Chapter 5 Overview: Applications of Integrals

Calculus, like most mathematical fields, began with trying to solve everyday problems. The theory and operations were formalized later. As early as 270 bc, Archimedes was working on the problem of finding the volume of a non-regular shapes. Beyond his bathtub incident that revealed the relationship between weight volume and displacement, he had actually begun to formalize the limiting process to explore the volume of a diagonal slice of a cylinder.

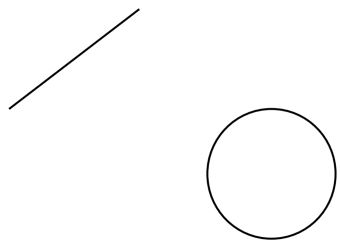
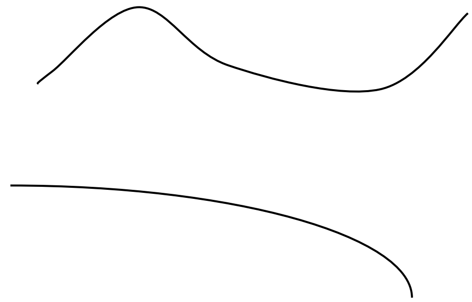
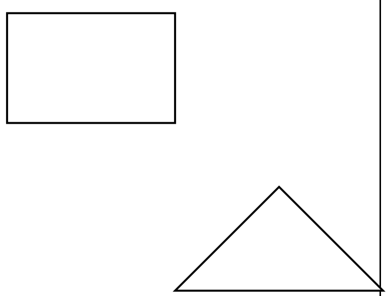
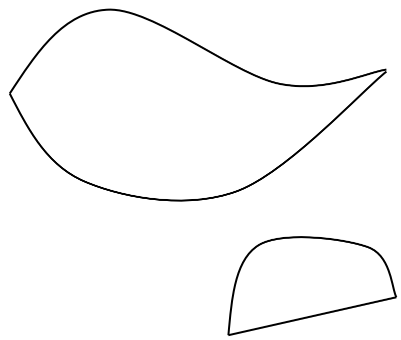
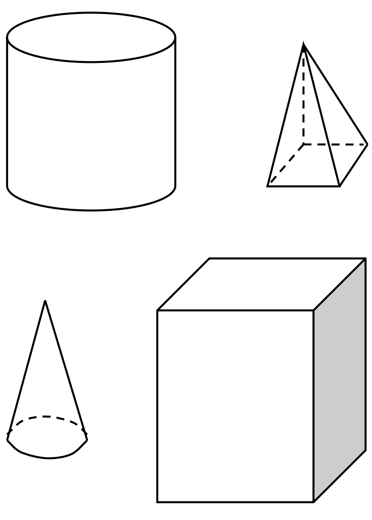
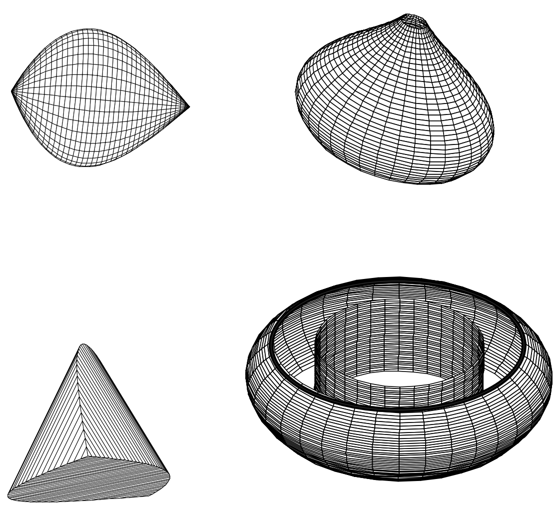
This is where Calculus can give us some very powerful tools. In geometry, we can find lengths of specific objects like line segments or arcs of circles, while in Calculus we can find the length of any arc that we can represent with an equation. The same is true with area and volume. In Geometry, area is limited to a few specific formulas for simple shapes, while Calculus is very open-ended in the problems it can solve. On the next page is an illustration of the difference.

In this chapter, we will investigate what have become the standard applications of the integral:

- Area
- Arc Length
- Volumes of Rotation
- Volume by Cross-Section

Though we will use these applications to reinforce the antidifferentiation skills, we will emphasize how the formulas relate to the geometry of the problems and the technological (graphing calculator) solutions rather than the algebraic solutions.

Below is an illustration of what we can accomplish with Calculus as opposed to geometry:

	Geometry	**Calculus**
Length		
Area		
Volume		

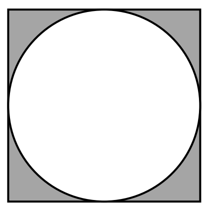
Calculus can also be used to generate surface areas for odd-shaped solids as well, but that is out of the scope of this class.

5.1 Area Between Two Curves

In a previous chapter, we learned how to find the area "under" a curve:

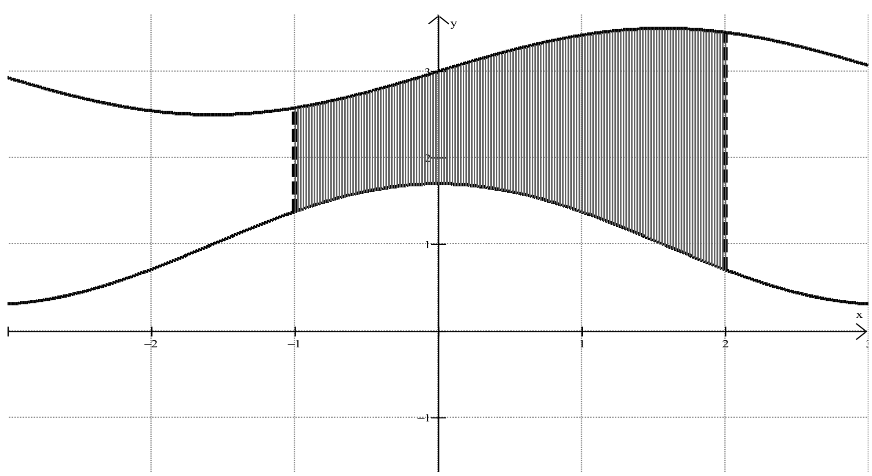
$$Area = \int_a^b \left| f(x) \right| \, dx$$

But what about the area between two curves? It turns out that it is a simple proposition, very similar to some problems we encountered in geometry. In geometry, if we wanted to know the area of a shaded region that was composed of multiple figures, like the illustration below, we find the area of the larger and subtract the area of the smaller.



The area for this figure would be $A_{Total} = A_{square} - A_{circle} = s^2 - \pi\left(\frac{s}{2}\right)^2$

Similarly, since the integral gets us a numeric value for the area between a curve and an axis, if we simply subtract the "smaller" curve from the "larger" one and integrate, we can find the area between two curves.



Objectives:

Find the area of the region between two curves.

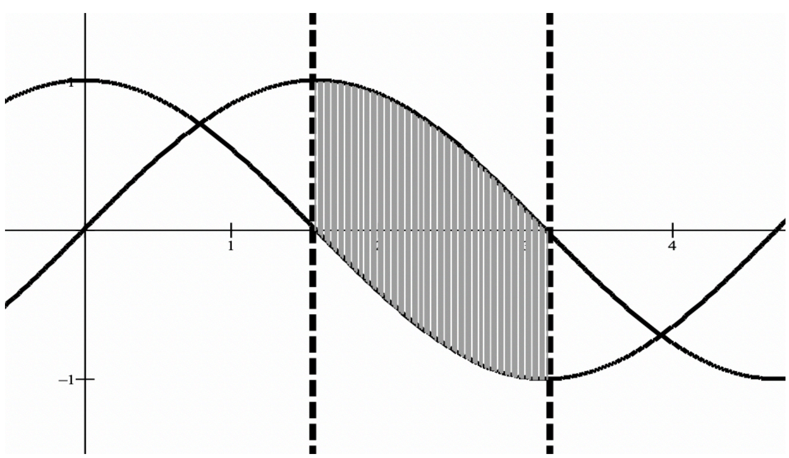
Unlike before, when we had to be concerned about positives and negatives from a definite integral messing up our interpretation of area, the subtraction takes care of the negative values for us (if one curve is under the axis, the subtraction makes the negative value of the integral into the positive value of the area.

Area Between Two Curves:
The area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ where f and g are continuous and $f \geq g$ for all x in $[a,b]$ is

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

You can also think of this expression as the '**top**' curve minus the '**bottom**' curve. If we associate integrals as 'area under a curve' we are finding the area under the top curve, subtracting the area under the bottom curve, and that leaves us with the area between the two curves.

Ex 1 Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and $x = \pi$



$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

$A=\int_a^b[\sin(x)-\cos(x)]dx$ On our interval $\sin(x)$ is greater than $\cos(x)$

$A=\int_{\pi/2}^{\pi}[\sin(x)-\cos(x)]dx$ Our region extends from $x=\frac{\pi}{2}$ to $x=\pi$

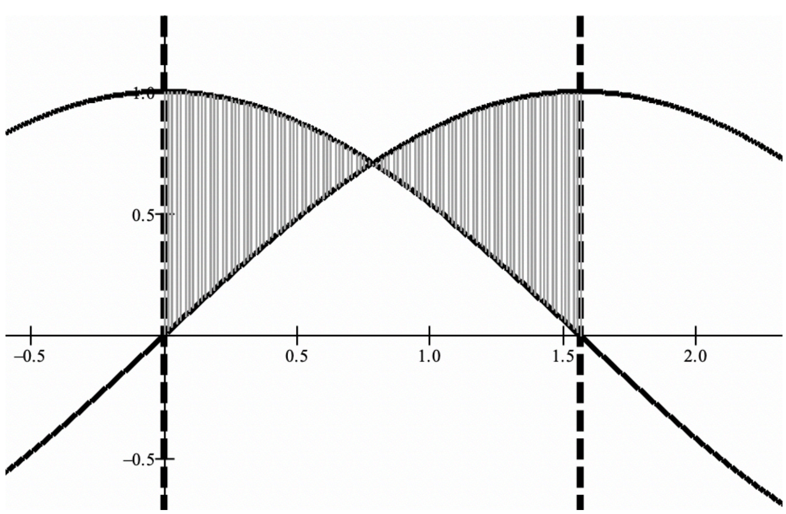
$$A=\left[-\cos(x)-\sin(x)\right]_{\pi/2}^{\pi}$$

$$A=\left(-\cos(\pi)-\sin(\pi)\right)-\left(-\cos\left(\frac{\pi}{2}\right)-\sin\left(\frac{\pi}{2}\right)\right)=2$$

Steps to Finding the Area of a Region:

1. Draw a picture of the region.
2. Sketch a Reimann rectangle – if the rectangle is vertical your integral will have a *dx* in it, if your rectangle is horizontal your integral will have a *dy* in it.
3. Determine an expression representing the length of the rectangle: (top – bottom) or (right – left).
4. Determine the endpoints the boundaries (points of intersection).
5. Set up an integral containing the limits of integration, which are the numbers found in Step 4, and the integrand, which is the expression found in Step 3.
6. Solve the integral.

Ex 2 Find the area of the region bounded by $y=\sin x$, $y=\cos x$, $x=0$, and $x=\frac{\pi}{2}$.



$$A=\int_0^{\pi/4}\left[\cos(x)-\sin(x)\right]dx+\int_{\pi/4}^{\pi/2}\left[\sin(x)-\cos(x)\right]dx=.828$$

With this problem, we had to split it into two integrals, because the "top" and "bottom" curves switch partway through the region. We could have also done this with an absolute value:

$$A=\int_0^{\pi/2}\left|\cos(x)-\sin(x)\right|dx=.828$$

If we are asked this question on an AP test or in a college classroom, they usually want to see the setup. The absolute value is just an easy way to integrate using the calculator.

NOTE: Any integral can be solved going dy or dx. It is usually the case though, that one is "easier" than the other.

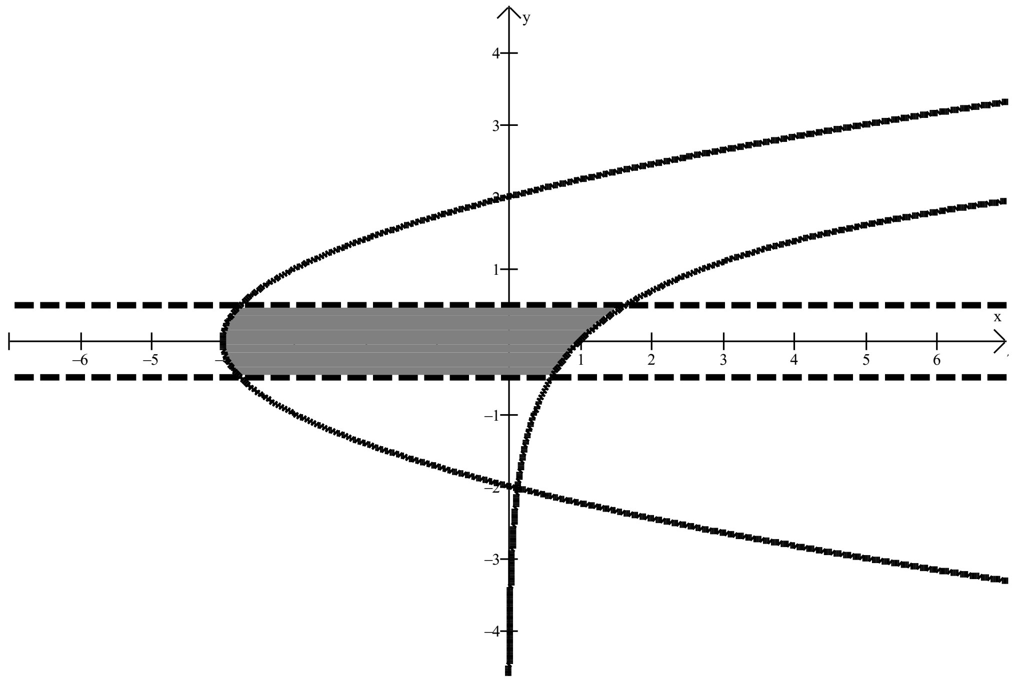
What if the functions are not defined in terms of *x* but in terms of *y* instead?

The area of the region bounded by the curves $x=f(y)$, $x=g(y)$, and the lines $y=c$ and $y=d$ where *f* and *g* are continuous and $f\geq g$ for all *y* in $[c,d]$ is

$$A=\int_c^d\left[f(y)-g(y)\right]dy$$

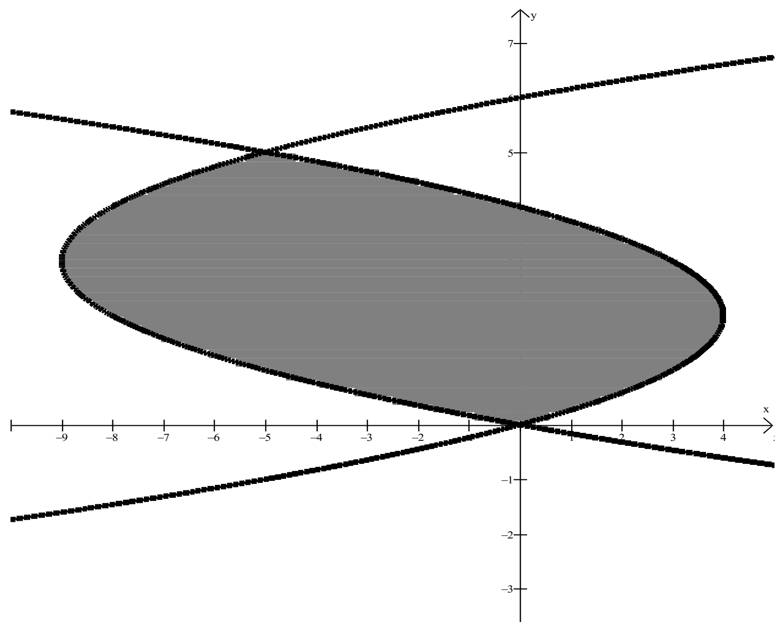
You can also think of this expression as the '**right**' curve minus the '**left**' curve.

