Chapter 7 Overview: Numerical Sequences and Series

In most texts, the topic of sequences and series appears, at first, to be a side topic. There are almost no derivatives or integrals (which is what most students think all Calculus is comprised of) and those that do appear do so at the end of the chapter. The sense of disconnectedness is heightened by the fact that most Calculus students have not seen series since freshman year (and, then, only briefly).

For the purposes of AP, this topic is broken into four basic subtopics:

- Numerical sequences and series
- Radius and interval of convergence for a Power Series
- Taylor polynomials
- Creating a new series from an old one

This is the order most texts use, and all topics are in one chapter, which can be overwhelming. The final subtopic is where the derivatives and integrals reappear, built on the theoretical foundations laid by the first three.

We are opting to break the topic of sequences and series into two chapters. This chapter covers convergence and divergence of numerical sequences and series. The other chapter will cover the material related to Power Series and occurs at the end of the curriculum.

What we will consider in Calculus is the issue of whether a sequence or series converges or not. There are several processes to test if a series is convergent or not. There are seven Tests to prove convergence or divergence of a numerical series.

1. The Divergence (nth Term) Test
2. The Comparison Test
3. The Limit Comparison Test
4. The Integral Test
5. The Ratio Test
6. The nth Root Test
7. The Alternating Series Test

Each may be easier or more difficult depending on the series being tested. Obviously, we will not cover #4 in this chapter, since we have not learned

Integration yet. The rest rely on Limits-at-Infinity, and they provide a good context within which to practice that material. Most of these Tests apply to series comprised of positive values. The Alternating Series Test applies to series in which the terms alternate signs.

Though each section introduces a new test, the homework in each section will apply to all the tests that were introduced previously.

NB. Something that Calculus students often find frustrating is that the Series Tests will often prove if a series has a sum or not—that is, the Series is either convergent or divergent—but they will not determine what the total is if there is one. Finding the total of a series is, for the most part, beyond the scope of this course.

7.1: Sequences and series—Preliminary Algebra

Let's have a quick review from Algebra 2:

A sequence is simply a list of numbers. Sequences are only interesting to a mathematician when there is a pattern within the numbers. For example:

$$1, 3, 5, 7, \ldots$$

$$14, 7, 0, -7, -14, \ldots$$

$$1, 3, 9, 27, \ldots$$

$$4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \ldots$$

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

In Algebra, a sequence is written as

$$a_1,\ a_2,\ a_3,\ a_4,\ a_5,\ \ldots,\ a_n,\ \ldots$$

The general terms in the sequence have a subscript indicating the cardinality--that is, which place they are in the sequence. a_1 is the first number in the sequence, a_2 is the second, and so on. a_n is the nth number. **Note that n was always a counting number. In calculus and physics, this will change in that the subscript may be tied to the exponent and the series will generally be written**

$$a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ \ldots,\ a_n,\ \ldots$$

Vocabulary
Sequence-- a function with the Natural Numbers as the domain.
Arithmetic Sequence-- A sequence in which one term equals a constant added to the preceding term, i.e. $a_{n+1} = a_n + d$.
Geometric Sequence-- A sequence in which one term equals a constant multiplied to the preceding term, i.e. $a_{n+1} = a_n r$.

OBJECTIVES

Identify Sequences and Series
Find Partial Sums of a given Series.
Find the sum of an infinite Geometric Series.

Ex 1 Find the next three terms and the general term in each of these sequences and classify them as an arithmetic sequence, a geometric sequence or neither.

a) 1, 3, 5, 7, …

b) 14, 7, 0, -7, -14, …

c) $4, 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$

d) 1, 1, 2, 3, 5, 8, 13, …

e) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

(f) $1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, \ldots$

a) 9, 11, 13, …, $1+2(n-1)$. It is arithmetic because we add 2 each time.

b) -21, -28, -35, …, $14-7(n-1)$. It is arithmetic because we add -7 each time.

c) $\frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, 2\left(\frac{1}{2}\right)^{n-1}$. It is geometric because we multiply by $\frac{1}{2}$ each time.

d) 21, 34, 55. This is neither arithmetic nor geometric. We do not multiply by nor add the same amount each step. It is called the Fibonacci Sequence.

e) $\frac{1}{36}, \frac{1}{49}, \frac{1}{64}, \ldots, \frac{1}{n^2}$. This is a p-Sequence with p=2.

f) $-\frac{1}{6}, \frac{1}{7}, -\frac{1}{8}, \ldots, \frac{(-1)^{n+1}}{n}$. This is the Alternating Harmonic Series.

Sequences are generally written Set Notation:

$$\{a_n\}$$

The two most common sequences and their algebraic formulas are:

Arithmetic Sequence	$a_n = a_1 + (n-1)d$
Geometric Sequence	$a_n = a_1 r^{n-1}$

In Algebra, a great deal of arithmetic is done with these formulas. In Calculus, we will be more interested in Series.

Vocabulary
Series--the sum of a sequence.
Partial Sum--Defn: the sum of the first *n* terms of a sequence.

Series are generally written with sigma notation:

$$\sum_{k=1}^{n} a_k$$

where a_k is the k^{th} term in the sequence and we are substituting the integers 1 through *n* in for *k*, getting the terms and adding them up. $\sum$ means sum. [There is a symbol for the product of a sequence (Π), but we will not go into that here.]

Ex 2 Find $\sum_{k=0}^{4}\left(1-k^2\right)$

$$\begin{aligned}\sum_{k=0}^{4}\left(1-k^2\right) &= \left(1-0^2\right)+\left(1-1^2\right)+\left(1-2^2\right)+\left(1-3^2\right)+\left(1-4^2\right)\\ &= 1+0-3-8-15\\ &= -25\end{aligned}$$

Ex 3 Find $\sum_{k=2}^{5}\frac{3}{k^2}$

$$\begin{aligned}\sum_{k=2}^{5}\frac{3}{k^2} &= 3\sum_{k=2}^{5}\frac{1}{k^2}\\ &= 3\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}\right)\\ &= 3\left(\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}\right)\\ &= 1.391\end{aligned}$$

In an arithmetic series, the sum of the first and last terms is the same as the sum of the second and second to last, which is the same as the sum of the third and third to last. There are half as many pairs of numbers in a sequence as there are numbers in a sequence, so the partial sum is

$$S_n = \frac{n}{2}\left(a_1 + a_n\right)$$

Since we have a formula $a_n = a_1 + (n-1)d$ to get any term in the sequence from the first term, we can create a formula for the sum of the first n terms.

Arithmetic Partial Sum

$$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$

Ex 4 Find S_{32} for an arithmetic sequence with $a_1 = 6$ and d = 3

$$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$
$$S_{32} = \frac{32}{2}\left(2(6)+(31)3\right)$$
$$= 1680$$

As with the arithmetic sequence formula $a_n = a_1 + (n-1)d$, $S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$ has four variables in it: $S_n,\ a_1,\ n,\ and\ d$. Given any three, we can find the fourth.

Ex 5 Find the first term in the arithmetic series where $d = -\frac{4}{3}$ and the 13th partial sum is -65.

$$S_n = \frac{n}{2}\left(2a_1 + (n-1)d\right)$$
$$-65 = \frac{13}{2}\left(2a_1 + (13-1)\left(\frac{-4}{3}\right)\right)$$
$$-10 = 2a_1 - 16$$
$$a_1 = 3$$

Just as with the arithmetic series, there is a formula for the nth partial sum of a geometric series.

Geometric Partial Sum

$$S_n = a_1 \frac{1-r^n}{1-r}$$

As with the arithmetic sequences, the geometric sequence formula $a_n = a_1 r^{n-1}$ clearly has four variables in it: $S_n,\ a_1,\ n,\ r$. Given any three, we can find the fourth.

Ex 6 Find S_{15} for the geometric sequence with $a_1 = 1024$ and $r = -\frac{1}{2}$

$$S_n = a_1 \frac{1-r^n}{1-r}$$
$$S_{15} = 1024 \frac{1-\left(\frac{-1}{2}\right)^{14}}{1-\left(\frac{-1}{2}\right)}$$
$$= 682.625$$

Ex 7 Find the first term in the geometric sequence where $r = -3$ and the 9th partial sum is 24605.

$$S_n = a_1 \frac{1-r^n}{1-r}$$
$$24605 = a_1 \frac{1-(-3)^9}{1-(-3)}$$
$$a_1 = 5$$

Ex 8 If $S_n = 1026$, $a_1 = 6$ and $r = -2$, find n.

$$S_n = a_1 \frac{1-r^n}{1-r}$$
$$1026 = (6)\frac{1-(-2)^n}{1-(-2)}$$
$$513 = 1-(-2)^n$$
$$-512 = (-2)^n$$

Since we know we cannot log a negative, the negatives in this equation must cancel, and

$$
\begin{aligned}
-512 &= (-2)^n \\
512 &= (2)^n \\
Ln\ 512 &= n\ Ln\ 2 \\
n &= \frac{Ln\ 512}{Ln 2} \\
n &= 9
\end{aligned}
$$

All of these examples are really preliminary to the perspecitive of Calculus on Series. Calculsu is less interested in **partial** sums and more interested in **infinite** series.

