a. Foil to apply the Power Rule
   b. Apply the Chain Rule
      i. Simple u-sub
      ii. Pythagorean Identities
      iii. Don't forget about Back Substitution
   c. Integration by Parts. (LIPET)
3. Rational Functions:
   a. Simplify to apply the Power Rule
   b. Polynomial Long Division
   c. Partial Fractions
   d. $\int \frac{1}{u} du = Ln|u| + C$
   e. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \cdot \tan^{-1} \left| \frac{u}{a} \right| + C$
4. Radical functions:
   a. Apply the Chain Rule.
   b. Apply Trig Substitution.



OBJECTIVE

Determine the correct technique to use and perform the integration.

Ex 1 $\int \frac{\cot^3 x}{\sin^3 x}\, dx$

$$\begin{aligned}\int \frac{\cot^3 x}{\sin^3 x}\, dx &= \int \cot^3 x \csc^3 x\, dx \\ &= -\int \cot^2 x \csc^2 x\,(-\cot x \csc x\, dx) \\ &= -\int \left(u^2+1\right)u^2\, du \\ &= -\int \left(u^4+u^2\right) du \\ &= -\frac{1}{5}u^5 - \frac{1}{3}u^3 + C \\ &= -\frac{1}{5}\csc^5 x - \frac{1}{3}\csc^3 x + C\end{aligned}$$

Ex 1 (again) $\int \frac{\cot^3 x}{\sin^3 x}\, dx$

$$\begin{aligned}\int \frac{\cot^3 x}{\sin^3 x}\, dx &= \int \frac{\cos^3 x}{\sin^3 x}\frac{1}{\sin^3 x}\, dx \\ &= \int \frac{\cos^3 x}{\sin^6 x}\, dx \\ &= \int \frac{\cos^2 x}{\sin^6 x}\cos x\, dx \\ &= \int \frac{1-u^2}{u^6}\, du \\ &= \int \left(u^{-6}-u^{-4}\right) du \\ &= -\frac{1}{5}u^{-5} - \frac{1}{3}u^{-3} + C \\ &= -\frac{1}{5}\sin^{-5} x - \frac{1}{3}\sin^{-3} x + C \\ &= -\frac{1}{5}\csc^5 x - \frac{1}{3}\csc^3 x + C\end{aligned}$$

Ex 2 $\int \frac{dx}{x\sqrt{\ln x}}$

This is one of the u-subs you need to remember: $\left\{ \begin{array}{c} u = \ln x \\ du = \frac{1}{x}dx \end{array} \right\}$

$$\begin{aligned} \int \frac{dx}{x\sqrt{\ln x}} &= \int u^{-1/2} du \\ &= \frac{u^{1/2}}{1/2} + C \\ &= 2\sqrt{\ln x} + C \end{aligned}$$

Ex 3 $\int \cos^{-1} x \ dx$

We saw this before with the $\tan^{-1} x$. We have no rule for the integration, but we can differentiate $\cos^{-1} x$.

$$\left\{ \begin{array}{ll} u = \cos^{-1} x & dv = dx \\ du = \frac{-1}{\sqrt{1-x^2}}dx & v = x \end{array} \right\}$$

$$\begin{aligned} \int \cos^{-1} x \ dx &= x\cos^{-1} x - \int \frac{-x}{\sqrt{1-x^2}} dx \\ &= x\cos^{-1} x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}} dx \\ &= x\cos^{-1} x - \frac{1}{2}\frac{\sqrt{1-x^2}}{1/2} + C \\ &= x\cos^{-1} x - \sqrt{1-x^2} + C \end{aligned}$$

Ex 4 $\int \sqrt{\frac{1-x}{1+x}}\,dx$

There are a couple of ways to start, but the simplest is to rationalize first.