Ex 4   Find the area of the region bounded by $x=e^y$, $x=y^2-4$, $y=\frac{1}{2}$, and $y=-\frac{1}{2}$.



$A = \int_c^d \left[f(y) - g(y) \right] dy$	Our pieces are perpendicular to the y – axis, so our integrand will contain dy
$A = \int_c^d \left[e^y - \left(y^2 - 4 \right) \right] dy$	On our interval e^y is greater than $y^2 - 4$
$A = \int_{-1/2}^{1/2} \left[e^y - \left(y^2 - 4 \right) \right] dy$	Our region extends from $y = -\frac{1}{2}$ to $y = \frac{1}{2}$
$A = 4.959$	

Ex 5 Sketch the graphs of $x = y^2 - 6y$ and $x = 4y - y^2$. If asked to find the area of the region bounded by the two curves which integral would you choose – an integral with a dy and an integral with a dx? Both will work but one is less time consuming.



We can see that if we went dx we would need three integrals (because the tops and bottoms are different on three different sections), but if we go dy we only need one integral. I like integrals as much as the next person, but I'll just do the one integral if that's OK right now.

$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

To find the boundaries, we need to set the equations equal to each other:

$$\begin{aligned} &4y - y^2 = y^2 - 6y \\ &0 = 2y^2 - 10y \\ &0 = 2y(y-5) \\ &y = 0, 5 \end{aligned}$$

$$A = \int_0^5 \left[\left(4y - y^2\right) - \left(y^2 - 6y\right) \right] dy$$

$$= \int_0^5 \left(10y - 2y^2\right) dy$$

$$= 5y^2 - \frac{2}{3} y^3 \Big]_0^5$$

$$= 125 - \frac{250}{3}$$

$$A = {}^{125}\!/_{3}$$

5.1 Free Response Homework

Draw and find the area of the region enclosed by the given curves.

1. $y = x+1,\ y = 9-x^2,\ x = -1,\ x = 2$

2. $y = \sin x,\ y = e^x,\ x = 0,\ x = \frac{\pi}{2}$

3. $y = x^2,\ y = 8-x^3,\ x = -3,\ x = 3$

4. $y = \sin 2x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$

5. $y = 5x - x^2,\ y = x$

6. $y = 1+\sqrt{x},\ y = \frac{x+7}{4}$

7. $x = e^y,\ x = y^2 - 2,\ y = -1,\ y = 1$

8. $x = 1 - y^2,\ x = y^2 - 1$

9. $y = \sqrt{x+2},\ y = \frac{1}{x^2},\ x = 1,\ x = 2$

10. $x = 2y - y^2,\ x = y^2 - 4y$

Use your calculator to sketch the regions described below. Find the points of intersection and find the area of the region described region.

11. $y = \sqrt{x},\ y = e^{-2x},\ x = 1$

12. $y = \ln\left(x^2+1\right),\ y = \cos x$

13. $y = x^2,\ y = 2^x$

5.1 Multiple Choice Homework

1. Let R be the region enclosed by the graph of $y = e^{\left(\frac{-1}{1+x^2}\right)}$, the x-axis, and the lines $x = -\frac{9}{8}$ and $x = \frac{9}{8}$. The closest integer approximation of R is

a) 0 b) 1 c) 2 d) 3 e) 4

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2. The area of the region enclosed by $y = x^2 - 4$ and $y = x - 4$ is given by

a) $\int_0^1 (x - x^2)\,dx$ b) $\int_0^1 (x^2 - x)\,dx$ c) $\int_0^2 (x - x^2)\,dx$

d) $\int_0^2 (x^2 - x)\,dx$ e) $\int_0^4 (x^2 - x)\,dx$

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3. What is the area enclosed by $y = x\ln(2x+1)$ and $y = 2\sin x$?

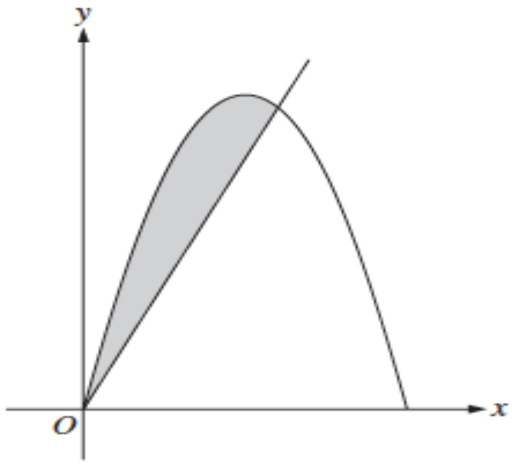
a) 0.334

b) 0.661

c) 3.526

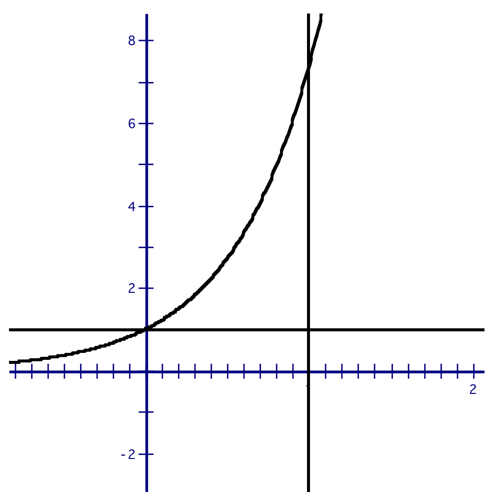
d) 0.825

e) 2.983



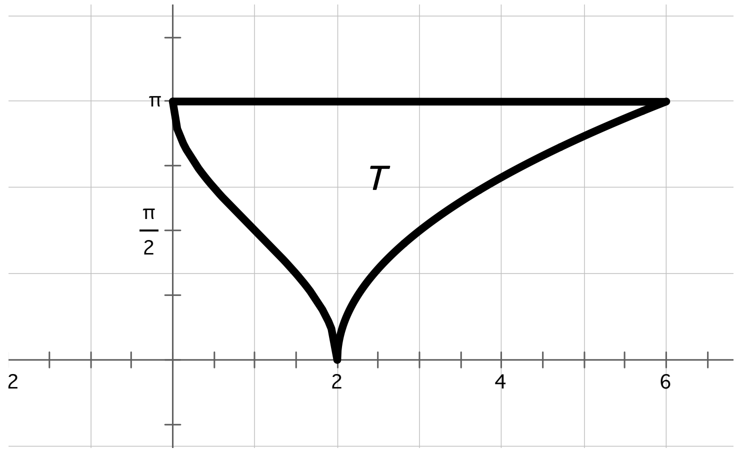
4. The figure above shows the graph of $y = 5x - x^2$ and the graph of the line $y = 2x$. What is the area of the shaded region?

a) $\frac{25}{6}$ b) $\frac{9}{2}$ c) $\frac{27}{2}$ d) $\frac{45}{2}$ e) 9



5. What is the area of the closed region bounded by the curve $y = e^{2x}$ and the lines $x = 1$ and $y = 1$?

a) $\frac{2-e^2}{2}$ b) $\frac{e^2-3}{2}$ c) $\frac{3-e^2}{2}$ d) $\frac{e^2-2}{2}$ e) $\frac{e^2-1}{2}$



6. Let ***T*** be the region above bounded by

$$y = \frac{\pi}{2} - \sin^{-1}(x-1),\;\; y = \frac{\pi}{2}\sqrt{x-2},\;\; \text{and } y = \pi$$

The area of ***T*** is

a) 6.225 b) 7.330 c) 39.111

d) 105.585 e) 108.895

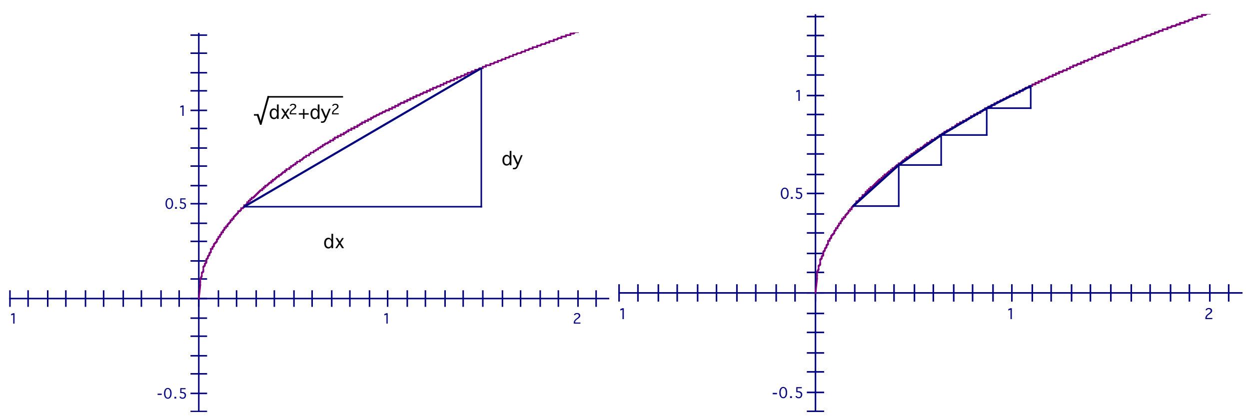
5.2 Arc Length

Back in your geometry days you learned how to find the distance between two points on a line. But what if we want to find the distance between to points that lie on a nonlinear curve?

Objectives:

Find the arc length of a function in Cartesian mode between two points.

Just as the area under a curve can be approximated by the sum of rectangles, arc length can be approximated by the sum of ever-smaller hypotenuses.



Here is our arc length formula:

Arc Length between Two Points:
Let $f(x)$ be a differentiable function such that $f'(x)$ is continuous, the length L of $f(x)$ over $[a,b]$ is given by

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad or \quad L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

As with the volume problems, we will use Math 9 in almost all cases to calculate arc length.

Ex 1 Find the length of the curve $y=1+e^x$ from $x=0$ to $x=3$.

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$ Start here

$$L=\int_0^3 \sqrt{1+\left(e^x\right)^2}\,dx$$ Substitute in for $\frac{dy}{dx}$

$$L=19.528$$ Math 9

Ex 2 Find the length of the curve $x=2y+y^3$ from $y=1$ to $y=4$.

$$L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$L=\int_1^4 \sqrt{1+\left(2+3y^2\right)^2}\,dy$$
$$L=69.083$$

Ex 3 Use right hand Riemann sums with n = 4 to estimate the arc length of $y=3+x\ln x$ from $x=1$ to $x=5$. How does this approximation compare with the true arc length (use Math 9)?

Let $f(x)=\sqrt{1+\left(1+\ln x\right)^2}$ Let $f(x)$ represent our integrand

$$L\approx\int_1^5 \sqrt{1+\left(1+\ln x\right)^2}\,dx$$
$$=1\cdot f(2)+1\cdot f(3)+1\cdot f(4)+1\cdot f(5)$$ Approximate using rectangles
$$=1.966+2.325+2.587+2.794$$
$$=9.672$$

Actual Value:

$$L = \int_1^5 \sqrt{1+(1+\ln x)^2} = 9.026$$

Note: The integral is the arc length. Some people get confused as to how "rectangle areas" can get us length. Remember that the Riemann rectangles are a way of evaluating an integral by approximation. Just like the "area under the curve" could be a displacement if the curve was velocity, in this case, the "area under the curve" $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ is the length of the arc along the curve y.

Ex 4 Find the length of the curve $f(x) = \int_2^{x^3} \sqrt{t+1}\,dt$ on $2 \le t \le 7$.

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_a^b \sqrt{1+\left(3x^2\sqrt{x^3+1}\right)^2}\,dx \\ &= 4235.511 \end{aligned}$$

5.2 Free Response Homework

Find the arc length of the curve.

1. $y = 1 + 6x^{3/2}$ on $x \in [0, 1]$

2. $y = \frac{1}{2}x^2 + \frac{1}{4}\ln x$ on $x \in [2, 4]$

3. $x = \frac{1}{3}\sqrt{y}(y-3)$ on $y \in [1, 9]$

4. $y = \ln(\sec x)$ on $x \in \left[0, \frac{\pi}{4}\right]$

5. $y = \ln x^{3/2}$ on $x \in [1, \sqrt{3}]$

6. $y = e^x$ on $x \in [0, 1]$

7. Find the length of the arc along $f(x) = \int_0^x \sqrt{\cos t}\, dt$ on $x \in \left[0, \frac{\pi}{2}\right]$.

8. Find the length of the arc along $f(x) = \int_{-2}^x \sqrt{3t^4 - 1}\, dt$ on $x \in [-2, 1]$.

9. Find the perimeter of the region R bounded by $y = (x-3)^2 - 1$, the x-axis, and the y-axis.

10. Find the perimeter of the region R bounded by $y = e^{-0.5x}$ and $y = x^2$.

11. Find the perimeter of the region R bounded by $y = \sqrt{2x}$ and $y = \frac{1}{8}x^4$.

12. Find the perimeter of each of the two regions bounded by $y = x^2$ and $y = 2^x$

5.2 Multiple Choice Homework

1. Compute the distance traveled by a particle along the curve $y=\frac{2}{3}x^{3/2}$ from $(0,\,0)$ to $\left(4,\,\frac{16}{3}\right)$.

a) $\frac{2}{3}\left(5\sqrt{5}-1\right)$ b) $\ln 4$ c) $\frac{1}{5\sqrt{5}}$

d) $\frac{14}{3}$ e) $\left(5\sqrt{5}-1\right)$

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2. Which of the following is the length of $y=t+\ln t$ between $1\le t\le 5$?

a) 1.609 b) 6.903 c) 16.047

d) 0.800 e) 148.413

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3. Which of the following is the length of $y=\ln(\cos x)$ between $0\le x\le\frac{\pi}{3}$?

a) $\ln\left(2+\sqrt{2}\right)$ b) $\ln\left(1+\sqrt{3}\right)$ c) $\ln\left(2-\sqrt{3}\right)$

d) $\ln\left(2+\sqrt{3}\right)$ e) $\ln\left(\sqrt{3}-1\right)$

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4. Which of the following integrals gives the length of the graph $y = \sin\sqrt{x}$ between $x = a$ and $x = b$ where $0 < a < b$?

a) $\int_a^b \sqrt{x + \cos^2\sqrt{x}}$

b) $\int_a^b \sqrt{1 + \cos^2\sqrt{x}}$

c) $\int_a^b \sqrt{\sin^2\sqrt{x} + \frac{1}{4x}\cos^2\sqrt{x}}$

d) $\int_a^b \sqrt{1 + \frac{1}{4x}\cos^2\sqrt{x}}$

e) $\int_a^b \sqrt{\frac{1 + \cos^2\sqrt{x}}{4x}}$

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5. Which of the following would represent the length of the graph of $\sqrt{x}$ between $x = a$ and $x = b$ where $0 < a < b$?