7.1 Free Response Homework

Determine whether the sequence is arithmetic, geometric, or neither. If arithmetic or geometric, define d or r.

1. $8, 13, 18, 23, \ldots$

2. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

3. $25, 50, 100, 200, \ldots$

4. $25, 75, 100, 125, \ldots$

5. $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

6. $\sqrt{5}, \sqrt[3]{5}, \sqrt[6]{5}, 1, \frac{1}{\sqrt[6]{5}}, \ldots$

Find the sum.

7. $\sum_{k=1}^{5} \frac{1}{k}$

8. $\sum_{k=1}^{4} (3k-5)$

9. $\sum_{k=1}^{5} (-1)^k (2k+1)$

10. $\sum_{k=3}^{7} 2^k$

11. $\sum_{k=0}^{5} (-1)^k \left(\frac{2}{3}\right)^{k+2}$

12. $\sum_{k=3}^{7} (2+k^2)$

Find the designated variable.

13. Find S_{10} for an arithmetic series with $a_1 = 35$ and d = -5.

14. Find S_{15} for an arithmetic series with $a_1 = -14$ and d = .5.

15. Find S_{13} for an arithmetic series with $a_2 = 4$ and d = -3.

16. Find n in the arithmetic series where $a_1 = 15$, $d = 4$ and $S_n = 3219$.

17. Find n in the arithmetic series where $a_1 = 8$, $d = 5$ and $S_n = 4859$.

18. Find the first terms in the arithmetic series where $d = 3$ and $S_{10} = 175$.

19. Find the first term in the arithmetic series where $d = -2$ and $S_{15} = 0$.

20. Find the sum of the first 12 terms of 2, 5, 8, …

21. Find the sum of the first 60 terms of 2, -6, -14, …

22. Find the sum of the first 12 terms of 2, 6, 18, …

23. Find the sum of the first 60 terms of 1, -2, 4, …

24. In the geometric sequence from $a_1 = 6$ to $a_n = -196608$ where $r = -2$, how many terms are there and what is the partial sum?

7.2: Convergence and Divergence of Infinite Series

Most of the convergence tests will involve finding a limit at infinity. It would be worth reviewing this topic from the Limits Chapter.

The Hierarchy of Functions

1. Logs grow the slowest.
2. Polynomials, Rationals and Radicals grow faster than logs and the degree of the End Behavior Model (EBM) determines which algebraic function grows fastest. For example, $y = x^{1/2}$ grows more slowly than $y = x^2$.
3. The trig inverses fall in between the algebraic functions at the value of their respective horizontal asymptotes.
4. Exponential functions grow faster than the others. (In BC Calculus, we will see the factorial function, $y = n!$, grows the fastest.)
5. **The fastest growing function in the combination determines the end behavior, just as the highest degree term did among the algebraics**.

Remember:

Convergent Integral--Defn: an improper integral that has a total.
Divergent Integral --Defn: an improper integral that does not have a total.

New Vocabulary

Infinite Sequence—Defn: a sequence with an infinite number of terms.

Convergent Sequence --Defn: an infinite sequence where a_n approaches a particular value when *n* goes to infinity.

Divergent Sequence --Defn: an infinite sequence where a_n does not approach a particular value when *n* goes to infinity.

Infinite Sum--Defn: the sum of all the terms of a sequence. This is not possible for an arithmetic series, but might be for a geometric series.

Convergent Series--Defn: an infinite series that has a total.

Divergent Series--Defn: an infinite series that does not have a total.

In some cases, a series of infinite terms can have a total, if the later terms get infinitely small. Much of the series work in Calculus is about finding whether an infinite series is convergent (has a total) or divergent (has total keeps getting bigger indefinitely). Geometric series are the easiest for which to find an infinite sum.

In other words,

A sequence is convergent if and only if $\lim_{n\to\infty} a_n = c$.

A sequence is divergent if and only if $\lim_{n\to\infty} a_n \neq c$.

For a series,

A series is convergent if and only if $\sum_{n=1}^{\infty} a_n = \text{c}$.

A series is divergent if and only if $\sum_{n=1}^{\infty} a_n \neq \text{c}$.

OBJECTIVES

Determine the convergence or divergence of a sequence.
Determine the divergence of a series.
Use the Alternating Series Test to check for convergence or divergence.

Ex 1 Which of the following sequences converge?

I. $\left\{\frac{4n}{3n-1}\right\}$ II. $\left\{\frac{e^{2n}}{2n}\right\}$ III. $\left\{\frac{e^{2n}}{e^{2n}-1}\right\}$

In each case, we just need to check the Limit at Infinity. That is, we need to check the end behavior.

I. $\lim_{n\to\infty}\frac{4n}{3n-1}=\frac{4}{3}$; therefore, this sequence converges.

II. $\lim_{n\to\infty}\frac{e^{2n}}{2n}=\infty$; therefore, this one diverges.

III. $\lim_{n\to\infty}\frac{e^{2n}}{e^{2n}-1}\overset{L'H}{=}\lim_{n\to\infty}\frac{e^{2n}(2)}{e^{2n}(2)}=1$; therefore, this converges.

So, only I and III converge.

This is a typical AP question. The key is CRITICAL READING. Sometimes, the question is about <u>Sequences</u>, sometimes it is about <u>Series</u> (which we will consider later it he chapter). Sometimes the question is about convergence and sometimes it is about divergence.

Ex 2 Which of the following sequences diverge?

I. $\left\{\dfrac{3n^2}{7n^3-1}\right\}$ II. $\left\{\dfrac{7\ln n}{2n}\right\}$ III. $\left\{\dfrac{4n^4}{2n^3-1}\right\}$

I. $\lim\limits_{n\to\infty} \dfrac{3n^2}{7n^3-1} = 0$; therefore, this sequence does not diverges.

II. $\lim\limits_{n\to\infty} \dfrac{7\ln n}{2n} \overset{L'H}{=} \lim\limits_{n\to\infty} \dfrac{7\left(\frac{1}{n}\right)}{2} = 0$; therefore, this one does not diverge.

III. $\lim\limits_{n\to\infty} \dfrac{4n^4}{2n^3-1} = \infty$; therefore, this diverges.

So, only III diverges.

When testing a numerical series for divergence or convergence, the two easiest tests are:

Divergence Test (nth term test): If $\lim\limits_{n\to\infty} a_n \neq 0 \to \sum\limits_{n=1}^{\infty} a_n$ diverges.

If $\lim\limits_{n\to\infty} a_n = 0 \to$ no conclusion

The Divergence Test basically says that if the terms at the end are not 0, the sum will just keep getting larger and larger (i.e., diverges). While easy to use, each has a disadvantage. The Divergence Test tells you if a function has terms approaching 0. If the terms at the upper end of the series do not approach 0, the series cannot converge. But, there are many series with end-terms approaching 0 that still diverge. So half the test is inconclusive.

Ex 3 For which of the following series is the Divergence Test inconclusive?

I. $\sum_{n=1}^{\infty}\frac{1}{2^n}$ II. $\sum_{n=1}^{\infty}\frac{n^2}{n(n+1)}$ III. $\sum_{n=1}^{\infty}\frac{1}{n}$

I. $\lim_{n\to\infty}\frac{1}{2^n}=0$; so the Divergence Test is inconclusive for I. Note that this is not enough to decide if it converges or not.

II. $\lim_{n\to\infty}\frac{n^2}{n(n+1)}=\lim_{n\to\infty}\frac{n^2}{n^2+n}=1$; therefore, II diverges.

III. $\lim_{n\to\infty}\frac{1}{n}=0$; therefore, the Divergence Test is inconclusive for III.

I and III pass the Divergence Test

Geometric and Arithmetic series are the ones we encountered in Algebra 2. There are several other kinds, some of which are important for our study of series. We will look at them more closely later.

Kinds of Series

Arithmetic Series: $a_{n+1} = a_1 + (n-1)d$

Geometric Series: $a_n = a_1 r^{n-1}$ [Convergent is r < 1]

p-series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ [Convergent is p > 1]

Alternating series: $\sum_{n=0}^{\infty} (-1)^n a_n$ or $\sum_{n=0}^{\infty} (\cos \pi n) a_n$ or any other series where the signs of the terms alternate between + and -.

Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$(Note that this is a p-series where p=1)

[Divergent because p = 1]

Alternating Harmonic Series: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

[Convergent by Alternating Series Test]

A similar test to the Divergence Test, in process, is the Alternating Series test:

The Liebnitz Alternating Series Test

If a series is Alternating Series and is $|a_n| \geq |a_{n+1}|$, then it is convergent if and only if

$$\lim_{n\to\infty} |a_n| = 0.$$

This means the non-alternating part of the series must be decreasing and pass the Divergence Test (that is, the sequence converges to 0).

Ex 4 Is the Alternating Harmonic Series convergent or divergent?