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}}\,dx &= \int \sqrt{\frac{1-x}{1+x}}\frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx\\ &= \int \frac{1-x}{\sqrt{1-x^2}}\,dx\\ &= \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx\\ &= \int \frac{1}{\sqrt{1-x^2}}\,dx + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx\\ &= \sin^{-1} x + \frac{1}{2}\frac{\sqrt{1-x^2}}{1/2} + C\\ &= \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

Ex 5 $\int_0^1 \frac{x^2+1}{x^2-1}\,dx$

There are three things to note about his problem:
1. The numerator has a higher degree, so long division is necessary;
2. The denominator is factorable, so it is a partial fractions problem;
3. It is an Improper Integral Type II.

$$\begin{aligned}\int_0^1 \frac{x^2+1}{x^2-1}\,dx &= \int_0^1 \left(1+\frac{2}{x^2-1}\right)dx\\ &= \int_0^1 dx + 2\int_0^1 \frac{1}{x^2-1}\,dx\\ &= \lim_{b\to 1^-}\left(\int_0^b dx + 2\int_0^b \frac{1}{x^2-1}\,dx\right)\end{aligned}$$

While it is true that the denominator is factorable, so it is a partial fractions problem, we also have a formula for this second integral.

$$\lim_{b\to 1^-}\left(\int_0^b dx + 2\int_0^b \frac{1}{x^2-1}\,dx\right) = \lim_{b\to 1^-}\left(x + \frac{1}{2}Ln\left|\frac{x-1}{x+1}\right|\right)_0^b$$

$$= \lim_{b\to 1^-}\left(b + \frac{1}{2}Ln\left|\frac{b-1}{b+1}\right|\right) - \left(0 + \frac{1}{2}Ln\left|\frac{0-1}{0+1}\right|\right)$$

$$= \left(0 + \frac{1}{2}Ln\left|\frac{0}{1}\right|\right) - 0$$

Since we cannot Ln 0, this integral is divergent.

8.8 Homework Set A

Decide which integration technique is appropriate for the following integrals:
A: *u*-sub
B: Integration by Parts
C: Trigonometric Identity
D: Partial Fractions
E: Memorized formula
AB: Divide first, then Integrate by Parts.
AC: Divide first, then use an Inverse Trigonometric Identity
AD: Divide first, then use Partial Fractions
AE: Foil and apply the Power Rule
BC: Back Substitution
BD: Integration by Parts
BE: Tabular integration

1. $\int e^x \sqrt{1+e^x}\, dx$

2. $\int_{-20}^{0} e^{\sqrt{x}}\, dx$

3. $\int \frac{d\theta}{\theta^4+\theta^2}$

4. $\int \arctan\left(\frac{1}{x}\right) dx$

5. $\int \sin^2(2\theta)\, d\theta$

6. $\int x^2 \tan^{-1} x\, dx$

7. $\int \frac{x^3}{(x+1)^3}\, dx$

8. $\int \sqrt{t}\ln(t)\, dt$

9. $\int x^{3/2} \ln x\, dx$

10. $\int \frac{e^{2x}}{e^{2x}+3e^x+2}\, dx$

11. $\int \frac{1}{x^2-6x+18}\, dx$

12. $\int \frac{1}{x^2-6x-8}\, dx$

13. $\int t\tan^{-1} t\, dt$

14. $\int \frac{du}{u\sqrt{u^2-a^2}}$

15. $\int x^5 e^{x^2}\, dx$

16. $\int \frac{1}{y^2-4y-12}\, dy$

17. $\int_3^{189} \frac{x}{x+2}\, dx$

18. $\int (x^3+3x^2+3x+1)(3x^4+6x+3)\, dx$

19. $\int (\cos^2 x+\sin^2 x)\, dx$

20. $\int \frac{x+1}{9x^2+6x+5}dx$ 21. $\int \tan^2 x \cos^2 x \, dx$ 22. $\int \frac{x^3}{(x+1)^{10}}dx$

Evaluate the following Integrals.