a) $\int_a^b \sqrt{x^2 + x}\ dx$

b) $\int_a^b \sqrt{x + \sqrt{x}}\ dx$

c) $\int_a^b \sqrt{x + \frac{1}{2\sqrt{x}}}\ dx$

d) $\int_a^b \sqrt{1 + \frac{1}{2\sqrt{x}}}\ dx$

e) $\int_a^b \sqrt{1 + \frac{1}{4x}}\ dx$

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6. The length of a curve $y = f(x)$ between $x=a$ to $x=b$ is given by $\int_a^b \sqrt{e^{2x} + 2e^x + 2}\,dx$. Therefore, $f(x) =$

a) $2e^{2x} + 2e^x$

b) $\frac{1}{2}e^{2x} + 2e^x + 2x$

c) $e^x - x + 3$

d) $e^x + 1$

e) $e^x + x - 2$

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7. The length of a curve from $x = 2$ to $x = 8$ is given by $\int_2^8 \sqrt{1+\frac{1}{4-x^2}}\, dx$. If the curve passes through the point $(1, 0)$,

a) $y = \sin^{-1} x$

b) $y = \sin^{-1} x - 1$

c) $y = \sin^{-1} x$

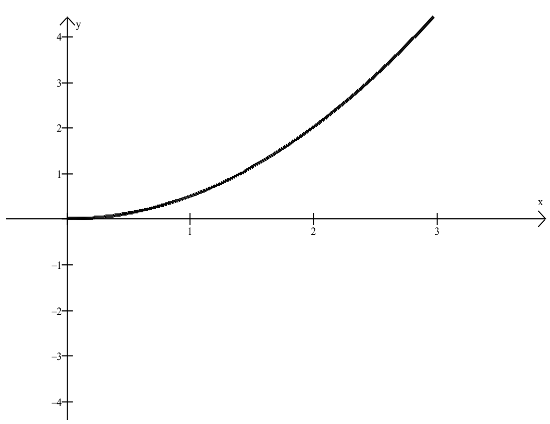
d) $y = \left(\sin^{-1} \frac{x}{2} \right) - 1$

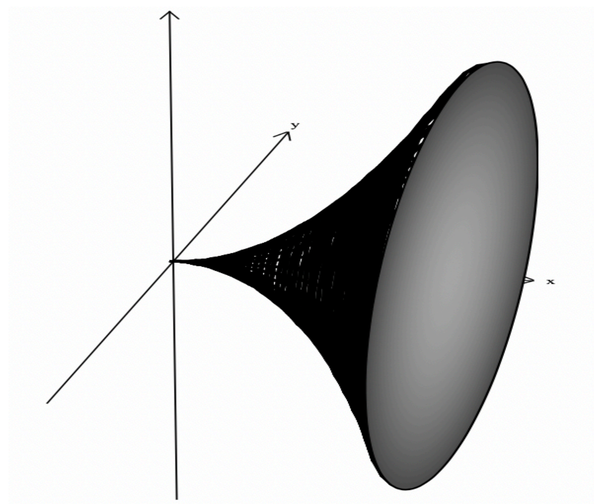
e) $y = \left(\sin^{-1} \frac{x}{2} \right) - \frac{\pi}{6}$

5.3 Volume by Rotation about the *X*-Axis

We know how to find the volume of many objects (remember those geometry formulas?) i.e. cubes, spheres, cones, cylinders … but what about non-regular shaped object? How do we find the volumes of these solids? Luckily, we have Calculus, but the basis of the Calculus formula is actually geometry.

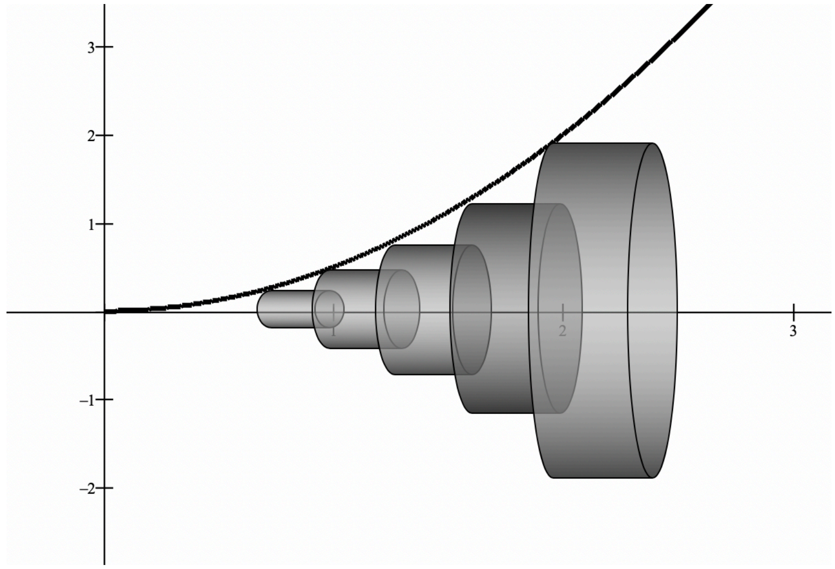
If we take a function (like the parabola below) and rotate it around the *x*-axis on an interval, we get a shape that does not look like anything we could solve with geometry.





When we rotate this curve about the axis, we get a shape like the one below:

This is sort of like a cone, but because the surface curves inward its volume would be less than a cone with the same size base. In fact, we cannot use a simple geometry formula to find the volume. But if we begin by thinking of the curve as having Reimann rectangles, and rotate those rectangles, the problem becomes a little more obvious:



Each of the rotated rectangles becomes a cylinder. The volume of a cylinder is

$$V = \pi r^2 h$$

Note that the radius (r) is the distance from the axis to the curve (which is $f(x)$) and the height is the change in x (Δx), giving us a volume of

$$V = \pi(f(x))^2 \Delta x$$

To find the total volume, we would simply add up all of the individual volumes on the interval at which we are looking (let's say from $x = a$ to $x = b$).

$$V_{total} = \sum_{x=a}^{b} \pi(f(x))^2 \Delta x$$

Of course, this would only give us an approximation of the volume. But if we made the rectangles very narrow, the height of our cylinders changes from Δx to dx, and to add up this infinite number of terms with these infinitely thin cylinders, the $\sum_{x=a}^{b}$ becomes $\int_a^b$ giving us the formula

$$V_{total} = \pi \int_a^b (f(x))^2 dx$$

What you may realize is that the $f(x)$ is simply the radius of our cylinder, and we usually write the formula with an r instead. The reason for this will become more obvious when we rotate about an axis that is not an x- or y-axis.

And since integration works whether you have dx or dy, this same process works for curves that are rotated around the y-axis and are defined in terms of y as well.

Volume by Disc Method (Part 1): The volume of the solid generated when the function $f(x)$ from $x = a$ and $x = b$, where $f(x) \geq 0$, is rotated about the x – axis is given by

$$V = \pi \int_a^b R^2 dx$$

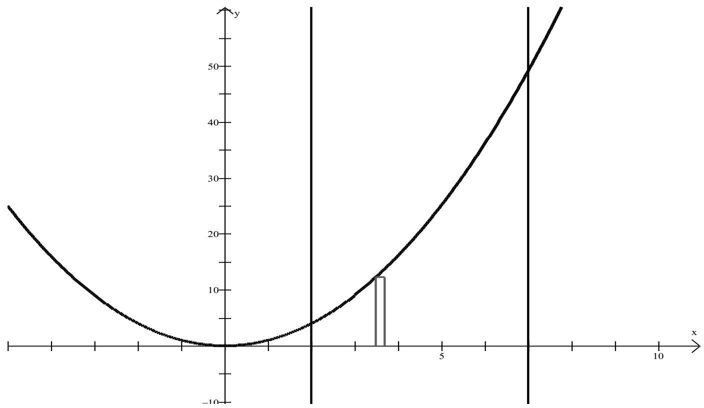
where r is the height of your Riemann rectangle.

NB. The Disk Method and Washer Method do not apply if the line of rotation is within the region being rotated.

Objective:

Find the volume of a solid rotated when a region is rotated about a given line.

Ex 1 Let R be the region bounded by the equations $y = x^2$, the x – axis, $x = 2$, and $x = 7$. Find the volume of the solid generated when R is rotated about the x – axis.



$V = \pi \int_a^b R^2 dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

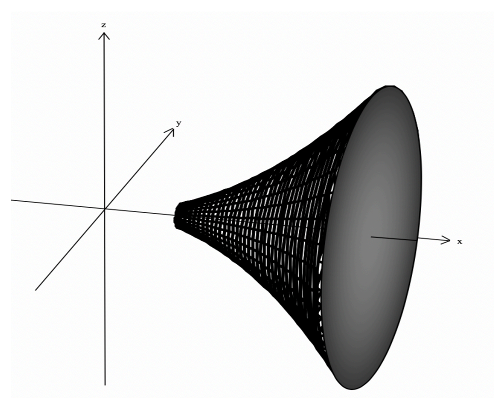
$V = \pi \int_a^b [y]^2 dx$ We cannot integrate y with respect to x so we will substitute out for y

$V = \pi \int_a^b [x^2]^2 dx$ The expression for y is x^2

$V = \pi \int_2^7 [x^2]^2 dx$ Our region extends from $x = 2$ to $x = 7$

$V = 3355\pi$

When the region, R, is rotated as in the example above, the solid generated would look like this:

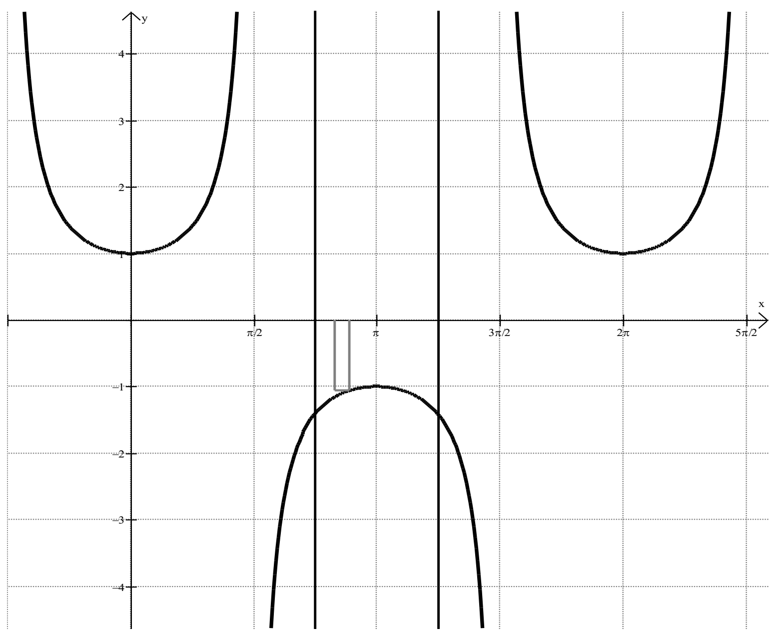


Note that knowing that this is what the rotated solid looks like has no bearing on the math. It makes an interesting shape, but being able to draw the image and being able to generate the volume by integration are two completely separate things. We are much more concerned with finding the volume.

Steps to Finding the Volume of a Solid about the *x*-axis:

1. Draw a picture of the region to be rotated.
2. Label the axis of rotation.
3. Sketch a Reimann rectangle. [Note: Your rectangle should always be sketched perpendicular to the axis of rotation.]
4. Determine an expression representing the height of the rectangle (in these cases, the height is the function value and is *R* in the volume formulas).
5. Determine the endpoints the region covers.
6. Set up an integral containing π outside the integrand, the limits of integration are the numbers found in Step 5, and the integrand is the expression found in Step 4.
7. Evaluate the integral

Ex 2 Let *R* be the region bounded by the curves $y = \sec x$, the *x* – axis, $x = \frac{3\pi}{4}$, and $x = \frac{5\pi}{4}$. Find the volume of the solid generated when *R* is rotated about the *x* – axis.



$$V = \pi \int_a^b R^2 dx$$

Our pieces are perpendicular to the *x* – axis, so our integrand will contain dx

$V = \pi \int_a^b y^2 dx$ We cannot integrate y with respect to x so we will substitute out for y

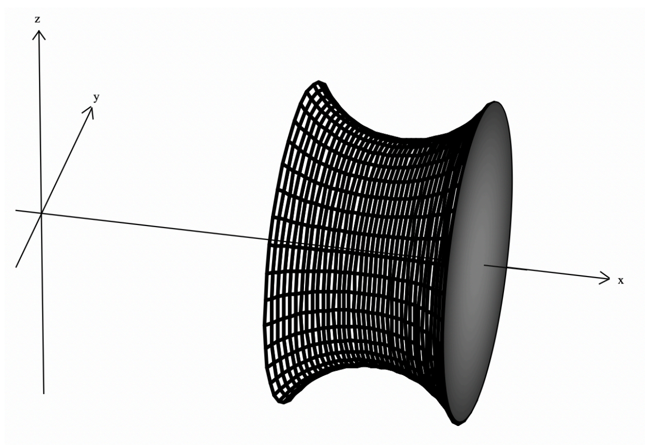
$V = \pi \int_a^b [\sec x]^2 dx$ The expression for y is $\sec x$

$V = \pi \int_{3\pi/4}^{5\pi/4} [\sec x]^2 dx$ Our region extends from $x = \dfrac{3\pi}{4}$ to $x = \dfrac{5\pi}{4}$

$$V = \pi[\tan x]_{3\pi/4}^{5\pi/4} = \pi\left[\tan\frac{5\pi}{4} - \tan\frac{3\pi}{4}\right] = \pi[1-(-1)] =$$

$$V = 2\pi$$

Again, when the region, *S*, is rotated as in the example above, the solid generated would look like this:

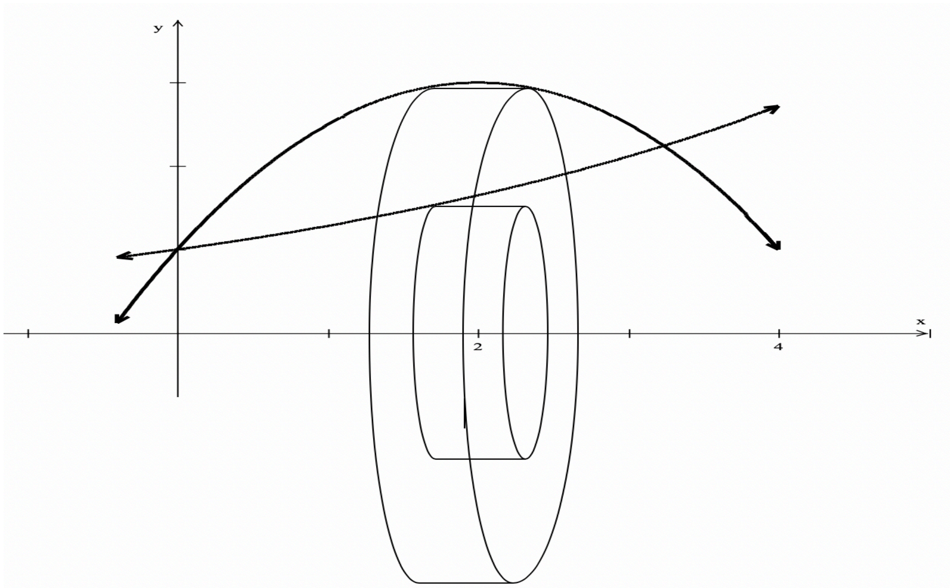


It's pretty cool-looking, but of no great consequence in terms of the math.