Remember that the Alternating Harmonic Series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

i) $$|a_n| = \left|\frac{(-1)^{n+1}}{n}\right| = \frac{1}{n} \text{ and } |a_{n+1}| = \left|\frac{(-1)^{n+2}}{n+1}\right| = \frac{1}{n+1}$$

$$\frac{1}{n} > \frac{1}{n+1}, \text{ so } |a_n| \geq |a_{n+1}|$$

ii) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

So $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges.

Ex 5 Is $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n!}$ convergent or divergent?

Remember that the Alternating Harmonic Series is.

i) $$|a_n| = \left|\frac{\cos(\pi n)}{n!}\right| = \frac{1}{n!} \text{ and } |a_{n+1}| = |a_n| = \left|\frac{\cos(\pi(n+1))}{(n+1)!}\right| = \frac{1}{(n+1)!}$$

$$\frac{1}{n!} > \frac{1}{(n+1)!}, \text{ so } |a_n| \geq |a_{n+1}|$$

ii) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{n}\right| = \lim_{n\to\infty}\frac{1}{n} = 0$$

So $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n!}$ converges.

As noted in the introduction, the Series Tests will often prove if a series has a sum or not (the Series is either convergent or divergent), but they will not determine what the total is if there is one. Finding the total of a series is, for the most part, beyond the scope of this course. The one kind of infinite series where the total is easy to find is the Infinite Geometric Series, for which we have a formula.

Sum of an Infinite Geometric Series

$$S = \frac{a}{1-r},$$

if and only if $|r| < 1$.

Ex 6 Find the infinite sum for the geometric sequence with $a_1 = 1024$ and $r = -\frac{1}{2}$

.

$$S = \frac{a_1}{1-r}$$
$$S = \frac{1024}{1-\left(\frac{-1}{2}\right)}$$
$$= 682\frac{2}{3}$$

Ex 7 Find the infinite sum for the geometric sequence with $a_1 = 24$ and $r = \frac{5}{4}$.

Since $r = \frac{5}{4} > 1$, the formula will yield an incorrect answer. This series is divergent.

Summary

Three of the tests for convergence use $\lim_{n\to\infty} a_n$. The conclusion in each case is determined by what is being tested:

1. A Sequence
   - The sequence converges if $\lim_{n\to\infty} a_n$ = any Real Number.
   - The sequence diverges if $\lim_{n\to\infty} a_n = \infty$.
2. An Alternating Series
   - An alternating series converges if $\lim_{n\to\infty} |a_n| = 0$.
   - An alternating series diverges if $\lim_{n\to\infty} |a_n| \neq 0$.
2. A Non-Alternating Series
   - A non-alternating series diverges if $\lim_{n\to\infty} |a_n| \neq 0$.
   - If $\lim_{n\to\infty} a_n = 0$ for a non-alternating series, **the test is inconclusive**.

7.2 Free Response Homework

Determine if the sequence converges or diverges.

1. $\{n(n-1)\}$

2. $\left\{\dfrac{3+2n^2}{n+n^2}\right\}$

3. $\left\{\dfrac{\sqrt{n}}{1+\sqrt{n}}\right\}$

4. $\left\{\dfrac{2^n}{3^{n+1}}\right\}$

5. $\left\{\dfrac{Ln\left(n^2\right)}{n}\right\}$

6. $\left\{n2^{-n}\right\}$

7. $\left\{\dfrac{n!}{(n+2)!}\right\}$

7. $\left\{\dfrac{n+3}{\sqrt{49n^2+1}}\right\}$

1. $\left\{\dfrac{\sqrt{2n^2+1}}{3n-5}\right\}$

2. $\left\{\sqrt{n^2+1}-\sqrt{n}\right\}$

3. $\left\{\tan^{-1}\left(n^2-n^3\right)\right\}$

4. $\left\{\dfrac{e^n-1-n}{n^3}\right\}$

5. $\{\cos n\}$

6. $\left\{\sqrt{16n^2+n}-4n\right\}$

Determine which series might be convergent by the Divergence Test.

8. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$

9. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{41}{3n^2-5}\right)$

10. $\displaystyle\sum_{n=1}^{\infty}3^{-n}(8)^{n+1}$

11. $\displaystyle\sum_{n=0}^{\infty}2^n$

12. $\sum_{n=0}^{\infty} \frac{4^{n+1}}{5^n}$

Test these series for convergence or divergence.

13. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n^3}}$

14. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2+1}$

15. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

16. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{Ln\ n}$

17. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

18. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{3/4}}$

19. $\sum_{n=1}^{\infty} \frac{(-1)^n}{4n+1}$

20. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^{5/4}}$

21. $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$

22. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n+1}$

9. $\sum_{n=0}^{\infty} 5\left(\frac{4}{3}\right)^n$

10. $\sum_{n=1}^{\infty} n^3 e^{-n^2}$

11. $\sum_{n=0}^{\infty} 180 \cdot 2^{-n}$

12. $\sum_{n=0}^{\infty} \frac{\sin n - n}{n^3}$

13. $\sum_{n=1}^{\infty} n^{n^2}$

14. $\sum_{n=0}^{\infty} \frac{n}{\tan^{-1}(4n)}$

23. Find the sum of the infinite geometric series where $a_1 = 15$ and $r = .4$.

24. Find the sum of the infinite geometric series where $a_1 = 8$ and $r = \frac{-3}{7}$.

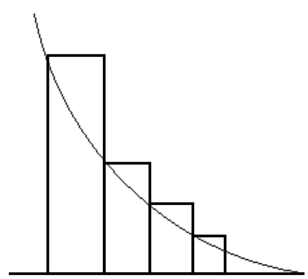
25. Find the sum of the infinite geometric series where $a_1 = -42$ and $r = \sqrt{2}$.

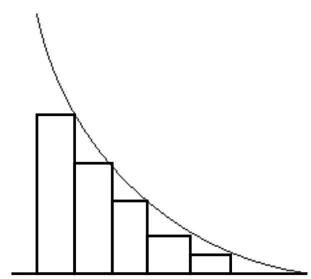
7.3: The Integral Test

When testing a numerical series for divergence or convergence, one of the easiest tests is:

Integral Test:	$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} x_n \, dx$	either both converge or both diverge.

The reason the Integral Test works is best explained visually. The series can be visualized as a sum of Riemann Rectangles with width = 1 and height = the term values. If the integral diverges (i.e., the sum is infinite), the rectangles can be drawn left-hand and their sum is larger than an infinite number. If the integral converges (i.e., the sum is finite), the rectangles can be drawn right-hand and their sum is smaller than a finite number.





In both situations, the function must be decreasing.

While easy to use, there is a disadvantage: The Integral Test only works on functions that can be integrated. Not all functions can be integrated.

Ex 1 Use the Integral Test to determine which of these series converge.

I. $\sum_{n=1}^{\infty} \frac{1}{n}$ II. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ III. $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$

I. $\int_1^{\infty} \frac{1}{x} dx = \ln x \Big|_1^{\infty} = dne$; therefore, I diverges.

II. $\int_1^{\infty} \frac{1}{x^2+1} dx = \tan^{-1} x \Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$; therefore, II converges.

III. $\int_2^{\infty} \frac{1}{x\ln^2 x} dx = \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \frac{-1}{u} \Big|_{\ln 2}^{\infty} = 0 - (\ln 2) = -\ln 2$; therefore, III converges.

II and III converge.

Two notes about this example. First, I. proves that the Harmonic Series always diverges. Second, III was a series that started at n = 2 instead of n = 1. This was to avoid the problem of the first term being transfinite.

Ex 2 For what values of p does the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converge?

$\lim_{n \to \infty} \frac{1}{n^p} = 0$ for all p>0, so p must be positive.

$\int_1^{\infty} \frac{1}{x^p} dx = \frac{x^{-p+1}}{-p+1} \Big|_1^{\infty}$. This integral will be finite as long as p>1, because this will leave *x* in the denominator so the limit can go to 0. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges for all p>1.}$$

We will use this fact later in this chapter, when applying the Comparison Tests.

Ex 3 Determine whether $\sum_{0}^{\infty} \frac{n}{1+n^2}$ is convergent or divergent.

$$\begin{aligned}\int_0^{\infty} \frac{x}{1+x^2}\,dx &= \lim_{b\to\infty} \int_0^{b} \frac{x}{1+x^2}\,dx \\ &= \frac{1}{2} \lim_{b\to\infty} \int_{x=0}^{x=b} \frac{du}{u} \\ &= \frac{1}{2} \lim_{b\to\infty} \ln\left(1+x^2\right)\Big|_0^b \\ &= \frac{1}{2} \lim_{b\to\infty} \left[\ln\left(1+b^2\right) - \ln 1\right] \\ &= \infty\end{aligned}$$

Since this integral diverges, the series $\sum_{0}^{\infty} \frac{n}{1+n^2}$ diverges.

7.3 Free Response Homework

Apply the Integral Test to each of these series to determine divergence or convergence. If the series is convergent, state the improper integral value.