23. $\int \frac{\cos x}{1+\sin^2 x}dx$

24. $\int \frac{1+\cos x}{\sin x}dx$

25. $\int_0^\infty \frac{e^{\arctan y}}{y^2+1}dy$

26. $\int \sin^2 x \cos^3 x dx$

27. $\int \frac{x}{\sqrt{1-x^2}}dx$

28. $\int_0^{1/\sqrt{2}} \frac{x^3}{\sqrt{1-x^2}}dx$

29. $\int_0^3 \frac{2t}{(t-3)^2}\, dt$

30. $\int \sin x \cos(\cos x)dx$

31. $\int e^{x+e^x}dx$

32. $\int t^3 e^{-2t} dt$

33. $\int \frac{3x^2-2}{x^2-2x-8}dx$

34. $\int \frac{3x^2-2}{x^3-2x-8}dx$

35. $\int_{-3}^{3} \left| t^3+t^2-2t \right| \, dt$

36. $\int \frac{4x^2+x-2}{x^3-5x^2+8x-4}dx$

37. $\int_0^5 \frac{3w-1}{w+2}dw$

38. $\int_0^{\pi/4} \tan^2\theta \cos^2\theta \; d\theta$

39. $\int_0^{\pi/4} \tan^3\theta \sec^4\theta \; d\theta$

40. $\int \frac{1}{x\sqrt{4x^2-1}}dx$

41. $\int \frac{x^4}{x^{10}+16}dx$

42. $\int \frac{x}{x^4+4x^2+3}dx$

43. $\int \frac{u^3+1}{u^3-u^2}dx$

44. $\int \frac{1}{1+2e^x-e^{-x}}dx$

8.8 Homework Set B

1. $\int \frac{dx}{\sqrt{3x+1}}$

2. $\int \frac{3x^2}{\left(1+x^3\right)^2}dx$

3. $\int x\sqrt{1-x^2}\,dx$

4. $\int \frac{t}{1+t^4}dt$

5. $\int xe^{x^2}\,dx$

6. $\int \sqrt{x^2-2x+1}dx$

7. $\int \frac{x}{x+2}dx$

8. $\int xe^{3x}\,dx$

9. $\int \frac{1}{x^2-4x-5}dx$

10. $\int \frac{1}{x^2-4x+5}dx$

11. $\int \frac{1}{x(\ln x)^2}dx$

12. $\int \frac{x^2}{\sqrt{1-x^2}}dx$

13. $\int \frac{\sec^2 x}{1+\tan x}dx$

14. $\int \frac{1}{\sqrt{x}\left(1-\sqrt{x}\right)}dx$

15. $\int e^{-x}dx$

16. $\int \frac{x^2+2x+9}{x^2+9}dx$

Integration Techniques Test

1. $\int \frac{4}{x^2-4x-12}\,dx=$

a) $\frac{1}{2}\ln\left|\frac{x+2}{x-6}\right|+c$

b) $\frac{1}{2}\ln\left|\frac{x-6}{x+2}\right|+C$

c) $\frac{1}{8}\ln\left|(x-6)(x+2)\right|+C$

d) $\frac{1}{8}\ln\left|(x-6)(x+2)\right|+C$

e) $\frac{1}{8}\ln\left|\frac{x-6}{x+2}\right|+C$

---

2. $\int xe^{2x}\,dx=$

a) $\frac{1}{4}e^{2x}(2x-1)+c$

b) $\frac{1}{2}e^{2x}(2x-1)+c$

c) $\frac{1}{4}e^{2x}(4x-1)+c$

d) $\frac{1}{2}e^{2x}(x-1)+c$

e) $\frac{1}{4}e^{2x}(x-1)+c$

---

3. What is the best method to evaluate $\int \frac{1}{x^2+4x+7}\,dx = ?$

a) Integration by Parts b) Substitution c) Partial Fractions

d) Completing the Square e) None of these

---

4. $\int \frac{e^{x^2}-2x}{e^{x^2}}dx =$

a) $-e^{-x^2}+c$
b) $-e^{x^2}+c$
c) $x-e^{x^2}+c$
d) $x+e^{-x^2}+c$
e) $x-e^{-x^2}+c$