Note the symmetry of the original region. We could have made out arithmetic easier by calculating half the volume and doubling it:

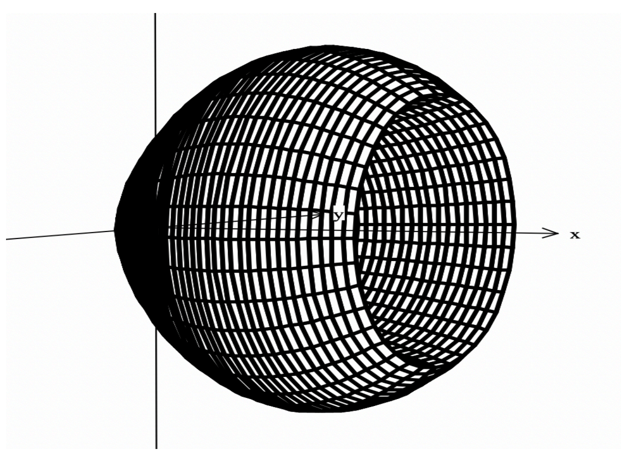
$$V = \pi \int_{3\pi/4}^{5\pi/4} [\sec x]^2 dx = 2\pi \int_{\pi/2}^{5\pi/4} [\sec x]^2 dx = 2\pi[\tan x]_{\pi/2}^{5\pi/4} = 2\pi[1-0] = 2\pi$$

Suppose we wanted to find the volume of a region bounded by two curves that was then rotated about an axis.

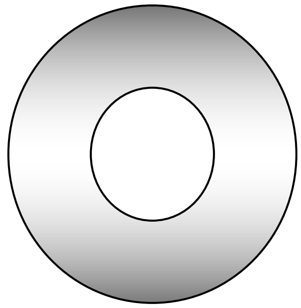


Again, think of taking a tiny strip of this region, a Riemann rectangle, and rotating it about the x – axis. Note that this gives us a cylinder (albeit a very narrow one) with another cylinder cut out of it. This is why the formula is what we stated above.

If we took all of those strips for the above example, the solid would look like this.



Notice that overall, this does not look like a “washer” per se, but if you cut a cross section, you would see washer shapes like the one illustrated below:



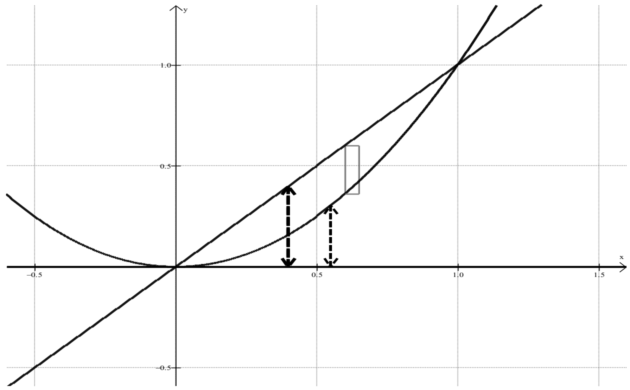
Volume by Washer Method: The volume of the solid generated when the region bounded by functions $f(x)$ and $g(x)$, from $x = a$ and $x = b$, where $f(x) \geq g(x)$, is rotated about the x – axis is given by

$$V = \pi \int_a^b \left(R^2 - r^2\right) dx$$

Where R = the distance from the axis of rotation to the further edge of the region and r = the distance from the axis of rotation to the closer edge of the region

Where R is the outer radius of your Riemann rectangle and r is the inner radius of your Riemann rectangle.

Ex 3 Let R be the region bounded by the equations $y = x^2$ and $y = x$. Find the volume of the solid generated when R is rotated about the x – axis.



$V = \pi \int_a^b (R^2 - r^2)\,dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

$V = \pi \int_a^b \left[(x)^2 - (x^2)^2\right]dx$ The expression for R_1 is x and the expression for R_2 is x^2

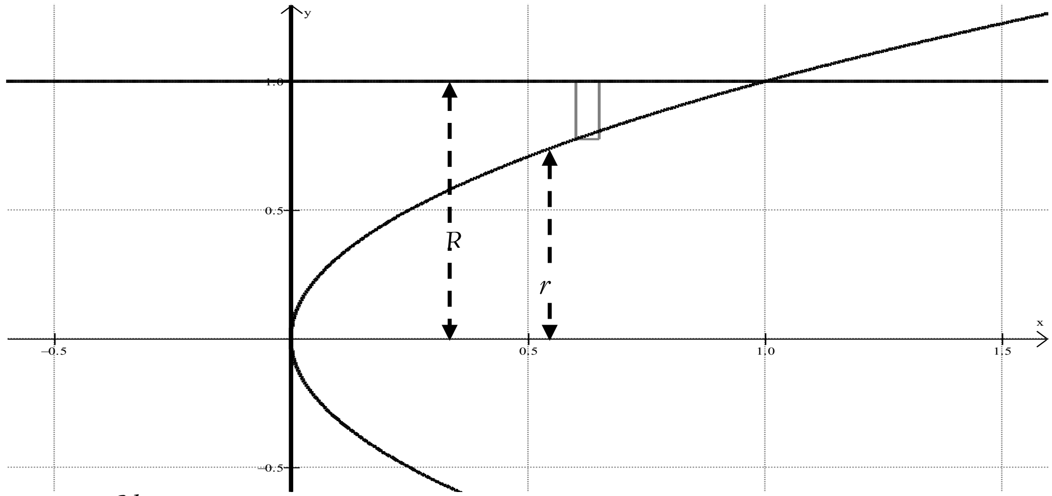
$V = \pi \int_0^1 \left[(x)^2 - (x^2)^2\right]dx$ Our region extends from $x = 0$ to $x = 1$

$V = .419$

<u>Steps to Finding the Volume of a Solid With the Washer Method:</u>

1. Draw a picture of the region to be rotated.
2. Draw the axis of rotation.
3. Sketch a Riemann rectangle. Your rectangle should always be sketched perpendicular to the axis of rotation.
4. Determine an expression representing the height of the rectangle – in the case of washers you will have an expression for the outer radius and an expression for the inner radius.
5. Determine the boundaries the region covers.
6. Set up an integral containing π outside the integrand, the limits of integration are the boundaries found in Step 5, and the integrand is the expression found in Step 4.

Ex 4 Let R be the region bounded by $x = y^2$, $y = 1$, and $x = 0$. Find the volume of the solid generated when R is rotated about the x – axis.



$V = \pi \int_a^b \left(R^2 - r^2\right) dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

$V = \pi \int_a^b \left[(1)^2 - \left(\sqrt{x}\right)^2\right] dx$ The expression for R_1 is 1and the expression for R_2 is $\sqrt{x}$

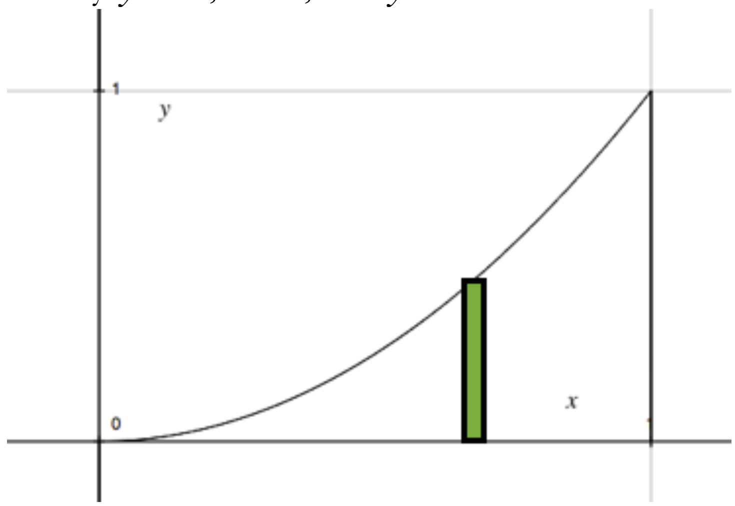
$V = \pi \int_0^1 \left[(1)^2 - \left(\sqrt{x}\right)^2\right] dx$ Our region extends from $x = 0$ to $x = 1$

$V = 1.571$

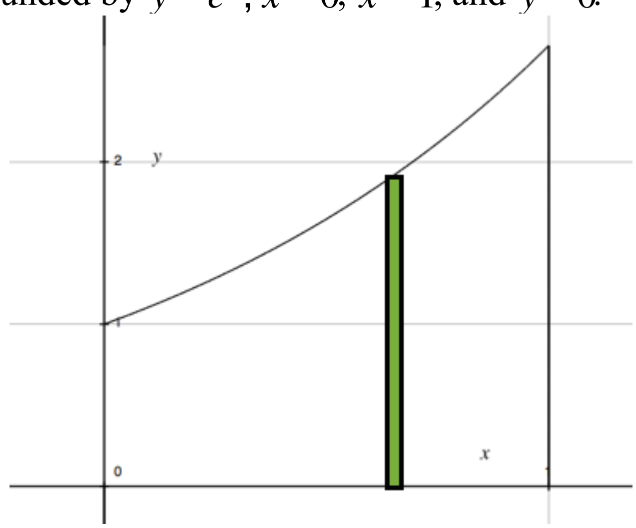
5.3 Free Response Homework

Find the volume of the solid formed by rotating the described regions about the x-axis.

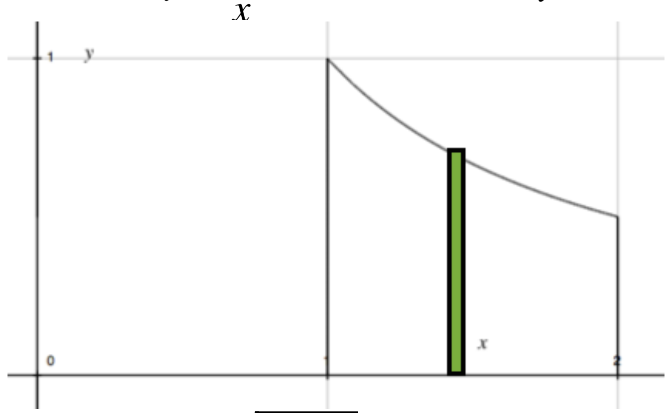
1. The region bounded by $y = x^2$, $x = 1$, and $y = 0$.



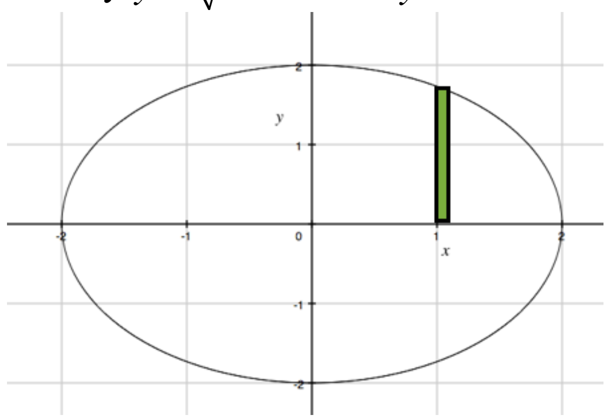
2. The region bounded by $y = e^x$, $x = 0$, $x = 1$, and $y = 0$.



3. The region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$.



4. The region bounded by $y = \sqrt{4 - x^2}$ and $y = 0$.



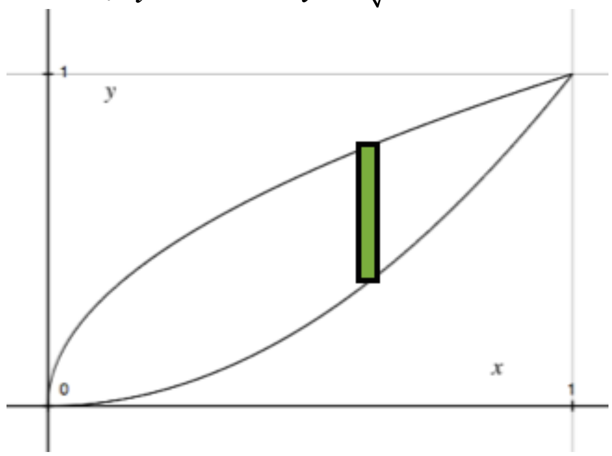
5. The region bounded by $y = \sqrt{\sin x}$ and $y = 0$ on $x \in [0, \ \pi]$.

6. The region bounded by $y = \frac{1 + 2x^2}{1 + x^2}$, $x = -3$, $x = 2$, and $y = 0$. [Solve by calculator.]

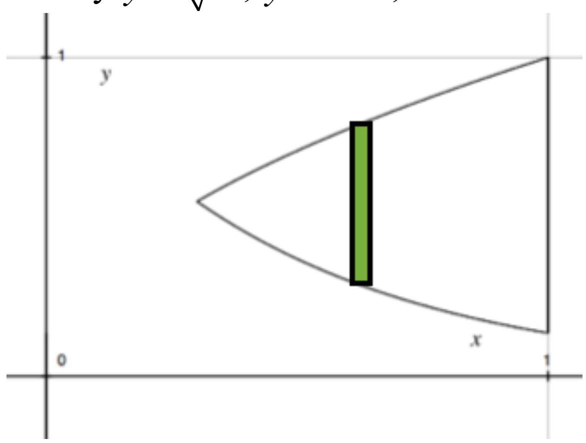
7. The region bounded by $y = \cos x$, $x = 0$, and $y = 0$.

8. The region bounded by $y = csc x$ and $y = 0$ on $x \in \left[\frac{\pi}{4}, \frac{3\pi}{4} \right]$.

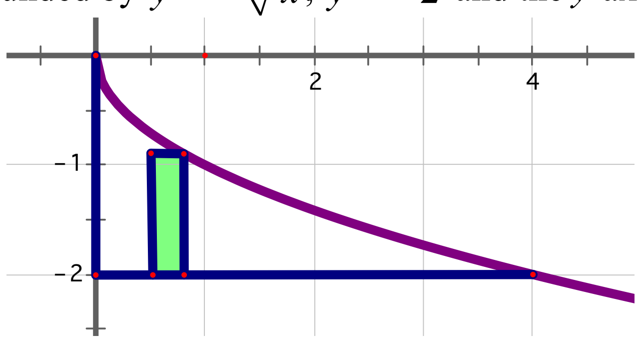
9. The region bounded by $y = x^2$ and $y = \sqrt{x}$.



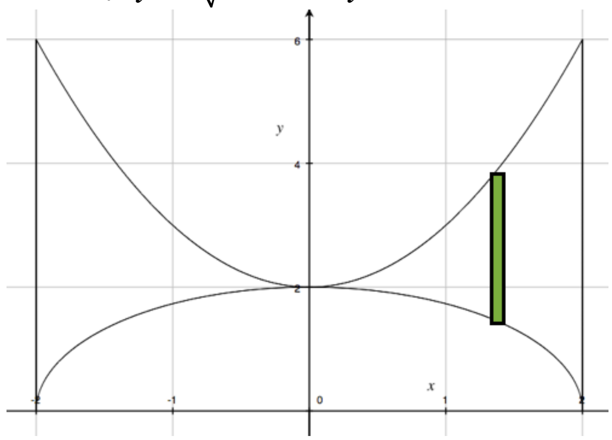
10. The region bounded by $y = \sqrt{x}$, $y = e^{-2x}$, and $x = 1$.



11. The region bounded by $y = -\sqrt{x}$, $y = -2$ and the y-axis.



12. The region bounded by $y=\sqrt{4-x^2}$, $y=2+x^2$, $x=-2$, and $x=2$.



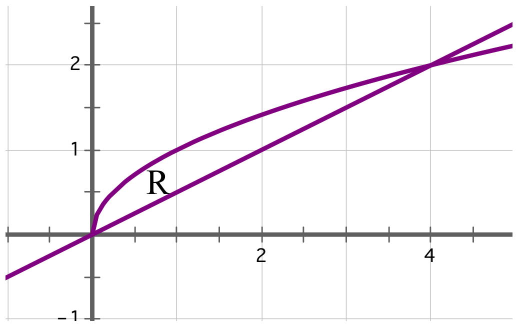
13. The region bounded by $y=x^3$, $y=8$, and the y-axis.