1. $\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^{3/2}}$

2. $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$

3. $\sum_{n=1}^{\infty} \frac{1}{3n+1}$

4. $\sum_{n=1}^{\infty} \frac{n+1}{n^4}$

5. $1+\frac{1}{8}+\frac{1}{27}+\frac{1}{64}+\frac{1}{125}+\ldots$

6. $\sum_{n=1}^{\infty} \frac{5-2\sqrt{n}}{n^3}$

7. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

9. $\sum_{n=1}^{\infty} \frac{1}{(5n+2)^4}$

10. $\sum_{n=1}^{\infty} ne^{-n^2}$

11. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

12. $\sum_{n=1}^{\infty} \frac{3n^2}{1+n^6}$

13. $\sum_{n=1}^{\infty} \frac{6n^5}{1+n^6}$

14. $\sum_{n=0}^{\infty} \frac{1}{(4n+3)^2}$

15. $\sum_{n=2}^{\infty} \frac{\ln n}{n}$

16. $\sum_{n=1}^{\infty} \sin n$

17. $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$

18. $\sum_{n=5}^{\infty} \frac{1}{\sqrt{n-4}}$

7.3 Multiple Choice Homework

1. Consider $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ and $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$. Based on the Integral Test, which of the following statements is true?

(A) $\sum_{n=2}^{\infty} \frac{1}{n\ln^2 n}$ converges.

(B) $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ converges.

(C) Both series converge.

(D) Neither series converges.

---

2. Consider $\sum_{n=2}^{\infty} \frac{1}{n^2+9}$ and $\sum_{n=2}^{\infty} \frac{n}{\sqrt{(n^2+9)^3}}$. Based on the Integral Test, which of the following statements is true?

(A) $\sum_{n=2}^{\infty} \frac{1}{n^2+9}$ converges.

(B) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{(n^2+9)^3}}$ converges.

(C) Both series converge.

(D) Neither series converges.

---

3. Consider $\sum_{n=2}^{\infty} \frac{1}{n+2}$ and $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^2+4}}$. Based on the Integral Test, which of the following statements is true?

(A) $\sum_{n=2}^{\infty} \frac{1}{n+2}$ converges.

(B) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^2+4}}$ converges.

(C) Both series converge.

(D) Neither series converges.

---

4. Consider $\sum_{n=2}^{\infty} ne^{3n^2}$ and $\sum_{n=2}^{\infty} \frac{n}{e^{3n^2}}$. Based on the Integral Test, which of the following statements is true?

(A) $\sum_{n=2}^{\infty} ne^{3n^2}$ converges.

(B) $\sum_{n=2}^{\infty} \frac{n}{e^{3n^2}}$ converges.

(C) Both series converge.

(D) Neither series converges.

7.4: The Direct and Limit Comparison Tests

The Direct and Limit Comparison Tests are much less algebraic than the Divergence and Integral Tests. They involve comparing a series to some series that we already know converges or diverges. So what do we know?

<u>REMINDER: Factual Knowledge for Comparisons</u>

Geometric series: 1. $\sum_{n=0}^{\infty} a_n r^n$ converges if $|r| < 1$ and diverges if $|r| \geq 1$

2. If $\sum_{n=0}^{\infty} a_n r^n$ converges, it converges to $\dfrac{a_0}{1-r}$

p-Series: 1. $\sum_{n=0}^{\infty} \dfrac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$

<u>Note Well:</u>

Harmonic Series: $\sum_{n=1}^{\infty} \dfrac{1}{n} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$ Diverges

Alternating Harmonic Series: $\sum_{n=1}^{\infty} \dfrac{1}{n} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots$ Converges

The Direct Comparison Test

When comparing a given series $\sum_{n=1}^{\infty} a_n$ to a known series $\sum_{n=1}^{\infty} b_n$,

1) if $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) if $a_n \ge b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

The use of this test is very verbal.

The Limit Comparison Test

When comparing a given series $\sum_{n=1}^{\infty} a_n$ to a known series $\sum_{n=1}^{\infty} b_n$,

1. If $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| =$ any positive Real number, then both converge or both diverge.

2. If $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = 0$, then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges.

3. If $\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = dne$, then $\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ diverges.

Key Idea I: What series to compare $\sum_{n=1}^{\infty} a_n$ to depends on where the variable is.

If n is in the base, compare to the p-Series.
If n is in the exponent, compare to the Geometric Series.
If n is in both the base and exponent (but not n^n), try either.
If n^n, use the Nth Root Test (section 5-4).

Key Idea II: Which test to use depends on the combination of the relative sizes and whether the known series converges or diverges.

	$\sum_{n=1}^{\infty} b_n$ converges	$\sum_{n=1}^{\infty} b_n$ diverges
$a_n \le b_n$	Direct Comparison Test	Limit Comparison Test
$a_n \ge b_n$	Limit Comparison Test	Direct Comparison Test

Key Idea III: If a_n is rational, we want to compare to the end behavior model of the corresponding function.

OBJECTIVE

Use the Comparison Tests to check for convergence or divergence.

Ex 1 Does $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ converge?

We can make a direct comparison between $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

$\frac{1}{n^3+2} < \frac{1}{n^3}$ because $\frac{1}{n^3+2}$ has a larger denominator. Since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (it is a p-Series with p>1) and $\frac{1}{n^3+2}$ is smaller than $\frac{1}{n^3}$, then $\sum_{n=1}^{\infty} \frac{1}{n^3+2}$ converges.

Ex 2 Does $\sum_{n=3}^{\infty} \frac{1}{n-2}$ converge?

We can make a direct comparison between $\sum_{n=1}^{\infty} \frac{1}{n-2}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$.

$\frac{1}{n-2} > \frac{1}{n}$ because $\frac{1}{n-2}$ has a smaller denominator. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it is the Harmonic Series--a p-Series with p=1) and $\frac{1}{n-2}$ is bigger than $\frac{1}{n}$, then $\sum_{n=3}^{\infty} \frac{1}{n-2}$ diverges.

Ex 3 Does $\sum_{n=1}^{\infty} \frac{1}{n^2-4}$ converge?

If we can try to make a direct comparison between $\sum_{n=1}^{\infty} \frac{1}{n^2-4}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, we find that $\frac{1}{n^2-4} > \frac{1}{n^2}$ (because $\frac{1}{n^2-4}$ has a larger denominator) but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it is a p-Series with p>1). So the Direct Comparison Test does not apply.

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n^2-4}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \left| \frac{n^2}{n^2-4} \right| = 1 > 0$$

This limit is greater than 0 and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2-4}$ converges also.

Ex 4 Does $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converge?

$\frac{1}{2^n - 1} > \frac{1}{2^n}$ because $\frac{1}{2^n - 1}$ has a smaller denominator. But $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges (it is a Geometric Series with r<1), so Direct Comparison will not work.

$$\lim_{n\to\infty} \left| \frac{\frac{1}{2^n - 1}}{\frac{1}{2^n}} \right| = \lim_{n\to\infty} \left| \frac{2^n}{2^n - 1} \right| \overset{L'H}{=} \lim_{n\to\infty} \left| \frac{2^n \ln 2}{2^n \ln 2} \right| = 1$$

This limit is greater than 0 and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges. Therefore, $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges also.

Ex 5 Does $\sum_{n=1}^{\infty} \frac{1}{2^n + 3}$ converge?

We can make a direct comparison between $\sum_{n=1}^{\infty} \frac{1}{2^n + 3}$ and $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

$\frac{1}{2^n + 3} < \frac{1}{2^n}$ because $\frac{1}{2^n + 3}$ has a larger denominator. Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and $\frac{1}{2^n + 3}$ is smaller than $\frac{1}{2^n}$, then $\sum_{n=1}^{\infty} \frac{1}{2^n + 3}$ converges.

Ex 6 Does $\sum_{n=1}^{\infty} \frac{1}{\left(n^3+2\right)^{1/4}}$ converge?

We can use the Limit Comparison Test between $\sum_{n=1}^{\infty} \frac{1}{\left(n^3+2\right)^{1/4}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$.

$$\lim_{n\to\infty} \left| \frac{\frac{1}{\left(n^3+2\right)^{1/4}}}{\frac{1}{n^{3/4}}} \right| = \lim_{n\to\infty} \left| \frac{n^{3/4}}{\left(n^3+2\right)^{1/4}} \right| = 1$$

$\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ diverges, therefore, $\sum_{n=1}^{\infty} \frac{1}{\left(n^3+2\right)^{1/4}}$ diverges.

The one formula from freshman year that will be useful is for the sum of an Infinite Geometric Series:

Sum of an Infinite Geometric Series

$$S = \frac{a}{1-r},$$

if and only if $|r| < 1$.

If $|r| \ge 1$, the series is divergent.

Ex 7 Find the infinite sum for the geometric sequence with $a_1 = 1024$ and $r = -\frac{1}{2}$

.

$$S = \frac{a_1}{1-r}$$
$$S = \frac{1024}{1-\left(\frac{-1}{2}\right)}$$
$$= 682\frac{2}{3}$$

Ex 8 Find the infinite sum for the geometric sequence with $a_1 = 24$ and $r = \frac{5}{4}$.

Since $r = \frac{5}{4} > 1$, the formula will yield an incorrect answer. This series is divergent.