---

5. What is the best method to evaluate $\int \frac{dx}{x\sqrt{4x^2-9}}?$

a) Integration by Parts b) Substitution c) Partial Fractions

d) Completing the Square e) None of these

---

6. The population $P(t)$ of a species satisfies the logistic differential equation $\frac{dP}{dt} = \frac{1}{4000} P(400 - P)$, where $P(0) = 100$. What is the **maximum rate of change** of $P(t)$?

a) 10

b) 100

c) 200

d) 400

e) 4000

---

7. Which of the following statements are true?

I. $\int (\sin^3 x \ \cos^2 x)\, dx = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + c$

II. $\int \sec 2x \, dx = 2 \sec 2x \tan 2x + c$

III. $\int \left(\frac{3x^2 + 6x - 4}{(x^3 + 3x^2 - 4x + 2)^2} \right) dx = \ln\left| x^3 + 3x^2 - 4x + 2 \right|^2 + c$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

7. Find the volume of the solid formed when the region bounded by $y = x^2 e^{-2x}$ and the x-axis on $x \in [-1, 0]$ is revolved about the x-axis. Show the anti-differentiation.

8. $\int x \cot^{-1} x^2 \, dx$

9. $\int \frac{2x^4 + 3x^3 - 14x^2 - 7x + 18}{x^3 - 7x + 6}\, dx$

Chapter 8 Answer Key

8.1 Free Response Answer Key

1. $\frac{1}{2}\ln\left(x^2-4x+5\right)+\frac{1}{2}\tan^{-1}(x-2)+c$

2. $\frac{1}{5}\tan^{-1}\frac{x}{5}+c$

3. $\frac{2}{\sqrt{7}}\tan^{-1}\frac{2x-1}{\sqrt{7}}+c$

4. $\frac{1}{\sqrt{7}}\tan^{-1}\frac{e^x}{\sqrt{7}}+c$

5. $\frac{1}{2}\ln\left(x^2+x+1\right)-\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c$

6. $\frac{x^2}{2}+2x+\frac{5}{4}\ln\left(2x^2-4x+3\right)-\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2}}\right)+C$

7. $\frac{x^2}{2}+2x+5\ln|x-2|+c$

8. $\frac{x^2}{2}-x+\frac{1}{2}\ln\left|x^2+x+1\right|+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}+c$

9. $\frac{x^2}{2}-4x+\ln\left|x^2+2x+5\right|+\frac{3}{2}\tan^{-1}\frac{x+1}{2}+c$

10. $\frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c$

11. $x+7\ln|x-7|+c$

12. $\frac{1}{2}\ln\left|x^2+2x+5\right|+2\tan^{-1}\left(\frac{x+1}{2}\right)+c$

13. $\frac{1}{2}\ln\left|x^2+6x+13\right|-\frac{3}{2}\tan^{-1}\left(\frac{x+3}{2}\right)+c$