14. The region bounded by $y=2+\cos(\pi x)$ and $y=x^2$.

15. The two regions bounded by $y=x^2$ and $y=2^x$.

16. The region bounded by $y=\sec x$, $x=-\dfrac{\pi}{3}$, $x=\dfrac{\pi}{3}$, and $y=1$.

<u>5.3 Multiple Choice Homework</u>



1. Region R is bounded by the curves $y=\sqrt{x}$ and $y=\dfrac{1}{2}x$. The volume of the solid formed by revolving region R about the x-axis is

a) $\dfrac{8}{3}$ b) $\dfrac{8\pi}{15}$ c) $\dfrac{16\pi}{3}$ d) $\dfrac{16}{3}$ e) $\dfrac{8\pi}{3}$

2. Let R be the region in Quadrant I bounded by the graph of $y = e^{-x^2/2}$, the line $x = 3$. The volume of the solid formed by revolving R about the x-axis is

a) 0.886 b) 2.784 c) 4.027 d) 1444.5 e) 4538.2

---

3. Let S be the region enclosed by the graph of $y = \sqrt{\ln x}$, the line $x = e$, and the x-axis. The volume of the solid formed by revolving S about the x-axis is

a) 1 b) 2.257 c) 3.142 d) 3.945 e) dne

---

4. A region is bounded by $y = \dfrac{1}{\sqrt[3]{x}}$, the x-axis, the line $x = m$, and the line $x = 2m$, where $m > 0$. A solid is formed by revolving the region about the x-axis. The volume of the solid

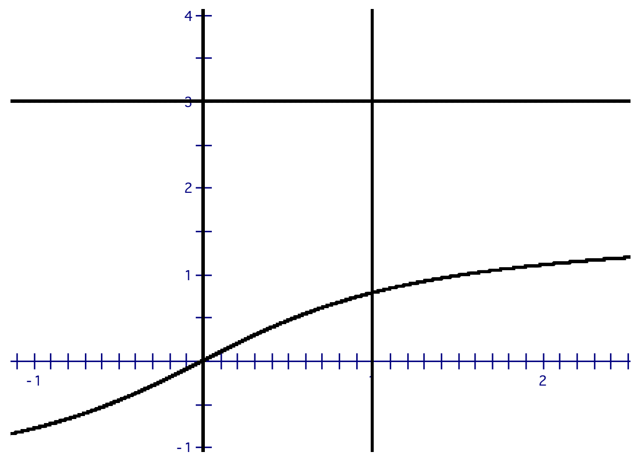
a) is independent of m.

b) increases as m increases.

c) decreases as m increases.

d) increases until $m = \dfrac{1}{2}$, then decreases.

e) is none of the above



5. A region bounded by $y=\tan^{-1}x$ $x=0$, $y=3$, and $x=1$. The volume of the solid formed by revolving the region about the *x*-axis is

a) 2.561 b). 6.612 c) 8.046 d) 8.755 e) 20.773

---

6. Let S be the region bounded by $y=2x+3$ and $y=x^2$. What is the volume of the solid formed by revolving S around the line $y=0$?

a) $\pi\int_{-1}^{3}(2x+3)^2-x^4\,dx$

b) $\pi\int_{-1}^{3}x^4-(2x+3)^2\,dx$

c) $\pi\int_{-1}^{3}\left(x^2-2x-3\right)^2dx$

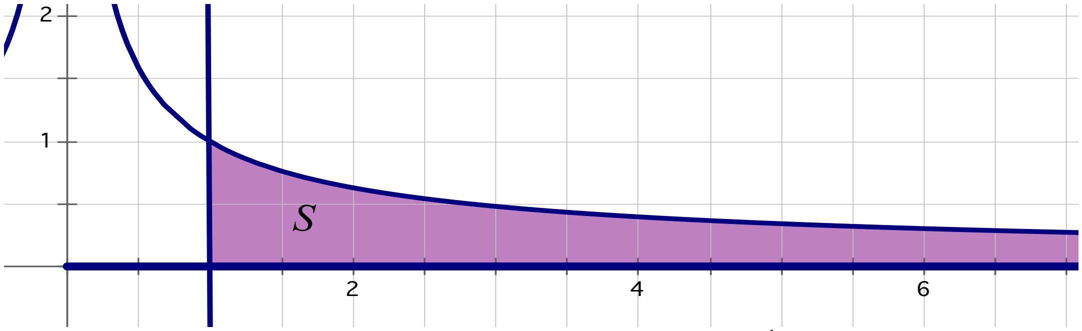
d) $\pi\int_{-1}^{9}y-\left(\frac{y-3}{2}\right)^2dy$

e) $\pi\int_{-1}^{9}\left(\frac{y-3}{2}\right)^2-y\,dy$

---

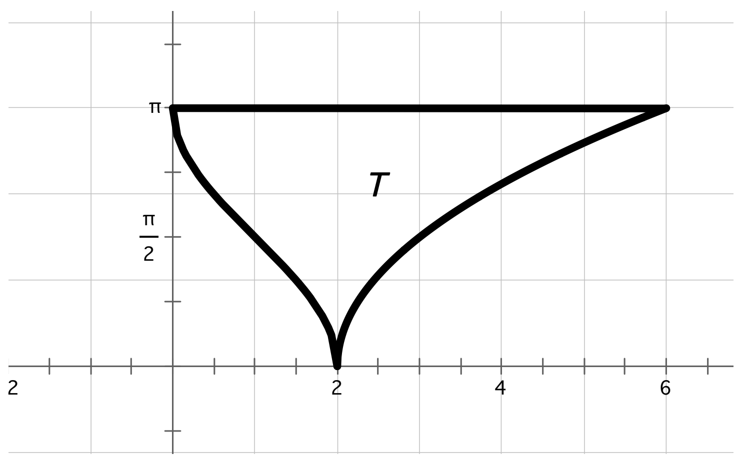
7. A region is enclosed by $x^2+4y^2=4$ in Quadrants I and II. The volume of the solid formed by revolving the region about the *x*-axis is

a) $\frac{8}{3}$ b) $\frac{8\pi}{3}$ c) $\frac{16}{3}$ d) $\frac{32}{3}$ e) $\frac{32\pi}{3}$



8. The region S (shown above) is bounded by $y = \dfrac{1}{\sqrt[3]{x^2}}$, the x-axis, and the line $x = 1$. There is not upper bound (i.e., $x \to \infty$). If the region S is revolved about the x-axis, the volume of the solid generated would be

a) 0 b) π c) 2π d) 3π e) undefined



9. Let $\boldsymbol{T}$ be the region above bounded by

$$y = \frac{\pi}{2} - \sin^{-1}(x-1), \quad y = \frac{\pi}{2}\sqrt{x-2}, \quad \text{and } y = \pi.$$

What is the volume of the solid formed by revolving $\boldsymbol{T}$ around the line $y = 0$?

a) 6.225 b) 7.330 c) 39.111

d) 105.585 e) 108.895

5.4 Volume by Rotation about the *y*-Axis

Objective:

Find the volume of a solid rotated when a region is rotated about a given line.

To find a volume of rotation about the *x*-axis, the Reiman rectangles must be horizontal, and the equations have to be in terms of *y*. If the equations of the region are in terms of *x* (that is, the *y* is isolated), the equations need to be rearranged to isolate *x*.

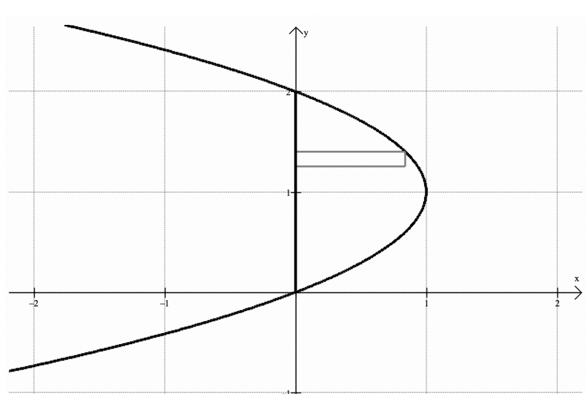
Volume by Disc Method (Part 2): The volume of the solid generated when the function $g(y)$ from $y = c$ and $y = d$, where $g(y) \geq 0$, is rotated about the y – axis is given by

$$V = \pi \int_c^d R^2 \, dy$$

where *r* is the length of your Riemann rectangle.

NB. When rotating about the *y*-axis, the formula will be in terms of *y* and *dy*.

Ex 1 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the y – axis.



$V = \pi \int_c^d r^2 dy$ Our pieces are perpendicular to the y – axis, so our integrand will contain dy

$V = \pi \int_c^d x^2 dy$ We cannot integrate x with respect to y so we will sub out for x

$V = \pi \int_c^d \left[2y - y^2\right]^2 dy$ The expression for x is $2y - y^2$

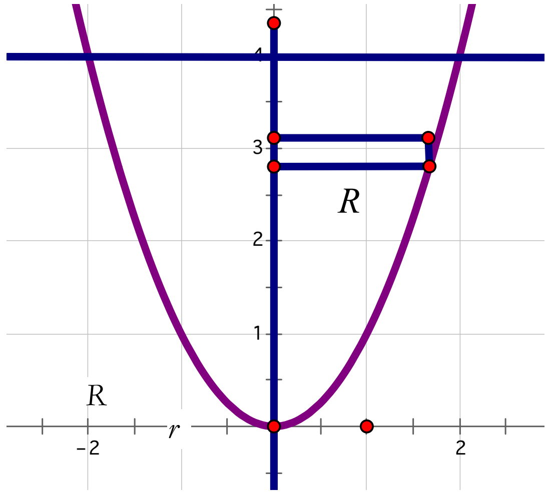
$V = \pi \int_0^2 \left[2y - y^2\right]^2 dy$ Our region extends from $y = 0$ to $y = 2$

$V = \dfrac{16\pi}{15}$

<u>Steps to Finding the Volume of a Solid of rotation about the *y*-axis:</u>

8. Draw a picture of the region to be rotated.
9. Draw the axis of rotation.
10. Sketch a Reimann rectangle – if the piece is vertical your integral will have a *dx* in it, if your piece is horizontal your integral will have a *dy* in it. Your rectangle should always be sketched perpendicular to the axis of rotation.
11. Determine an expression representing the radius of the rectangle (in these cases, the radius is the function value).
12. Determine the endpoints the region covers.
13. Set up an integral containing π outside the integrand, the limits of integration are the numbers found in Step 5, and the integrand is the expression found in Step 4.

As noted in step 4, sometimes the given equations need to be rearranged. That is, if rotating about the *y*-axis, the equations must be in terms of *y* (*x* must be isolated).Ex 2 Let *R* be the region bounded by $y = x^2$, $y = 4$, and $x = 0$. Find the volume of the solid generated when *R* is rotated about the y – axis.



Note that the equations have y isolated, but we are rotating about the *y*-axis, which requires that *x* be isolated.

$$V = \pi \int_c^d \left(R^2\right) dy$$ Our pieces are perpendicular to the $y-$axis, so our integrand will contain dy

$$V = \pi \int_c^d \left[\left(\sqrt{y}\right)^2\right] dy$$ The expression for R is the expression $x = \sqrt{y}$

$$V = \pi \int_0^4 y\,dy$$ Our region extends from $y = 0$ to $y = 4$

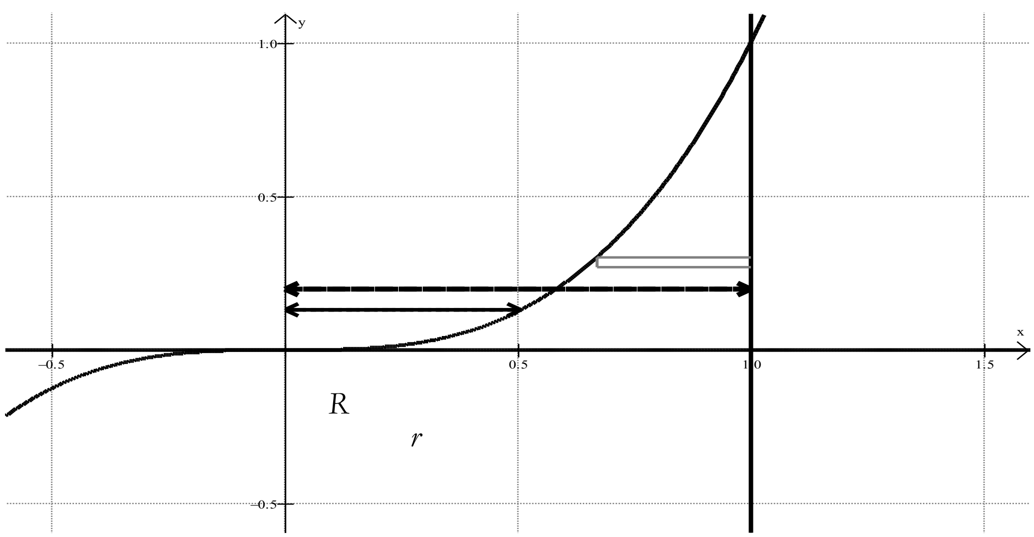
$$V = \pi \left[\frac{y}{2}\right]_0^4 = 8\pi$$

As with rotating about the *x*-axis, if the Reiman Rectangles are perpendicular and unattached to the axis of rotation, we need to use the Washer Method.

Volume by Washer Method: The volume of the solid generated when the region bounded by functions $f(x)$ and $g(x)$, from $x = a$ and $x = b$, where $f(x) \geq g(x)$ [or $f(y)$ and $g(y)$, from $y = c$ and $y = d$, where $f(y) \geq g(y)$], is rotated about the x – axis is given by

$$V = \pi \int_a^b \left(R^2 - r^2\right) dx \quad \text{or} \quad V = \pi \int_c^d \left(R^2 - r^2\right) dy$$

Ex 3 Let R be the region bounded by $y = x^3$, $x = 1$, and $y = 0$. Find the volume of the solid generated when R is rotated about the y – axis.



$V = \pi \int_c^d \left(R^2 - r^2\right) dy$ Our pieces are perpendicular to the y – axis, so our integrand will contain dy

$V = \pi \int_c^d \left[(1)^2 - \left(\sqrt[3]{y}\right)^2\right] dy$ The expression for R_1 is 1and the expression for R_2 is $\sqrt[3]{y}$

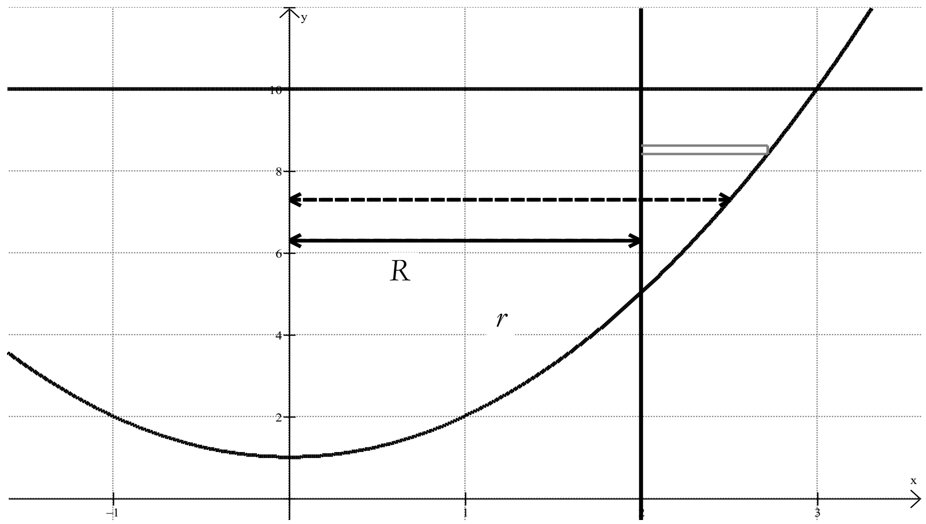
$V = \pi \int_0^1 \left[(1)^2 - \left(\sqrt[3]{y}\right)^2\right] dy$ Our region extends from $y = 0$ to $y = 1$

$V = 1.257$

Steps to Finding the Volume of a Solid With the Washer Method:

7. Draw a picture of the region to be rotated.
8. Draw the axis of rotation.
9. Sketch a Riemann rectangle – if the piece is vertical your integral will have a *dx* in it, if your piece is horizontal your integral will have a *dy* in it. Your rectangle should always be sketched perpendicular to the axis of rotation.
10. Determine an expression representing the radius of the rectangle – in the case of washers you will have an expression for the outer radius and an expression for the inner radius.
11. Determine the boundaries the region covers.
12. Set up an integral containing π outside the integrand, the limits of integration are the boundaries found in Step 5, and the integrand is the expression found in Step 4.