7.4 Free Response Homework

Test these series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+n+1}$

2. $\sum_{n=1}^{\infty} \frac{5}{2+3^n}$

3. $\sum_{n=1}^{\infty} \frac{n+1}{n^2}$

4. $\sum_{n=1}^{\infty} \frac{3}{n2^n}$

5. $\sum_{n=1}^{\infty} \frac{n^3+1}{n^3-1}$

6. $\sum_{n=1}^{\infty} \frac{n}{(n+1)2^n}$

7. $\sum_{n=1}^{\infty} \frac{3+\cos n}{3^n}$

8. $\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$

9. $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$

10. $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

1. $\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^3+1}$

2. $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^3-1}}$

3. $\sum_{n=1}^{\infty} \frac{1}{n!}$

4. $\sum_{n=1}^{\infty} \frac{3n^2+2n}{4^n\left(2n^2+7n-1\right)}$

5. $\sum_{n=1}^{\infty} \frac{n-5}{(n+2)^4}$

6. $\sum_{n=1}^{\infty} \frac{4+2n}{\left(1+n^3\right)^2}$

7. $\sum_{n=1}^{\infty} \frac{1+2\sin n}{n^4}$

8. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+4}$

9. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+4)^2}$

10. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+4}}$

7.4 Multiple Choice Homework

1. For what integer k, $k > 1$, will both $\sum_{n=1}^{\infty} \frac{(-1)^{1+kn}}{n}$ and $\sum_{n=1}^{\infty} \left(\frac{k}{3}\right)^n$ converge

a) 2 b) 3 c) 4 c) 5 e) 6

---

2. For what values of p does $\sum_{n=1}^{\infty} \frac{n}{n^{1-p}}$ converge?

a) $p > 0$ b) $p < 0$ c) $p \leq -1$ d) $p < -1$ e) $p < -2$

---

3. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} 5^{-n} 6^{n-1}$ II. $\sum_{n=3}^{\infty} \frac{3n^5}{7n^4 - 1}$ III. $\sum_{n=2}^{\infty} \frac{1}{5^n}$

a) I only b) III only c) I and II only d) II and III only e) I, II, and III

---

4. For which values of k does the infinite series $\sum_{n=1}^{\infty} 6k^{-n}$ converge?

a) $k < 1$ b) $|k| < 1$ c) $k > 0$

d) $k > 1$ e) $|k| > 1$

---

5. Which of the following series converge?

I. $\sum_{n=1}^{\infty} \frac{n}{n+1}$ II. $\sum_{n=3}^{\infty} \frac{\pi^n}{3^n}$ III. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

6. Which of the following three tests will establish that $\sum_{n=1}^{\infty} \frac{3n^2 - 5n}{n^3 + n - 1}$ diverges?

I. Direct Comparison to $\sum_{n=1}^{\infty} 3n^{-1}$ II. Limit Comparison to $\sum_{n=1}^{\infty} n^{-1}$

III. The Divergence Test

a) I only b) II only c) I and II only

d) I and III only e) I, II, and III

---

7. $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} =$

a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) 1 d) 2 e) ∞

---

8. Consider $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$. Based on the Direct Comparison Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ converges

(B) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converges

(C) Both $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converge.

(D) Neither $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converge.

---

9. Consider $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$. Based on the Limit Comparison Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ converges

(B) $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converges

(C) Both $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converge.

(D) Neither $\sum_{n=3}^{\infty} \frac{1}{n^2+4}$ and $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$ converge.

---

10. Consider $\sum_{n=3}^{\infty} \frac{1}{n+4}$ and $\sum_{n=5}^{\infty} \frac{1}{n-4}$. Based on the Direct Comparison Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{1}{n+4}$ diverges

(B) $\sum_{n=5}^{\infty} \frac{1}{n-4}$ diverges

(C) Both series diverge.

(D) Neither series diverges.

---

11. Consider $\sum_{n=3}^{\infty} \frac{1}{n+4}$ and $\sum_{n=5}^{\infty} \frac{1}{n-4}$. Based on the Limit Comparison Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{1}{n+4}$ diverges

(B) $\sum_{n=5}^{\infty} \frac{1}{n-4}$ diverges

(C) Both series diverge.

(D) Neither series diverges.

---

12 . Consider $\sum_{n=3}^{\infty} \frac{n^2+4}{5^n}$ and $\sum_{n=3}^{\infty} \frac{n^2-4}{5^n}$. Based on the Direct Comparison Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{n^2+4}{5^n}$ converges

(B) $\sum_{n=3}^{\infty} \frac{n^2-4}{5^n}$ converges

(C) Both series converge.

(D) Neither series converges.

7.5: Absolute and Conditional Convergence

The four tests in the previous two sections only apply to series of positive values. There is a separate test for alternating series called the Liebnitz Alternating Series Test.

> **The Alternating Series Test (AST)**
>
> If a series is Alternating Series and is $|a_n| \geq |a_{n+1}|$, then it is convergent if and only if
>
> $$\lim_{n\to\infty} |a_n| = 0.$$

This test was introduced in Section 6.2.

Vocabulary

Absolute Convergence—Defn: When an alternating series and its absolute value are both convergent.

Conditional Convergence—Defn: When an alternating series is convergent but its absolute value are divergent.

> OBJECTIVE
>
> Use the Alternating Series Test to check for convergence or divergence.

Ex 1 Is $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3}$ convergent or divergent?

Remember that the Alternating Harmonic Series is.

i) $|a_n| = \frac{\cos(\pi n)}{n^3} = \frac{1}{n^3}$ and $|a_{n+1}| = |a_n| = \left|\frac{\cos(\pi(n+1))}{(n+1)^3}\right| = \frac{1}{(n+1)^3}$

$$\frac{1}{n^3} > \frac{1}{(n+1)^3}, \text{ so } |a_n| \geq |a_{n+1}|$$

ii) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{n^3}\right| = \lim_{n\to\infty}\frac{1}{n^3} = 0$$

So $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}$ converges.

Ex 2 Is $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}$ absolutely or conditionally convergent?

Ex 1 proved that $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}$ is convergent by the AST. To demonstrate conditional vs. absolute convergence, we test the series without the alternator:

$$\left|\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}\right| = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

$\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a p-series with $p > 1$, therefore, since both the alternating and non-alternating versions of the series converge,

$\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n^3}$ converges absolutely.

Ex 3 Is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$ convergent or divergent? If convergent, is it absolutely or conditionally convergent?

i) $|a_n| = \left|\frac{(-1)^{n+1}(n+2)}{n(n+1)}\right| = \frac{n+2}{n^2+n}$ and $|a_{n+1}| = \left|\frac{(-1)^{n+1}((n+1)+2)}{(n+1)((n+1)+1)}\right| = \frac{n+3}{n^2+3n+2}$

$\frac{n+2}{n^2+n} > \frac{n+3}{n^2+3n+2}$, so $|a_n| \geq |a_{n+1}|$

ii) $$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)}{n(n+1)}\right| = \lim_{n\to\infty}\frac{n+2}{n^2+n} = 0$$

So $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$ converges.

To test for absolute vs. conditional convergence, we need to test $\sum_{n=1}^{\infty}\frac{n+2}{n(n+1)}$.

We can do the Limit Comparison Test against the end behavior model $\sum_{n=1}^{\infty}\frac{1}{n}$.

$$\lim_{n\to\infty}\frac{\frac{n+2}{n(n+1)}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n+2}{n(n+1)}\cdot\frac{n}{1} = \lim_{n\to\infty}\frac{n^2+2n}{n^2+n} = 1$$

This limit is greater than 0, therefore, both do the same thing--namely, they diverge. So,

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n+2)}{n(n+1)}$ is conditionally convergent.

Ex 4 Is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}$ convergent or divergent? If convergent, is it absolutely or conditionally convergent?

i) $|a_n| = \left|\frac{(-1)^{n+1}}{2^n}\right| = \frac{1}{2^n}$ and $|a_{n+1}| = \left|\frac{(-1)^{n+2}}{2^{n+1}}\right| = \frac{1}{2\cdot 2^n}$

$\frac{1}{2^n} > \frac{1}{2\cdot 2^n}$, so $|a_n| \geq |a_{n+1}|$

ii) $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{2^n}\right| = \lim_{n\to\infty}\frac{1}{2^n} = 0$ So $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n}$ converges.

Furthermore, $\frac{1}{2^n}$ converges because it is a Geometric series with r = $\frac{1}{2}$. So,

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^n} \text{ converges absolutely.}$$

7.5 Free Response Homework

Test these series for convergence or divergence. If it is convergent, determine if it has absolute or conditional convergence.

1. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n+1}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+1}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\ln n}$

5. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\ln n}{n}$

6. $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^{3/4}}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2+1}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$

9. $\sum_{n=1}^{\infty} \frac{(-3)^{n+1}}{2^{4n}}$

10. $\sum_{n=2}^{\infty} \frac{(-1)^n (n+5)}{4^n}$

11. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

12. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n+1}$

13. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}+1}$

14. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n!}$

15. $\sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{2} n\right)}{n}$

16. $\sum_{n=1}^{\infty} \left(-\frac{n}{6}\right)^n$

17. $\sum_{n=1}^{\infty} (-1)^n \arctan n$

18. $\sum_{n=1}^{\infty} (-1)^n \arctan\left(\frac{\pi}{n}\right)$

19. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{e^n}$

20. $\sum_{n=1}^{\infty} 2^n \sin\left(\frac{\pi}{2} n\right)$

7.5 Multiple Choice Homework

1. Which of the following series are conditionally convergent?

I. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n\sqrt{n}}$ II. $\sum_{n=0}^{\infty}\frac{e^n}{(n+5)!}$ III. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$

a) I only b) II only c) III only

d) I and II only e) I and III only

---

2. Which of the following series are conditionally convergent?

I. $\sum_{n=1}^{\infty}\frac{5n\cos(\pi n)}{n^2}$ II. $\sum_{n=0}^{\infty}\frac{(-n)^2}{5^n}$ III. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/4}}$

a) I only b) II only c) III only

d) I and III only e) I, II, and III

---

3. Which of the following series is/are absolutely convergent?

I. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ II. $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ III. $\sum_{n=1}^{\infty}\frac{1}{n^3}$

a) I only b) II only c) I and II only

d) I and III only e) I, II, and III

---

4. Which of the following series are conditionally convergent?

I. $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ II. $\sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1}}{e^n}$ III. $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{5n}$

a) I only b) II only c) I and II only

d) II and III only e) I, II, and III

---

5. Which of the following series converge absolutely?

a) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{2n}$ b) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$ c) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n+1}$

d) $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{2}\right)^n$ e) None of these

---

7.6: The Ratio and Nth Root Tests

The Ratio Test is one of the more easily used tests and it works on most series, especially those that involve factorials and those that are a combination of geometric and p-series. Unfortunately, it is sometimes inconclusive, especially with alternating series.

Cauchy Ratio Test: If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \rightarrow$ the series converges;

If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1 \rightarrow$ the series diverges;

If $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \rightarrow$ the test is inconclusive.

The Nth Root Test is best for series with n is both the base and the exponent.

The Nth Root Test: If $\lim_{n \to \infty} \sqrt[n]{a_n} < 1 \rightarrow$ the series converges;

If $\lim_{n \to \infty} \sqrt[n]{a_n} > 1 \rightarrow$ the series diverges;

If $\lim_{n \to \infty} \sqrt[n]{a_n} = 1 \rightarrow$ the test is inconclusive.

OBJECTIVE

Use the Ratio and Nth Root Tests to check for convergence or divergence.

Since the Cauchy Ratio Test involves Limits at Infinity, It might be a good idea to review the Hierarchy of functions:

The Hierarchy of Functions

1. Logs grow the slowest.
2. Polynomials, Rationals and Radicals grow faster than logs and the degree of the EBM determines which algebraic function grows fastest. For example, $y = x^{1/2}$ grows more slowly than $y = x^2$.
3. The trig inverses fall in between the algebraic functions at the value of their respective horizontal asymptotes.
4. Exponential functions grow faster than the others. (In BC Calculus, we will see the factorial function, $y = n!$, grows the fastest.)
5. The fastest growing function in the combination determines the end behavior, just as the highest degree term did among the algebraics.

Since the Ratio Test works well with factorials, we should review them.

Def'n: $n! = n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1)$

So, $(n+1)! = (n+1) \cdot (n) \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1) = (n+1) \cdot (n!)$.
Also, $(n-1)! = (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1)$
and $(n-2)! = (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1)$
etc.

So, factorials cancel from the back end of the list rather than the front:

$$\frac{(n+1)!}{n!} = \frac{(n+1) \cdot (n) \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1)}{(n) \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot \ \cdot \ \cdot \ \cdot (1)} = \frac{(n+1) \cdot (n!)}{(n!)} = (n+1)$$

And

$$\frac{(n-2)!}{(n-1)!} = \frac{(n-2)\cdot(n-3)\cdot\ \cdot\ \cdot\ \cdot(1)}{(n-1)\cdot(n-2)\cdot(n-3)\cdot\ \cdot\ \cdot\ \cdot(1)} = \frac{1}{n-1}$$

Notice that $(n-2)$ is a smaller number than $(n-1)$.

Ex 1 Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!}$$

$$= \lim_{n\to\infty}\frac{n!}{(n+1)n!} = \lim_{n\to\infty}\frac{1}{n+1}$$

$$= 0$$

Since this limit is <1, $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges.

Ex 2 Does $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converge?

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}\right|$$

$$= \lim_{n\to\infty}\frac{(n+1)2^n}{2^{n+1}n}$$

$$= \lim_{n\to\infty}\frac{n+1}{2n}$$

$$= \frac{1}{2}$$

Since this limit is <1, $\sum_{n=1}^{\infty} \frac{n}{2^n}$ converges.

Ex 3 Does $\sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}$ converge?

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\frac{(n+1)+2}{(n+1)((n+1)+1)}}{\frac{n+2}{n(n+1)}} \right|$$

$$= \lim_{n \to \infty} \frac{n+3}{(n+1)(n+2)} \cdot \frac{n(n+1)}{n+2}$$

$$= \lim_{n \to \infty} \frac{n^2+3n}{n^2+4n+4}$$

$$= 1$$

So the Ratio Test is inconclusive.

The easiest test would be the Limit Comparison Test:

Compare $\sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}$ to $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$\lim_{n \to \infty} \frac{\frac{n+2}{n(n+1)}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n(n+2)}{n(n+1)} = 1$$

Since the Limit = 1, both series converge or both diverge. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, therefore,

$\sum_{n=1}^{\infty} \frac{n+2}{n(n+1)}$ diverges also.

Ex 4 Does $\sum_{n=1}^{\infty} \frac{1}{\ln^n(n+2)}$ converge?

This series has *n* to an *n* power; therefore, the Root Test is appropriate.

$$\lim_{n \to \infty} \sqrt[n]{\frac{1}{\ln^n(n+2)}} = \lim_{n \to \infty} \frac{1}{\ln(n+2)} = 0$$

Since this limit is <1, $\sum_{n=1}^{\infty} \frac{1}{\ln^n(n+2)}$ converges.

Ex 5 Does $\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{e^{-n}}$ converge?

This series has n to an n power; therefore, the Root Test is appropriate.

$$\lim_{n \to \infty} \sqrt[n]{\frac{\left(1+\frac{1}{n}\right)^n}{e^{-n}}} = \lim_{n \to \infty} \frac{\left(1+\frac{1}{n}\right)}{e^{-1}} = \lim_{n \to \infty} \frac{(n+1)e}{n} = e > 1$$

Since this limit is >1, $\sum_{n=1}^{\infty} \frac{\left(1+\frac{1}{n}\right)^n}{e^{-n}}$ diverges.

7.6 Free Response Homework

Test these series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2}{2n}$

2. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

3. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{5+n}$

6. $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$

7. $\sum_{n=1}^{\infty} \frac{\cos\frac{\pi n}{3}}{n!}$

8. $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

9. $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

10. $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+3n}}$

11. $\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 4 \cdot 7 \cdot 10 \ldots \cdot (3n+1)}$

12. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \ldots \cdot (2n+1)}{n!}$

13. For which of the following series is the Ratio Test inconclusive. What test would you use instead?

a) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ b) $\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{\sqrt{n}}$ c) $\sum_{n=1}^{\infty} \frac{n}{2^n}$ d) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{1+n^2}$

14. For which of the following series is the Ratio Test inconclusive? What test would you use instead?

a) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+2n}$ b) $\sum_{n=1}^{\infty} \frac{1}{(3n)!}$ c) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^4}$ d) $\sum_{n=1}^{\infty} \frac{12^n}{(n+2)3^{3n+1}}$

7.6 Multiple Choice Homework

1. What is the most appropriate test to prove convergence or divergence for $\sum_{n=1}^{\infty} \frac{2^n \cdot n^3}{n!}$?

(A) The Divergence Test

(B) The Direct Comparison Test

(C) The Limit Comparison Test

(D) The Ratio Test

---

2. What is the most appropriate test to prove convergence or divergence for $\sum_{n=1}^{\infty} \frac{n^4}{4^n}$?

(A) The Divergence Test

(B) The Direct Comparison Test

(C) The Limit Comparison Test

(D) The Ratio Test

---

3. What is the most appropriate test to prove convergence or divergence for $\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$?

(A) The Divergence Test

(B) The Direct Comparison Test

(C) The Limit Comparison Test

(D) The Ratio Test

---

4. What is the most appropriate test to prove convergence or divergence for $\sum_{n=2}^{\infty} \frac{4^n}{n^4}$?

(A) The Divergence Test

(B) The Direct Comparison Test

(C) The Limit Comparison Test

(D) The Ratio Test

---

5. Consider $\sum_{n=3}^{\infty} \frac{n^2}{2^n}$ and $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$. Based on the Ratio Test, which of the following statements is true?

(A) $\sum_{n=3}^{\infty} \frac{n^2}{2^n}$ converges

(B) $\sum_{n=1}^{\infty} \frac{1}{(2n)!}$ converges

(C) Both series converge.

(D) Neither series converge.

---

6. Consider $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ and $\sum_{n=3}^{\infty} \frac{1}{n+4}$. Based on the Ratio Test, which of the following statements is true?