14. $3\ln\left|x^3+3x^2+5\right|+c$

15. $\dfrac{-1}{x^2+2x+2}+c$

16. $\dfrac{3x^2}{2}+x+\tan^{-1}(x+2)+c$

8.1 Multiple Choice Answer Key

1. C 2. D 3. D 4. D 5. D 6. C

7. A 8. E

8.2 Free Response Answer Key

1. $\ln\dfrac{(x+5)^2}{|x-2|}+c$

2. $\dfrac{1}{5}\ln\left|\dfrac{t-1}{t+4}\right|+c$

3. $\dfrac{1}{2}x^2-5x+25\ln|x+5|+c$

4. $x+\ln\left|\dfrac{(x-1)^2}{x}\right|+c$

5. $\dfrac{2}{3}\ln|x-5|+\dfrac{1}{3}\ln|x+1|+c$

6. $\dfrac{1}{2}\ln\left|x^2-4x+5\right|+\tan^{-1}(x-2)+c$

7. $\dfrac{1}{2}x^2+\dfrac{1}{7}\ln\left|\dfrac{x-3}{x+4}\right|+c$

8. $\ln\left|\dfrac{e^x+1}{e^x+2}\right|+c$

9. $2\ln|x|+\dfrac{9}{5}\ln|x+2|+\dfrac{1}{5}\ln|x-3|+c$

10. $\dfrac{1}{5}\ln\left|\dfrac{x-2}{x+3}\right|+c$

11. $\dfrac{1}{2}x^2+x-\ln|x|+\dfrac{1}{2}\ln|(x+1)(x-1)|+c$

12. $x-\ln|x-2|-\dfrac{1}{7}\ln|x+2|+\dfrac{50}{7}\ln|x-5|+c$

13. $\frac{5}{2}\ln\left|\frac{x-2}{3x-4}\right|+c$

14. $-\frac{3}{2}\ln|2x+1|+2\ln|x-4|+c$

15. $\ln\frac{9}{2}$ 16. $\frac{1}{3}\ln 5$ 17. $\frac{\pi\ln 3}{2}$ 18. 13.208

8.1 Multiple Choice Answer Key

1. B 2. E 3. A 4. D 5. A

8.3 Free Response Answer Key

1a) 180 lbs. b) 150 lbs.

c) $W=180-30e^{-\frac{1}{175}t}$ d) 192.257 days

2a) $A-10=0.45(t-0)$ b) 15.4 days.

c) $A=100-90e^{-.005t}$

(d) The population is increasing at an increasing rate.

3a) $A=100-100e^{-kt}$ b) $A=100-100e^{-0.1732t}$

(d) 17.693%

4a) $y=20+80e^{-0.144t}$ b) $58.960°C$ c) 9.634min

5a) $y = 37 - 38e^{-0.005t}$. b) $y = 37°F$ c) 29.391min

6a) 41 $C°$ b) underestimate

c) $b(t) = 25 + \left(\frac{12-t}{3}\right)^3$; $52°C$

7a) $\lim_{t \to \infty} F = 100$ 7b) 100.

7c) $\frac{A}{2} = 50$ 7d) $F = 100 - 90e^{-.004t}$

8a) $\lim_{t \to \infty} P = 1000$ 8b) $\lim_{t \to \infty} P = 1000$

8c) $P = \frac{1000}{1 + 9e^{-.00008t}}$ 8d) $\frac{A}{2} = 500$

9a) $\lim_{t \to \infty} G = A = 12$ 9b) $\frac{A}{2} = 6$

9c) $G(t) = \frac{12}{1 + 11e^{-0.3t}}$ 9d) $G = 12 - 11e^{-0.3t}$

10a) The decomposition rate was decreasing by 10.8 pounds per day per day at $t = 7$.

10b) 171.65 *lbs* 10c) 100 10d) $W(t) = \int_0^t 50e^{-.2(x-6)^2} dx$

11a) 500 11b) $M(t) = \frac{1000}{1 + 99e^{-0.000256t}}$

11c) $M = 1100 - 1090e^{-.256t}$ 11d) 1100

12a) $A = 120$ 12b) $M = 130 - 110e^{-\frac{1}{40}t}$ 12c) 130

13a) 150 13b) $\lim_{t \to \infty} P = 300$

13c) $\lim_{t \to \infty} P = 300.$ 13d) $\left(1 - \frac{P}{300}\right)$ 13e. $P = \frac{300}{1 + 299e^{-5.7t}}$