Ex 4 Let R be the region bounded by $y = 1 + x^2$, $x = 2$, and $y = 10$. Find the volume of the solid generated when R is rotated about the $y-$axis.



$$V = \pi \int_c^d \left(R^2 - r^2\right) dy$$

$$= \pi \int_5^{10} \left[\left(\sqrt{y-1}\right)^2 - (2)^2\right] dy$$

$$= 39.270$$

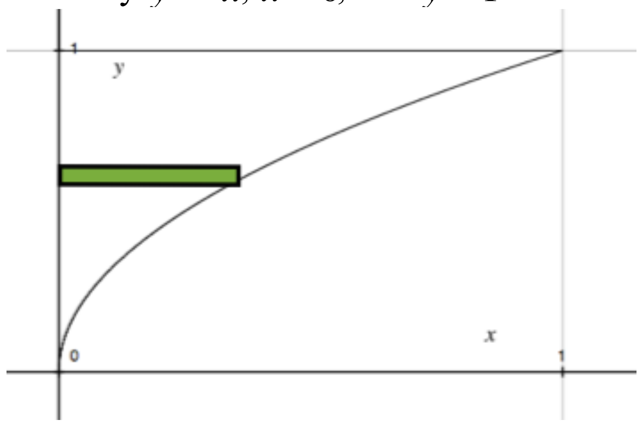
Summary

	Rectangles Vertical (about the x-axis)	Rectangles Horizontal (about the y-axis)
Rectangles perpendicular and attached (Disk Method)	$V = \pi \int_a^b [f(x)]^2 dx$	$V = \pi \int_c^d [g(y)]^2 dy$
Rectangles perpendicular and not attached (Washer Method)	$V = \pi \int_a^b \left(R^2 - r^2\right) dx$	$V = \pi \int_c^d \left(R^2 - r^2\right) dy$

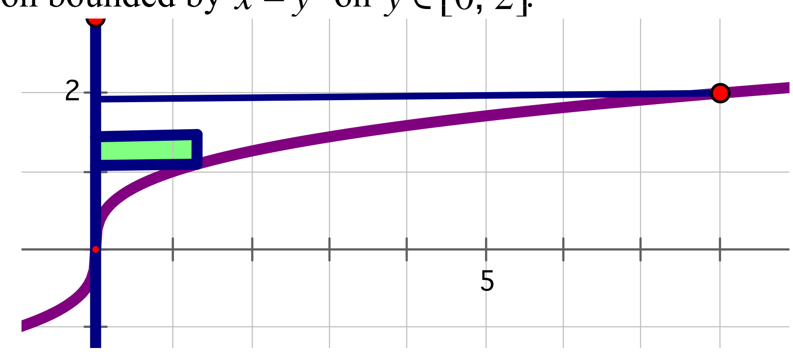
5.4 Free Response Homework

Find the volume of the solid formed by rotating the described regions about the *y*-axis.

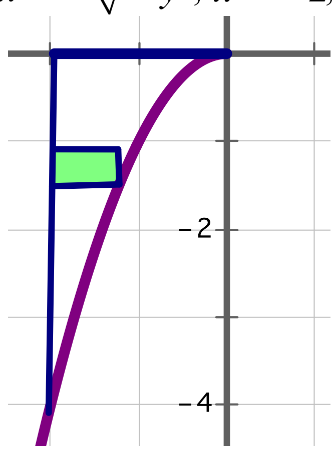
1.(12) The region bounded by $y^2 = x$, $x = 0$, and $y = 1$.



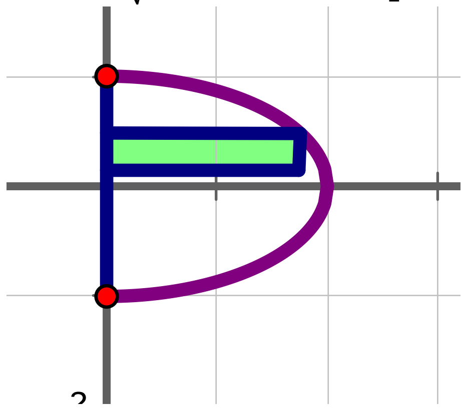
2. The region bounded by $x = y^3$ on $y \in [0, 2]$.



3. The region bounded by $x = -\sqrt{-y}$, $x = -2$, and $y = 0$.



4. The region bounded by $x = \sqrt{4 - 4y^2}$ on $y \in [-1, 1]$ in Quadrants I and IV.



5. The region bounded by $x = e^y - 1$, $x = 0$, and $y = \ln 3$.

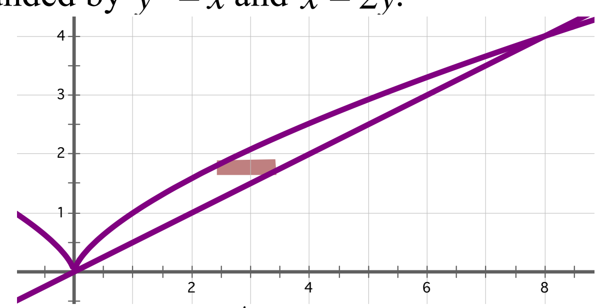
6. The region bounded by $y = x^{2/3}$ in Quadrant I on $y \in [0, 4]$.

7. The region bounded by $x^2 + 4y^2 = 36$ and the y-axis on $y \in [-3, 3]$ in Quadrants I and IV.

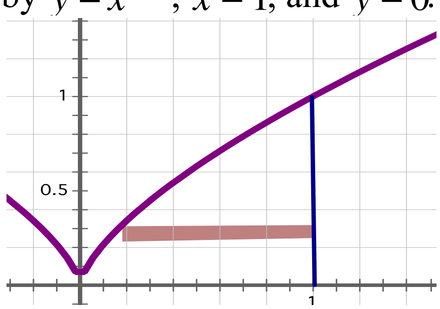
8. The region bounded by $y = 2\sin x$ and the y-axis on $y \in \left[0, \frac{\pi}{2}\right]$.

[Solve by calculator.]

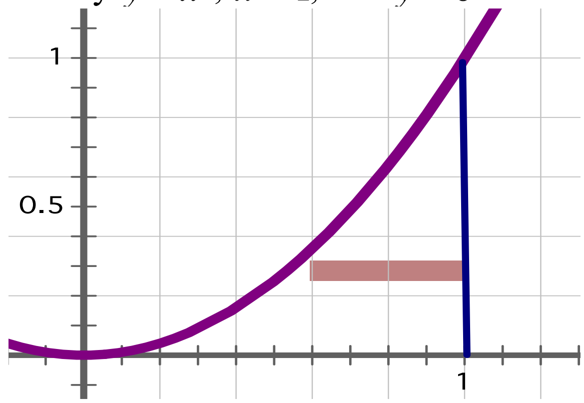
9. The region bounded by $y^2 = x$ and $x = 2y$.



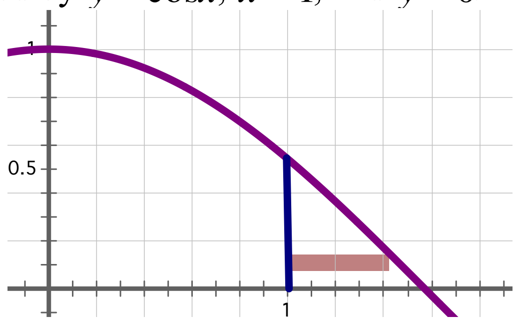
10. The region bounded by $y = x^{2/3}$, $x = 1$, and $y = 0$.



11. The region bounded by $y = x^2$, $x = 1$, and $y = 0$.



12. The region bounded by $y = \cos x$, $x = 1$, and $y = 0$.



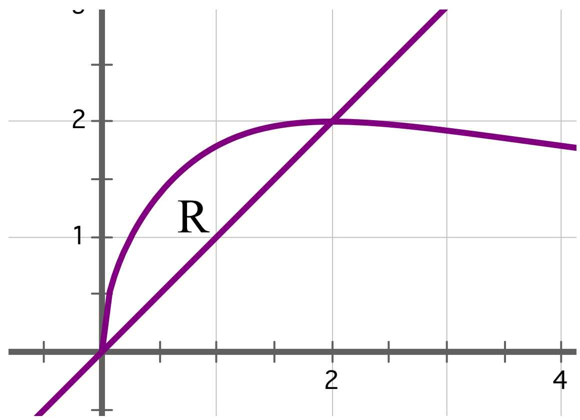
13. The region bounded by $y = x^2$ and $x = y^2$.

14. Let R be the region in Quadrant I bounded by the graphs of $x = \frac{1}{2}y^3$ and $x = 4-(y-2)^2$. The curves intersect at $(0,0)$ and $(4,2)$.

15. Let R be the region bounded by $y = x^3$, $x = 2$, and $y = 0$.

16. Let R be the region bounded by $y = \sqrt{\frac{4}{3}x}$ and $y = \frac{2}{9}x^2$.

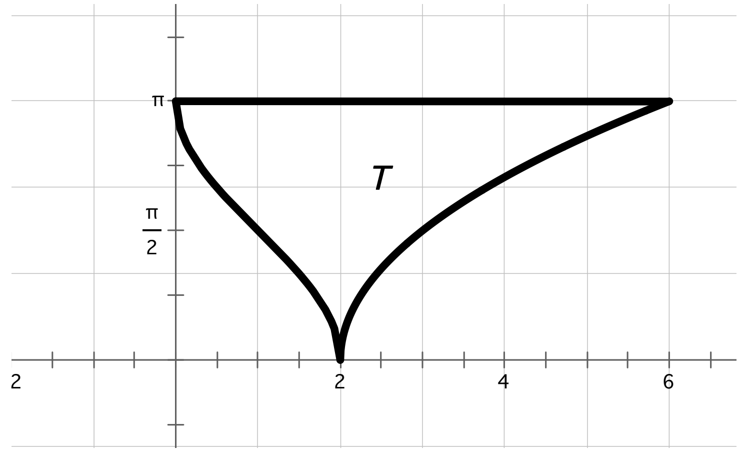
5.4 Multiple Choice Homework



1. Region R is bounded by the curves $y=\sqrt{\dfrac{16x}{x^2+4}}$ and $y=x$. The volume of the solid formed by revolving region R about the y-axis is

a) $8\ln 2-\dfrac{8}{3}$ b) $16\ln 2-\dfrac{8}{3}$ c) $16\pi\ln 2-\dfrac{8}{3}$

d) $16\pi\ln 2-\dfrac{8\pi}{3}$ e) $8\pi\ln 2-\dfrac{8\pi}{3}$



2. Let ***T*** be the region above bounded by

$$y = \frac{\pi}{2} - \sin^{-1}(x-1),\ y = \frac{\pi}{2}\sqrt{x-2},\ \text{and } y = \pi$$

What is the volume of the solid formed by revolving ***T*** around the line $x = 0$?

a) 6.225 b) 7.330 c) 39.111

d) 105.585 e) 108.895

---

3. The region enclosed by the graphs of $y = x^3 - 1$ and $y = x - 1$ is revolved about the y-axis. The volume of this solid is

(a) 0.360 (b) 0.972 (c) 1.944 (d) 3.032 (e) 6.462

---

4. The region enclosed by the graphs of $y = 3e^x$ and $y = 3$ on $0 \le x \le 1$ is revolved about the y-axis. The volume of this solid is

(A) $\pi \int_3^{3e} \left(\ln \frac{y}{3} \right)^2 dy$

(B) $\pi \int_0^1 (3e^x - 3)^2 dx$

(C) $\pi \int_3^{3e} \left(1 - \ln^2 \frac{y}{3} \right) dy$

(D) $\pi \int_0^3 \left(1 - \ln \frac{y}{3} \right)^2 dy$

(E) $\pi \int_3^{3e} \left(1 - \ln \frac{y}{3} \right)^2 dy$

---

5. Let R be the region in the first quadrant bounded by $y = \sin^{-1} x$, the y-axis, and $y = \frac{\pi}{2}$. Which of the following integrals gives the volume of the solid generated when R is rotated about the y-axis?

(a) $\pi \int_0^{\pi/2} y^2 dy$

(b) $\pi \int_0^1 \left[\left(\frac{\pi}{2} \right)^2 - (\sin^{-1} x)^2 \right] dx$

(c) $\pi \int_0^{\pi/2} (\sin^{-1} x)^2 dx$

(d) $\pi \int_0^{\pi/2} (\sin y)^2 dy$

(e) $\pi \int_0^1 (\sin y)^2 dy$

---

6. The region in Quadrant IV enclosed by the graphs of $y = x^3 - 1$ and $y = x - 1$ is revolved about the x-axis. The volume of this solid is

(A) 0,267 (B) 0.838 (C) 0.190 (D) 3.142 (E) 6.462

---

7. The region in Quadrant I is bounded by the graphs of $y = 2e^x$ and $y = 4$ is revolved about the *x*-axis. The volume of this solid is

(a) $\pi \int_0^{\ln 2} (4 - 2e^x)^2 dx$

(b) $\pi \int_2^4 \left(4 - \ln\left(\frac{y}{2}\right)\right)^2 dy$

(c) $\pi \int_2^{\ln 2} \left[1 - \ln^2\left(\frac{y}{2}\right)\right] dy$

(d) $\pi \int_2^4 \left[\ln^2\left(\frac{y}{2}\right)\right] dy$

(e) $\pi \int_0^{\ln 2} \left(16 - 4e^{2x}\right) dx$

---

8. The region in Quadrant I is bounded by the graphs of $y = 2e^x$ and $y = 4$ is revolved about the *y*-axis. The volume of this solid is

(a) $\pi \int_0^{\ln 2} (4 - 2e^x)^2 dx$

(b) $\pi \int_2^4 \left(4 - \ln\left(\frac{y}{2}\right)\right)^2 dy$

(c) $\pi \int_2^{\ln 2} \left[1 - \ln^2\left(\frac{y}{2}\right)\right] dy$

(d) $\pi \int_2^4 \left[\ln^2\left(\frac{y}{2}\right)\right] dy$

(e) $\pi \int_0^{\ln 2} \left(16 - 4e^{2x}\right) dx$

5.5 Volume by Rotation about a Line not an Axis

In the last section we learned how to find the volume of a non-regular solid. This section will be more of the same, but with a bit of a twist. Instead of rotating about an axis, let the region be revolved about a line not the origin.

Volume by Disc Method (Form 2): The volume of the solid generated when the function $f(x)$ from $x = a$ and $x = b$, where $f(x) \geq 0$, is rotated about the line $y = k$ [or $g(y)$ from $y = c$ and $y = d$, where $g(y) \geq 0$, is rotated about the line $x = h$ is given by

$$V = \pi \int_a^b [f(x) - k]^2 dx \quad \text{or} \quad V = \pi \int_c^d [g(y) - h]^2 dy$$

or

$$V = \pi \int_a^b R^2 dx$$

where R is the Length of your Riemann rectangle from the curve to the line of rotation.