(A) $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^3}$ diverges

(B) $\sum_{n=3}^{\infty} \frac{1}{n+4}$ diverges

(C) Both series diverge.

(D) Neither series diverges.

7.7 General Series Summary

A series is convergent if and only if $\sum_{n=1}^{\infty} a_n = c$.

A series is divergent if and only if $\sum_{n=1}^{\infty} a_n \neq c$.

Previously, we investigated the seven tests that will help determine if a given series converges or not. Those tests were:

1. **Divergence Test** (nth term test): If $\underset{n\to\infty}{Lim}\, a_n \neq 0 \rightarrow$ it diverges.
   If $\underset{n\to\infty}{Lim}\, a_n = 0 \rightarrow$ no conclusion

2. **Integral Test**: If *f(x)* is a decreasing function, then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} x_n dx$ either both converge or both diverge.

3. **Cauchy Ratio Test**: If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1 \rightarrow$ it converges;
   If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1 \rightarrow$ it diverges;
   If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1 \rightarrow$ the test is inconclusive.

4. **The Alternating Series Test:** An Alternating Series is convergent if and only if

   i) $|a_n| \geq |a_{n+1}|$,

   and ii) $\lim_{n\to\infty} |a_n| = 0$.

5. **The Nth Root Test**: If $\lim_{n\to\infty} \sqrt[n]{a_n} < 1 \to$ it converges;

If $\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \to$ it diverges;

If $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \to$ the test is inconclusive.

6. **The Direct Comparison Test:** When comparing a given series $\sum_{n=1}^{\infty} a_n$ to a known series $\sum_{n=1}^{\infty} b_n$,

i) if $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

ii) if $a_n \ge b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

7. **The Limit Comparison Test:** When comparing a given series $\sum_{n=1}^{\infty} a_n$ to a known series $\sum_{n=1}^{\infty} b_n$,

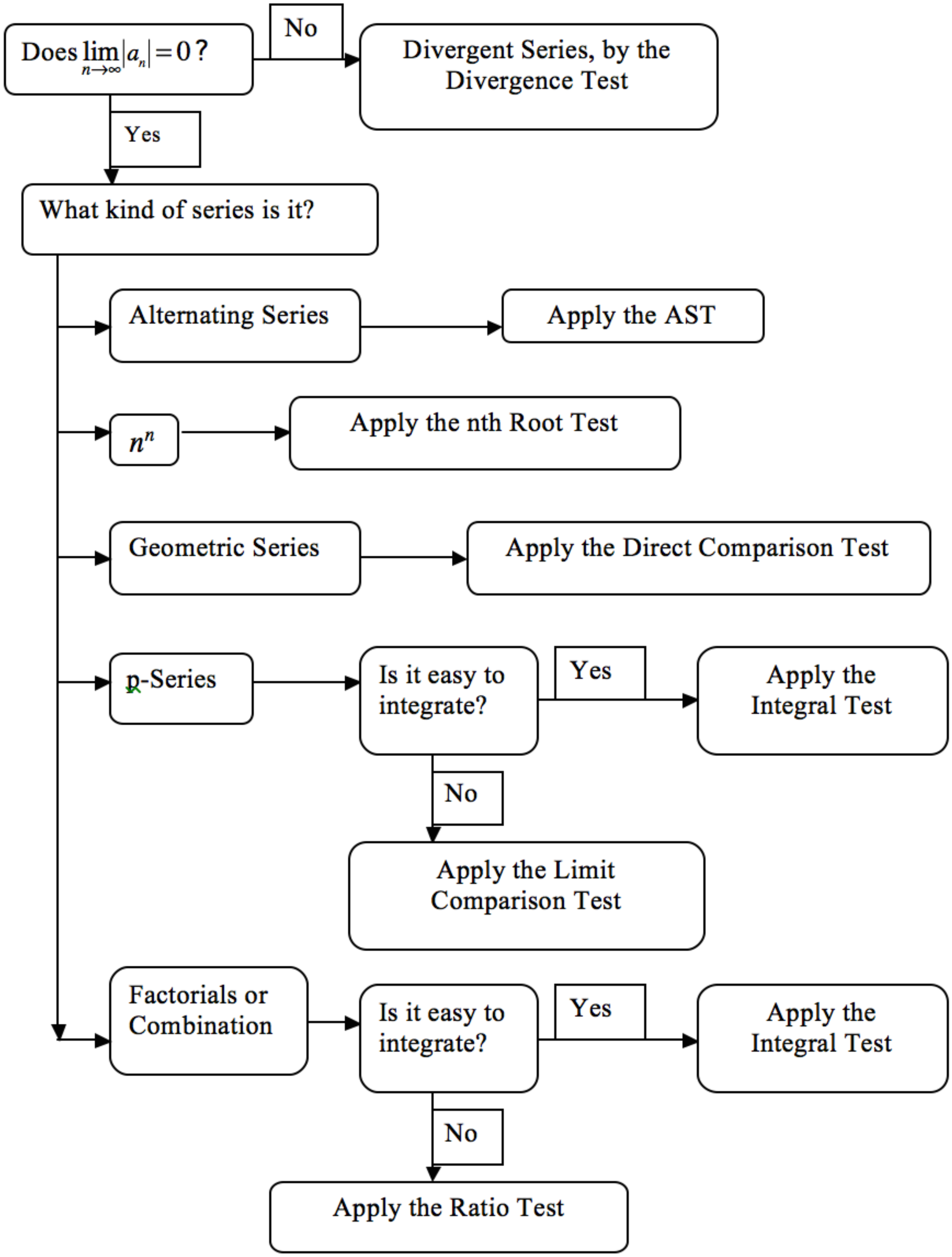
i. If $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| > 0$ then both converge or both diverge.

ii. If $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = 0$, then $\sum_{n=1}^{\infty} a_n$ converges if $\sum_{n=1}^{\infty} b_n$ converges.

iii. If $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges if $\sum_{n=1}^{\infty} b_n$ diverges.

OBJECTIVE

Find whether a given numerical series converges or diverges.



Does $\lim_{n\to\infty}|a_n| = 0$?
No
Divergent Series, by the Divergence Test
Yes
What kind of series is it?
Alternating Series
Apply the AST
n^n
Apply the nth Root Test
Geometric Series
Apply the Direct Comparison Test
p-Series
Is it easy to integrate?
Yes
Apply the Integral Test
No
Apply the Limit Comparison Test
Factorials or Combination
Is it easy to integrate?
Yes
Apply the Integral Test
No
Apply the Ratio Test

7.7 Homework

1. Which of the following sequences converge?

I. $\left\{\dfrac{3n^5}{7n^4-1}\right\}$ II. $\left\{\dfrac{\cos n}{\pi^n}\right\}$ III. $\left\{\dfrac{n!}{e^{2n}}\right\}$

a) I only b) II only c) I and III only

d) II and III only e) III only

2. Which of the following sequences diverge?

I. $\left\{\dfrac{\ln n}{5n}\right\}$ II. $\left\{\left(\dfrac{5\cos 3}{\sqrt{e}}\right)^n\right\}$ III. $\left\{\dfrac{n!}{(n+2)!}\right\}$

a) II only b) III only c) I and II only

d) I and III only e) I, II, and III

3. Which of the following series diverge?

I. $\displaystyle\sum_{n=1}^{\infty} 9^{-n}8^{n+1}$ II. $\displaystyle\sum_{n=3}^{\infty}\frac{\ln 5}{\cos(2\pi n)}$ III. $\displaystyle\sum_{n=2}^{\infty}\frac{2^n 3^{n+1}}{(2e)^n}$

a) I only b) III only c) I and II only

d) II and III only e) I, II, and III

4. What are all values of k for which the infinite series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{k}{5}\right)^n$ converges?