14a) A=666 14b) $\frac{A}{2} = 333$ 14c) $Z = \frac{666}{1 + 665e^{-0.45t}}$

14d) $K = 666 - 665e^{-299.7t}$

15. See AP Central

8.3 Multiple Choice Answer Key

1.	C	2.	A	3.	D	4.	D	5.	C	6.	B
7.	B	8.	B	9.	A	10.	A				

8.4 Free Response Answer Key

1. $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

2. $-t^3e^{-t} - 3t^2e^{-t} - 6te^{-t} - 6e^{-t} + c$

3. $\frac{1}{4}x^2\sin 4x + \frac{1}{8}x\cos 4x + \frac{1}{32}\sin 4x + c$

4. $\frac{1}{2}m^2e^{2m} - me^{2m} + \frac{1}{2}e^{2m} + c$

5. $-\left(x^3+1\right)e^{-x}-3x^2e^{-x}-6xe^{-x}+6e^{-x}+c$

6. $x^7e^x-7x^6e^x+42x^5e^x-210x^4e^x+840x^3e^x-2520x^2e^x+5040xe^x-5040e^x+c$

7. $x(\ln x)^2-2x(\ln x)-2x+c$

8. $x\cos^{-1}x-\sqrt{1-x^2}+c$

9. $-\frac{2}{5}e^{-\theta}\sin 2\theta-\frac{1}{5}e^{-\theta}\cos 2\theta+c$

10. $\frac{1}{3}x^2\sin 3x+\frac{2}{9}x\cos 3x-\frac{2}{27}x\sin 3x+c$

11. $4\ln 2-\frac{3}{2}$

12. $\frac{2}{13}e^{2\theta}\sin 3\theta-\frac{3}{13}e^{2\theta}\cos 3\theta+c$

13. $\frac{1}{2}x^2\tan^{-1}x-\frac{1}{2}x+\frac{1}{2}\tan^{-1}x+c$

14. $\frac{1}{2}x(\cos(\ln x)+\sin(\ln x))+c$

15. $x\ln(3x+2)-x+\frac{2}{3}\ln(3x+2)+c$

16. $\frac{1}{6}\left(3x^2+2\right)\ln\left(3x^2+2\right)-\frac{1}{6}\left(3x^2+2\right)+c$

17. $2\sqrt{x}\sin\sqrt{x}+2\cos\sqrt{x}+c$

18. $x\sin^{-1}x+\left(1-x^2\right)^{1/2}+c$

19. $f(x)=12x^3$

20. Converges

21. Converges

22. 1

23. $\frac{2}{3}(\tan x)^{3/2}+c$

24. $\frac{9}{2}\ln 3-\frac{13}{9}$

25. 2.103

8.4 Multiple Choice Answer Key

1. C 2. B 3. A 4. A 5. E

8.5 Free Response Answer Key

1. $\frac{1}{3}\left(x^2+9\right)^{3/2}-9\sqrt{x^2+9}+C$

2. $\frac{\sqrt{x^2-9}}{3x}+c$

3. $=-81\left(\frac{\sqrt{9-y^2}}{3}\right)^3+\frac{243}{5}\left(\frac{\sqrt{9-y^2}}{3}\right)^5+C$

4. .836

5. $-\frac{1}{3}\left(r^2+4\right)^{-3/2}+c$

6. $\frac{1}{\sqrt{3}}\ln\left|\frac{x^2+3+\sqrt{3}}{x}\right|+c$

7. $\frac{1}{2}\sin^{-1}(x-1)-\frac{1}{2}(x-1)\sqrt{1-(x-1)^2}$

8. $\frac{9}{2}\left(\sin^{-1}\frac{e^t}{3}-\frac{e^t}{9}\sqrt{9-e^{2t}}\right)+c$