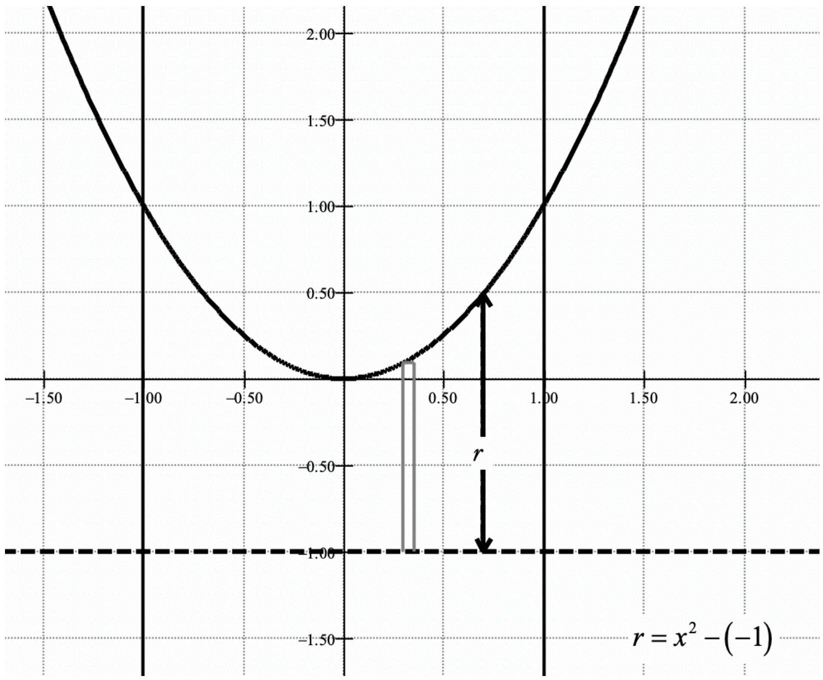
Volume by Washer Method (Part 2): The volume of the solid generated when the region bounded by functions $f(x)$ and $g(x)$, from $x = a$ and $x = b$, where $f(x) \geq g(x)$ [or $f(y)$ and $g(y)$, from $y = c$ and $y = d$, where $f(y) \geq g(y)$], is rotated about the line $y = k$ is given by

$$V = \pi \int_c^d \left(R^2 - r^2\right) dy$$

where R is the distance from the line of rotation to the further edge of the region and r is the distance from the line of rotation to the closer edge of the region.

NB. The Disk Method and Washer Method do not apply of the line of rotation is within the region being rotated.

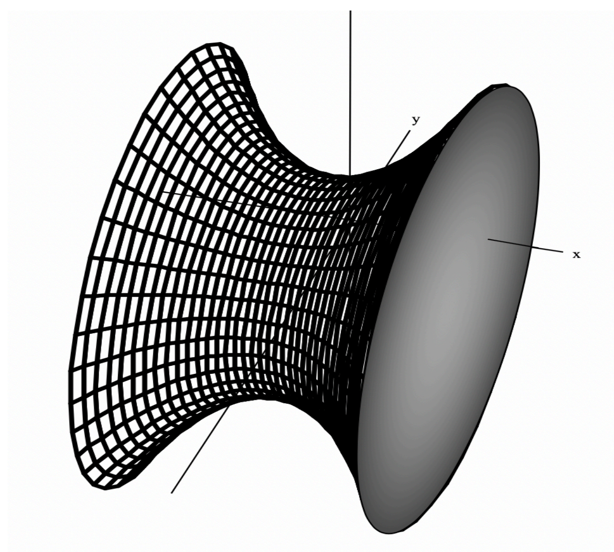
Ex 1 Let R be the region bounded by the equations $y = x^2$, $y = -1$, $x = -1$, and $x = 1$. Find the volume of the solid generated when R is rotated about the line $y = -1$.



$$V = \pi \int_a^b r^2 dx$$

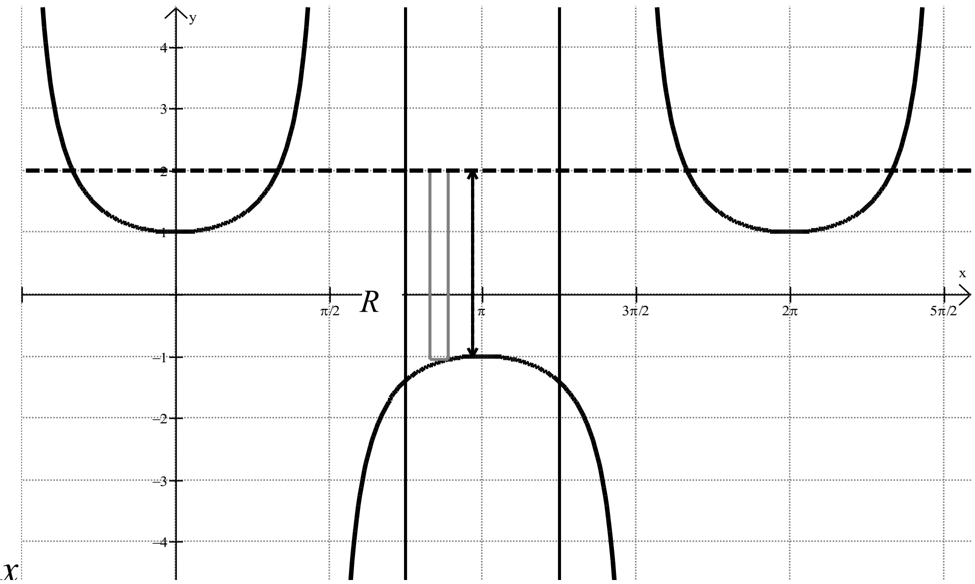
$$= \pi \int_2^7 [x^2 + 1]^2 dx$$

$$= 3583.333$$



As in the previous section, this makes an interesting shape, but is of very little use to us in actually solving for the volume. You could easily have found the value for the volume never knowing what the shape actually looked like.

Ex 2 Let R be the region bounded by the curves $y = \sec x,\ y = 2,\ x = \frac{3\pi}{4}$, and $x = \frac{5\pi}{4}$. Find the volume of the solid generated when R is rotated about the line $y = 2$.



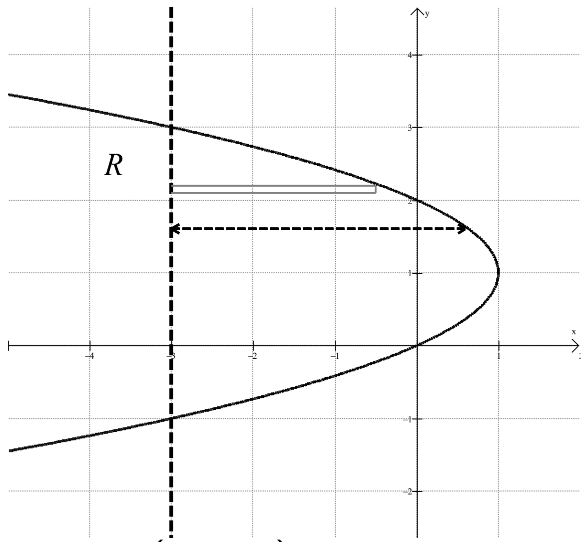
$R = 2 - \sec x$

$$V = \pi \int_a^b [2 - y]^2 dx$$

$$= \pi \int_{3\pi/4}^{5\pi/4} [2 - \sec x]^2 dx$$

$$= 48.174$$

Ex 3 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the line $x = -3$ about the line $x = -3$.



$$R = (2y - y^2) - (-3)$$

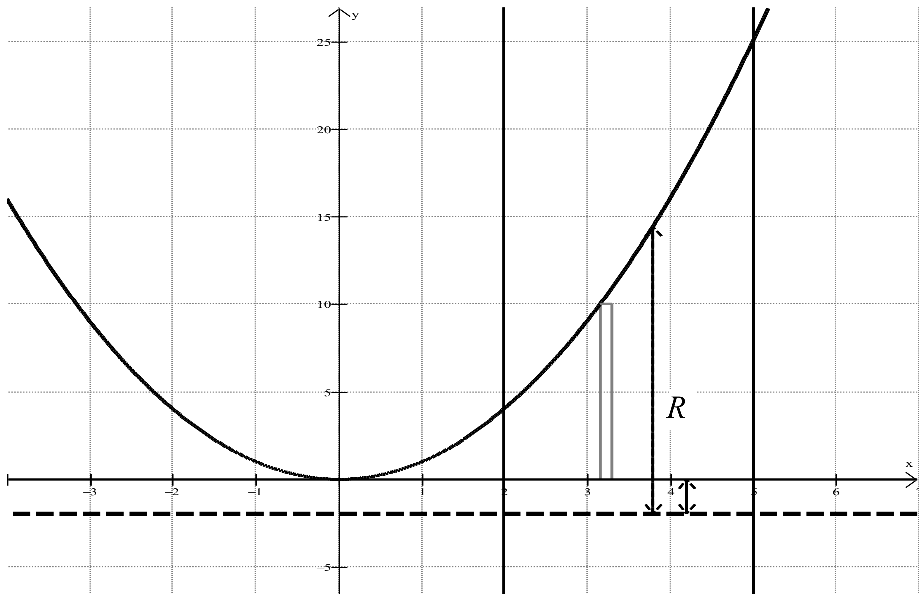
$$V = \pi \int_c^d [3 + x]^2 dy$$

$$= \pi \int_{-1}^{3} [3 + 2y - y^2]^2 dy$$

$$= 961.746$$

What if you were asked to take the region bounded by $y = x$ and $y = x^2$ and rotate it about the x – axis? How is this different from the previous problems?

Ex 4 Let R be the region bounded by the equations $y = x^2$, the x – axis, $x = 2$, and $x = 5$. Find the volume of the solid generated when R is rotated about the line $y = -2$.



$$V = \pi \int_a^b \left(R^2 - r^2\right) dx$$

$$= \pi \int_2^5 \left[\left(x^2 + 2\right)^2 - (0 - (-2))^2\right] dx$$

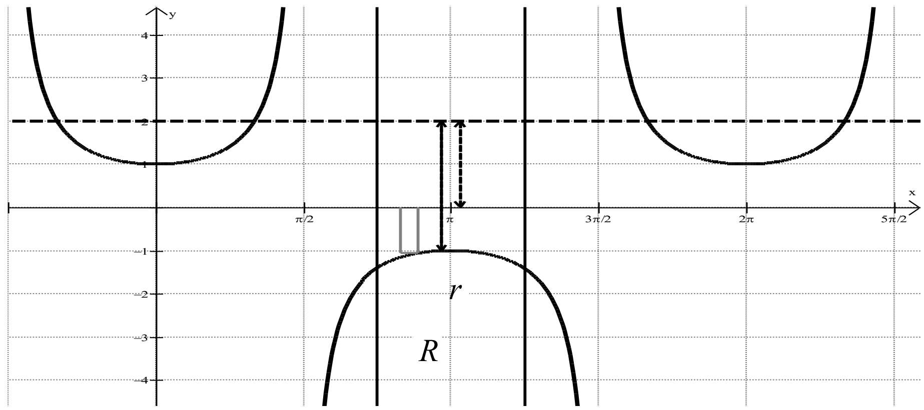
$$= 2433.478$$

Ex 5 Let R be the region bounded by the curves $y = \sec x$, the x – axis, $x = \frac{3\pi}{4}$, and $x = \frac{5\pi}{4}$. Find the volume of the solid generated when R is rotated about the line $y = 2$.

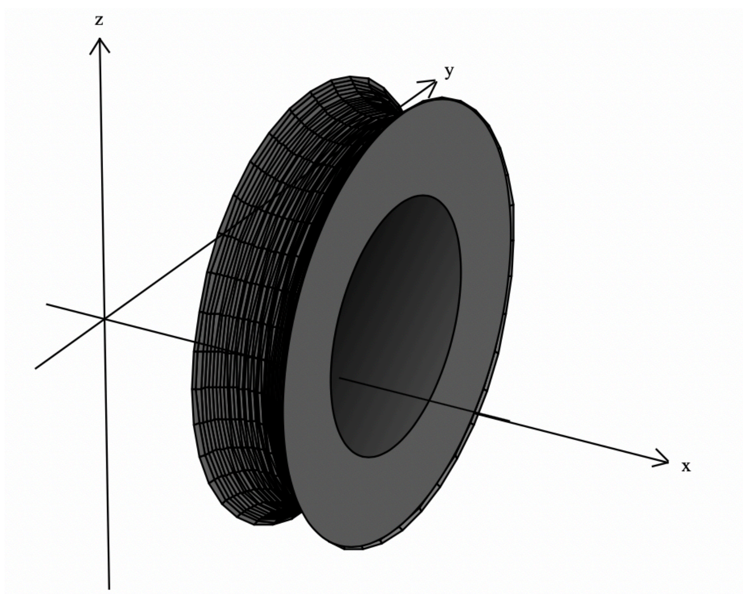
$$V = \pi \int_a^b \left(R^2 - r^2\right) dx$$

$$= \pi \int_{3\pi/4}^{5\pi/4} \left[(2 - \sec x)^2 - (2)^2\right] dx$$

$$= 28.435$$



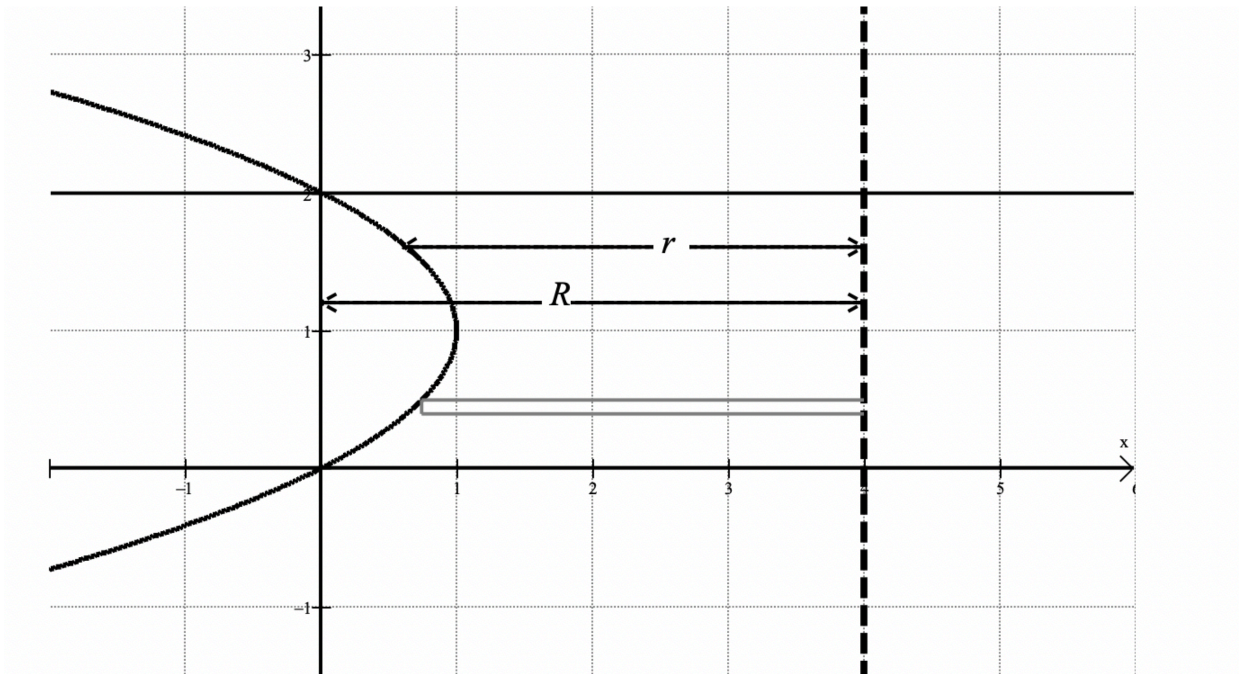
The rotated figure looks like this:



It would be difficult to sketch this by hand, and we definitely do not need it to solve the problem.