a) $|k|<5$ b) $|k|\le 5$ c) $|k|<1$

d) $|k|\le 1$ e) $k=0$

5. If $f(x)=\sum_{n=0}^{\infty}\left((1-\sin x)^2\right)^n$, then $f\left(\frac{\pi}{6}\right)=$

a) 2 b) $\frac{4}{3}$ c) $\frac{1}{2}$ d) $\frac{3}{2}$ e divergent

6. Which of the following series converge?

I. $\sum_{n=1}^{\infty}\frac{2n+3}{n^2+3n+6}$ II. $\sum_{n=1}^{\infty}\frac{n+1}{e^n}$ III. $\sum_{n=1}^{\infty}\frac{2n}{1+n^4}$

a) I only b) III only c) I and III only

d) II and III only e) I and II only

7. Which of the following series are divergent?

I. $\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1}$ II. $\sum_{n=0}^{\infty}\frac{e^n}{(n+5)!}$ III. $\sum_{n=1}^{\infty}\frac{n!}{3^n}$

a) I only b) II only c) III only

d) I and II only e) II and III only

8. Which of the following series are conditionally convergent?

I. $\sum_{n=0}^{\infty}\frac{(-1)^n}{n\sqrt{n}}$ II. $\sum_{n=0}^{\infty}\frac{e^n}{(n+5)!}$ III. $\sum_{n=1}^{\infty}\frac{\cos(\pi n)}{n}$

a) I only b) III only c) I and II only

d) I and II only e) I and III only

9. Which of the following series are divergent?

I. $\sum_{n=0}^{\infty}(-1)^n \frac{1}{3n+1}$ II. $\sum_{n=0}^{\infty}\frac{(n+2)!}{n!}$ III. $\sum_{n=1}^{\infty}\frac{n!}{3^n}$

a) I and IIIonly b) II only c) III only

d) I and II only e) II and III only

10. What are all values of p for which the infinite series $\sum_{n=1}^{\infty} 5n^{1+p}$ diverges?

a) $p > -2$
b) $p \geq -2$
c) $p < -2$
d) $p \leq -2$
e) All real values of p

Numerical Series Test
Calculator Allowed

1. Which of the following sequences converge?

I. $\left\{\dfrac{\ln n}{\ln 2n}\right\}$ II. $a_n = n(n-1)$ III. $\left\{\dfrac{(-1)^n n^3}{3^3+2n^2+1}\right\}$

a) I only b) III only c) I and II only

d) I and III only e) I, II, and III

2. Which of the following Series diverge?

I. $\sum_{n=1}^{\infty} 5^{-n}6^{n-1}$ II. $\sum_{n=1}^{\infty} \dfrac{3n^5}{7n^4-1}$ III. $\sum_{n=1}^{\infty} \dfrac{1}{5^n}$

a) I only b) III only c) I and II only

d) I and III only e) I, II, and III

3. Which of the following are series divergent?

I. $\sum_{n=1}^{\infty} \dfrac{e^n}{(n+5)!}$ II. $\sum_{n=3}^{\infty} \dfrac{\ln 5}{\cos(2\pi n)}$ III. $\sum_{n=2}^{\infty} \dfrac{2^n 3^{n+1}}{(2e)^n}$

a) I only b) III only c) I and III only

d) II and III only e) I, II, and III

4. What are all values of p for which the infinite series $\sum_{n=1}^{\infty} 5n^{1+p}$ converges?

a) $p > -2$ b) $p \geq -2$ c) $p < -2$

c) $p \leq -2$ e) All real values of p

5. Which of the following are series convergent?

I. $\sum_{n=1}^{\infty} \frac{5}{n^3+3}$ II. $\sum_{n=1}^{\infty} \frac{1}{4+n^2}$ III. $\sum_{n=2}^{\infty} (3n+7)$

(A) I only (B) III only (C) I and II only

(D) I and III only (E) II and III only

6. Which of the following series diverge?

I. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n+1}$ II. $\sum_{n=1}^{\infty} (-1)^n \frac{\cos n}{3^n}$ III. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

a) I only b) III only c) I and II only

d) I and III only e) None of these

7. If $\sum_{n=1}^{\infty} |a_n|$ converges, then which of the following is true?

I. $\sum_{n=1}^{\infty} a_n$ diverges II. $\sum_{n=1}^{\infty} a_n$ converges absolutely

III. $\sum_{n=1}^{\infty} 4a_n$ converges

a) I only b) II only c) III only

d) II and III only e) I, II, and III

8. Which of the following series are conditionally convergent?

I. $\sum_{n=2}^{\infty} \frac{(-1)^n}{(Ln\ n)}$ II. $\sum_{n=1}^{\infty} \frac{(-1)^n}{(\tan^{-1} n)}$ III. $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$

a) I only b) I and III only c) II and III only

d) I and III only e) I, II, and III

9. $\sum_{n=1}^{\infty} \left(\sin \frac{\pi}{6} \right)^n =$

a) 1 b) 2 c) $1-\frac{\sqrt{3}}{2}$ d) $\frac{\frac{\sqrt{3}}{2}}{1-\frac{\sqrt{3}}{2}}$ e) divergent

10. Which of the following three tests will establish that the series $\sum_{n=1}^{\infty} \frac{n}{\sqrt{2n^5+1}}$ converges?

I. Direct Comparison Test with $\sum_{n=1}^{\infty} n^{-5/2}$

II. Limit Comparison Test with $\sum_{n=1}^{\infty} n^{-3/2}$

III. Direct Comparison Test with $\sum_{n=1}^{\infty} n^{-1/2}$

a) I only b) II only c) I and II only

d) II and III only e) I, II, and III

Numerical Series Test
Calculator Allowed

1. Use the Integral Test to determine if $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent or divergent.

2. Use the Ratio Test to determine if $\sum_{n=1}^{\infty} \frac{2^n \cdot n^3}{n!}$ is convergent or divergent.

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+4)^2}$ converges or diverges.

Chapter 7 Answer Key

7.1 Free Response Key

1.	arithmetic; d = 5	2.	geometric; $r = \frac{1}{3}$
3.	geometric; $r = 2$	4.	neither
5.	neither	6.	geometric; $r = 5^{-1/6}$

7.	$\frac{137}{60}$	8.	10	9.	-7	10.	248
11.	.243	12.	145	13.	125	14.	-157.5
15.	-143	15.	37	16.	43	18.	4
19.	14	20.	222	21.	-14040	22.	531,440
23.	-3.843×10^{17}	24.	n = 16, S = -131070				

7.2 Free Response Key

1.	Divergent	2.	Convergent	3.	Convergent
4.	Convergent	5.	Convergent	6.	Convergent
7.	Convergent	8.	Might Converge	9.	Might Converge
10.	Divergent	11.	Divergent	12.	Might Converge
13.	Convergent	14.	Convergent	15.	Convergent
16.	Divergent	17.	Convergent	18.	Convergent
19.	Convergent	20.	Convergent	21.	Divergent

22. Divergent 23. 25 24. 5.6

25. Divergent

7.3 Free Response Key

1. Convergent, $\frac{1}{\sqrt{2}}$
2. Convergent, $\frac{1}{\ln 2}$
3. Divergent
4. Convergent, $\frac{1}{\sqrt{2}}$
5. Convergent, $\frac{1}{2}$
6. Convergent, $\frac{7}{6}$
7. Convergent, $\frac{\pi}{4}$
8. Divergent
9. Convergent, .0002
10. Convergent, $\frac{1}{2e}$
11. Convergent, e
12. Convergent, $\frac{\pi}{2}$
13. Divergent
14. Convergent, $\frac{1}{12}$
15. Divergent
16. Divergent
17. Divergent
18. Divergent

7.3 Multiple Choice Key

1. A 2. C 3. D 4. B

7.4 Free Response Key

1. Converges by LCT
2. Converges by LCT
3. Diverges by LCT
4. Converges by the Ratio Test
5. Diverges by the Divergence Test
6. Converges by the Ratio Test
7. Converges by DCT
8. Diverges by LCT

9. Diverges by LCT
10. Diverges by LCT
11. Converges by DCT
12. Diverges by LCT
13. Converges by DCT
14. Converges by LCT
15. Converges by DCT
16. Converges by LCT
17. Converges by DCT
18. Diverges by LCT
19. Converges by LCT
20. Diverges by LCT

7.4 Multiple Choice Key

1.	A	2.	D	3.	C	4.	E	5.	E	6.	B
7.	A	8.	A	9.	C	10.	B	11.	C	12.	D

7.5 Free Response Key

1. Conditionally Convergent
2. Divergent
3. Conditionally Convergent
4. Divergent
5. Conditionally Convergent
6. Conditionally Convergent
7. Absolutely Convergent
8. Divergent
9. Absolutely Convergent
10. Conditionally Convergent
11. Divergent
12. Conditionally Convergent
13. Conditionally Convergent
14. Absolutely Convergent
15. Conditionally Convergent
16. Divergent
17. Divergent
18. Divergent

19. Absolutely Convergent 20. Divergent

7.5 Multiple Choice Key

1. C 2. A 3. D 4. B 5. D

7.6 Free Response Key

1. Divergent by Divergence Test
2. Converges by the Integral Test
3. Divergent by AST
4. Converges by AST
5. Divergent by Divergence Test
6. Converges by Ratio Test
7. Converges by Ratio Test
8. Converges by Ratio Test
9. Converges by Root Test
10. Diverges by Root Test
11. Converges by Ratio Test
12. Diverges by Ratio Test

13a. Ratio Test inconclusive; use the p-Series

13b. Ratio Test works

13c. Ratio Test works

13d. Ratio Test inconclusive; use the LCT

14a. Ratio Test inconclusive; use the Divergence Test

14b. Ratio Test works

14c. Ratio Test inconclusive; use the AST

14d. Ratio Test works

7.6 Multiple Choice Key

1.	D	2.	D	3.	D	4.	A	5.	C	6.	A

7.7 Homework Key

1.	B	2.	A	3.	D	4.	A	5.	B	6.	D
7.	E	8.	C	9.	E	10.	B				

Numerical Series Practice Test

1.	D	2.	C	3.	D	4.	C	5.	C	6.	E
7.	D	8.	A	9.	A	10.	D				

1. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges

2. $\sum_{n=1}^{\infty} \frac{2^n \cdot n^3}{n!}$ is convergent.

3. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(n+4)^2}$ is convergent.