9. $\ln\left|\frac{x-3+\sqrt{x^2-6x+14}}{\sqrt{5}}\right|+c$

8.6 Free Response Answer Key

1. $\frac{1}{2(x+1)^2}-\frac{1}{x+1}+c$

2. $\frac{1}{36}\ln\left|\frac{t-1}{t+5}\right|-\frac{1}{6(t+5)}+c$

3. $\frac{1}{25}\ln\left|\frac{y-3}{y+2}\right|+\frac{1}{5(y+2)}+c$

4. $\frac{1}{x}+\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c$

5. $\frac{-80}{1083}\ln|3x-1|+\frac{147}{361}\ln|x+6|-\frac{30}{19(x+6)}+c$

6. $-\frac{216}{7}\ln|3z+5|-\frac{12751}{63}(3z+5)^{-1}-\frac{85}{18}(3z+5)^{-2}-12\ln|z+2|+C$

7. $x-3\ln|x+1|-\frac{3}{x+1}+\frac{1}{2(x+1)^2}+c$

8. 4.068

8.7 Free Response Answer Key

1. $\frac{1}{2}x^2 + \frac{1}{2}\ln\left(x^2+1\right) + c$

2. $\ln|x| + \frac{1}{x^2+1} + c$

3. $2\ln|y-1| + \frac{1}{2}\ln\left(y^2+1\right) - 3\tan^{-1}y + c$

4. $\frac{1}{2}\ln\left(t^2+1\right)\left(t^2+2\right) - \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}} + c$

5. $\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

6. $x + \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln\left(x^2+x+1\right) - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$

7. $\frac{1}{2}\ln\left(x^2+4\right) - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + \tan^{-1}x + c$

8.8 Free Response Answer Key

1. A	2. B	3. D	4. B	5. E	6. B
7. AD	8. B	9. B	10. AD	11. BD	
12. BD	13. B	14. E	15. B	16. D	
17. AD	18. AE	19. AC	20. A	21. C	22. BC

23. $\tan^{-1}(\sin x)+c$

24. $\ln|\csc x-\cot x|-\ln|\sin x|+c$

25. $e^{\pi/2}-1$

26. $\frac{1}{3}\sin^3 x+\frac{1}{5}\sin^5 x+c$

27. $\sqrt{1-x^2}+c$

28. $\frac{-5}{6\sqrt{2}}+\frac{2}{3}$

29. divergent

30. $-\sin(\cos x)+c$

31. $e^{e^x}+c$

32. $-\frac{1}{2}t^3e^{-2t}+\frac{3}{4}t^2e^{-2t}-\frac{3}{4}te^{-2t}-\frac{3}{8}e^{-2t}+c$

33. $3x+\frac{23}{3}\ln|x-4|-\frac{5}{3}\ln|x+2|+c$

34. $\ln\left|x^3-2x-8\right|+c$

35. $\frac{86}{3}$

36.

$3\ln|x-1|-\frac{16}{x-2}+\ln|x-2|+c$

37. $15+7\ln\frac{2}{7}$

38. $\frac{\pi-2}{8}$

39. $\frac{5}{12}$

40. $\sec^{-1}(2x)+c$

41. $\frac{1}{20}\tan^{-1}\left(\frac{x^5}{4}\right)+c$

42.

$=-\frac{1}{4}\ln(x^2+3)+\frac{1}{4}\ln(x^2+1)+C$

43. $u-\ln|u|+u^{-1}+2\ln|u-1|+c$

44. $\frac{1}{2}\ln\left|2e^x-1\right|-\frac{1}{3}\ln\left|e^x+1\right|+c$

Chapter 8 Practice Test Key

1. B 2. A 3. D 4. D 5. E 6. A

7. A

8. 20.027

9. $\frac{1}{2}x^2\cot^{-1}x^2 + \frac{1}{4}\ln\left(x^4+1\right) + c$

10. $x^2 + 3x - x^3 - 7x - \frac{3}{10}\ln|x+3| - \frac{1}{2}\ln|x-1| + \frac{4}{5}\ln|x-2| + c$