Ex 6 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the y – axis about line $x = 4$.

Given the information above and the sketch of $x = 2y - y^2$ below, sketch your boundaries, the axis of rotation, your Riemann rectangle, and your inner and outer radii.



Set up the integration:

$$V = \pi \int_c^d \left(R^2 - r^2\right) dy$$

$$= \pi \int_0^2 \left[(4)^2 - \left(4 - 2y + y^2\right)^2\right] dy$$

$$= 30.159$$

5.5 Free Response Homework

Find the volume of the solid formed by rotating the described region about the given line. Sketch the graph determine whether the Disk Method or the Washer Method applies.

1. The region bounded by $y = \frac{1}{2}x^2$ and the line $y = 8$, revolved about the line $y = -1$.

2. The region bounded by $y = \frac{1}{2}x^2$ and the line $y = 8$, revolved about the line $y = 8$.

3. The region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 3$, and the line $y = 0$, revolved about the line $y = -1$.

4. The region bounded by $y = 2 + \sin x$, $x = 0$, $x = \pi$, and the line $y = 2$, revolved about the line $y = 3$.

5. The region bounded by $y = \sqrt{x}$ and $y = x^3$, revolved about the line $y = 1$.

6. The region bounded by $y = \sqrt{x}$ and $y = x^3$, revolved about the line $x = 1$.

7. The region bounded by the x-axis, $y = \sqrt{x+1}$, and the line $x = 3$, revolved about the line $x = 3$.

8. The region bounded by $y^2 = x$ and the line $x = 1$, revolved about the line $x = 1$.

9. The region bounded by $y^2 = x$ and the line $x = 1$, revolved about the line $x = 3$.

10. The region bounded by $y = \pm\sqrt{x}$ and the line $x = 1$, revolved about the line $x = -3$.

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid formed by rotating the described region about the given line.

11. The region bounded by $y = \ln\left(x^2 + 1\right)$ and $y = \cos x$, revolved about the line $y = 1$.

12. The region bounded by $f(x) = \sqrt{6x - x^2}$ and $g(x) = -1 + e^{-0.5x}$, revolved about the line $y = 3$.

13. The region bounded by $f(x) = \sqrt{6x - x^2}$ and $g(x) = -1 + e^{-0.5x}$, revolved about the line $x = -1$.

14. The region is bounded by $y = 3 - \frac{1}{2}x^2$ and $y = \frac{1}{x}$, revolved about the line $x = 3$.

<u>5.5 Multiple Choice Homework</u>

1. Let *S* be the region enclosed by the graphs of $y = 2x$ and $y = 2x^2$ for $0 \le x \le 1$. What is the volume of the solid generated when *S* is revolved about the line $y = 3$?

a) $\pi \int_0^1 \left((3 - 2x^2)^2 - (3 - 2x)^2\right) dx$

b) $\pi \int_0^1 \left((3 - 2x)^2 - (3 - 2x^2)^2\right) dx$

c) $\pi \int_0^1 \left(4x^4 - 4x^2\right) dx$

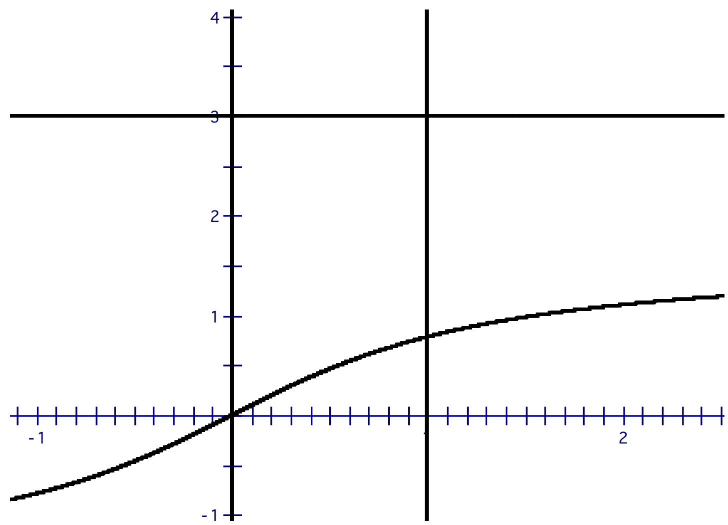
d) $\pi \int_0^2 \left(\left(3 - \frac{y}{2}\right)^2 - \left(3 - \sqrt{\frac{y}{2}}\right)^2\right) dx$

e) $\pi \int_0^2 \left(\left(3 - \sqrt{\frac{y}{2}}\right)^2 - \left(3 - \frac{y}{2}\right)^2\right) dx$

2. Let R be the region in the first quadrant that is enclosed by the graph of $f(x) = \ln(x+1)$, the x-axis, and the line $x = e$. What is the volume of the solid generated when R is rotated about the line $y = -1$?

a) 5.037 b) 6.545 c) 10.073

d) 20.146 e) 28.686



3. The region above is bounded by $y = \tan^{-1} x$ $x = 0,$ $y = 3,$ and What is the volume of the solid generated when S is revolved about the line $y = 3$?

a) 2.561 b) 6.612 c) 8.046 d) 8.755 e) 20.773

---

4. The region $\boldsymbol{R}$ is bounded by the lines $y = 2x - 4$, $x = 3$, and $y = 0$. Which of these expressions gives the volume of the solid formed by revolving $\boldsymbol{R}$ around the line $x = 5$?

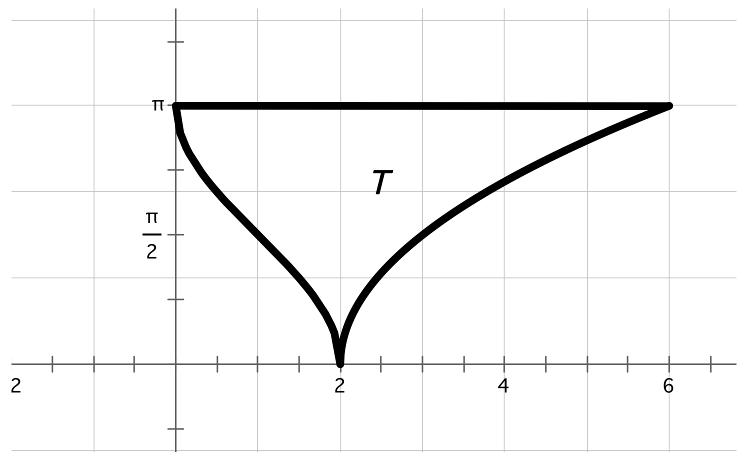
a) $\int_2^3 \left((2x-4)^2 - 3^2\right) dx$

b) $\int_2^3 \left((2x-9)^2 - 2^2\right) dx$

c) $\int_0^2 \left(\left(\frac{y+4}{2}\right)^2 - 3^2\right) dy$

d) $\int_0^2 \left(\left(\frac{y-6}{2}\right)^2 - 2^2\right) dy$

e) $\int_0^6 \left(\left(\frac{y+4}{2}\right)^2 - 3^2\right) dy$



5. Let $\boldsymbol{T}$ be the region above bounded by

$$y = \frac{\pi}{2} - \sin^{-1}(x-1), \quad y = \frac{\pi}{2}\sqrt{x-2}, \quad \text{and } y = \pi$$

What is the volume of the solid formed by revolving $\boldsymbol{T}$ around the line $y = \pi$?

a) 6.225 b) 7.330 c) 39.111

d) 105.585 e) 108.895

5.6 Volume by Cross Sections

In the last sections, we have learned how to find the volume of a solid created when a region was rotated about a line. In this section, we will find volumes of solids that are not made though revolutions, but cross sections.

Volume by Cross Sections Formula:

The volume of solid made of cross sections with area A is

$$V = \int_a^b A(x)\,dx \text{ or } V = \int_c^d A(y)\,dy$$

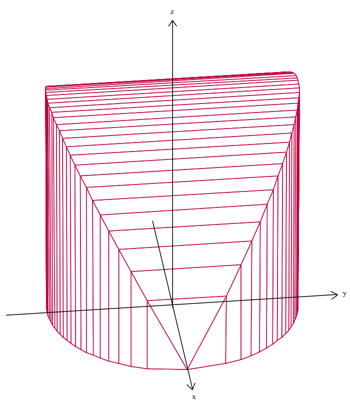
Here are the most common cross-sections and their areas:

Cross-section	**Area**		**Cross-section**	**Area**
Square	$A = s^2$		Triangle	$A = \frac{1}{2}bh$
Rectangle	$A = bh$		Equilateral Triange	$A = \frac{\sqrt{3}}{4}b^2$
Circle	$A = \pi r^2$		Isosceles Right Triangle	$A = \frac{1}{2}s^2$
Semi-Circle	$A = \frac{1}{2}\pi r^2$		Trapezoid	$A = \frac{1}{2}(b_1 + b_2)h$

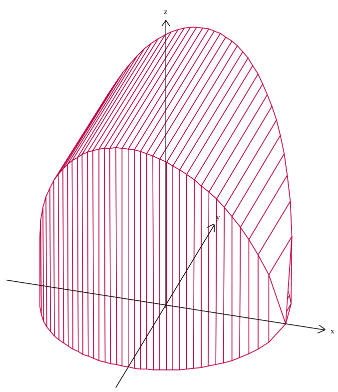
Objectives:

Find the volume of a solid with given cross sections.

Ex 1   The base of solid S is $x^2 + y^2 = 1$. Cross sections perpendicular to the x – axis are squares. What is the volume of the solid?



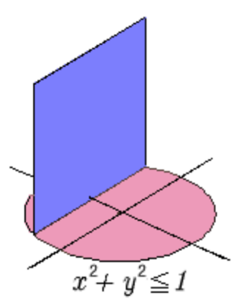
You might view of the solid generated with the given cross-sections from the perspective of the x-axis coming out of the page.

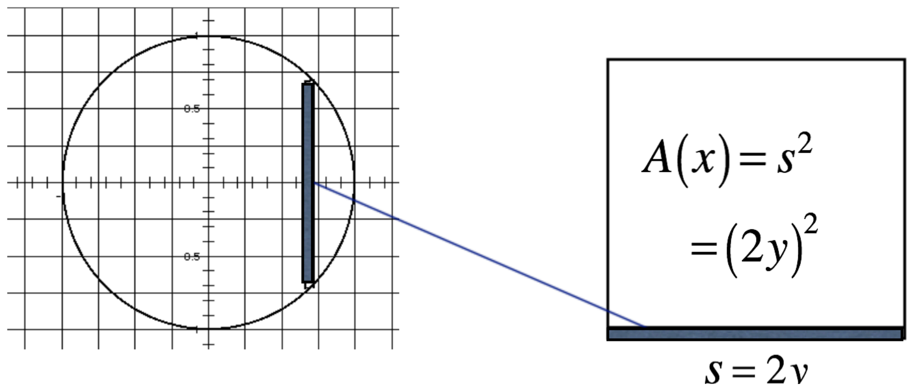


You might view of the solid generated with the given cross-sections from the perspective of the x-axis coming out of the page.

But you are actually better off not trying to view the 3D shape at all.

Ex 1   The base of solid S is $x^2 + y^2 = 1$. Cross-sections perpendicular to the x – axis are squares. Find the volume of the solid.





$$V = \int_a^b A(x)\,dx$$ Start here

$$V = \int_a^b (\text{side})^2\,dx$$ Area formula for our (square) cross section

$$V = \int_{-1}^1 (\text{side})^2\,dx$$ Endpoints of the region of our solid

$$V = 2\int_0^1 (\text{side})^2\,dx$$ Simplify the integral

$$V = 2\int_0^1 (2y)^2\,dx$$ Length of the side of our cross section

$$V = 2\int_0^1 4y^2\,dx$$ We cannot integrate y with respect to x so we will sub out for y

$$V = 2\int_0^1 4\left(1-x^2\right)dx$$

$$V = 5.333$$

Steps to Finding the Volume of a Solid by Cross Sections:

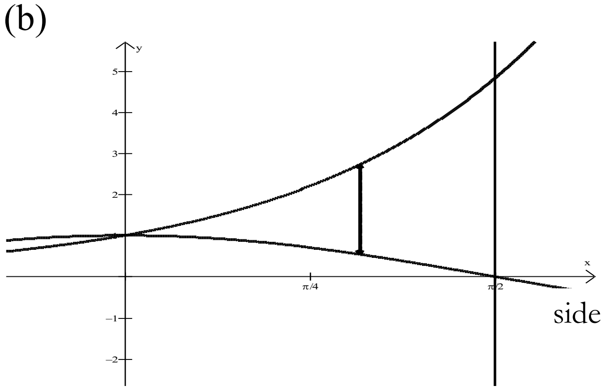
1. Draw a picture of the region of the base.
2. Draw a representative Riemann Rectangle and determine the general length, based on the equations.
   a. Be sure the rectangle goes in the desired direction.
   b. Rearrange the equations if necessary.
3. Sketch a sample cross section separate from the base region.
4. Determine an expression representing the area of the sample cross section
   a. Be careful when using semi-circles – the radius is **half** the distance of the base).
5. Substitute the expression for the Riemann Rectangle into the cross-section area formula.
6. Determine the endpoints of the base region.
7. Set up an integral expression for the volume.
8. Solve the integral

Ex 2 Let R be the region in the first quadrant bounded by $y = \cos x$, $y = e^x$, and $x = \dfrac{\pi}{2}$.

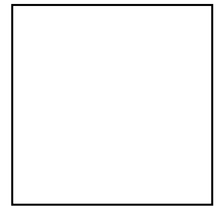
(a) Find the area of region R.
(b) The region R is the base of s solid. For this solid, each cross section perpendicular to the x – axis is a square. Find the volume of the solid.

(a) $A = \int_0^{\pi/2} \left(e^x - \cos x\right) dx = 2.810$

(b)



$e^x - \cos x \qquad V = \int_0^{\pi/2} (\text{side})^2 dx$



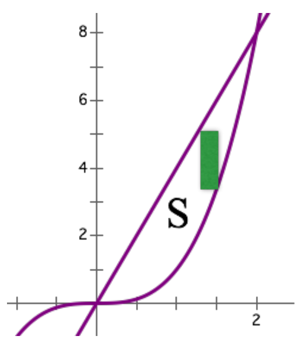
$$V = \int_0^{\pi/2} \left(e^x - \cos x\right)^2 dx$$

$$V = 8.045$$

$e^x - \cos x$

<u>What you may have noticed is that the base of your cross-section is always either "top curve" – "bottom curve" or "right curve" – "left curve"</u>.

Ex 3: The base of solid S is bounded by $y = x^3$ and $y = 4x$ in the first Quadrant. Find the volumes of the solid where region S is the base and the cross-sections perpendicular to the x-axis are rectangles that are twice as tall as they are wide.



$$A = s(2s) = 2s^2$$

The rectangles are vertical, so the volume equation is $V = \int_a^b A(x)dx$, where the boundaries are the x-coordinates of the intersections of the curves—namely, $a = 0$ and $b = 2$.

Note that $A = 2s^2$ and $s = x^3 - 4x$. Therefore,

$$A = 2\left(x^3 - 4x\right)^2 \text{ and } V = 2\int_0^2 \left(x^3 - 4x\right)^2 dx.$$

$$V = 2\int_0^2 \left(x^3 - 4x\right)^2 dx$$

$$= 2\int_0^2 \left(x^6 - 8x^4 + 16x^2\right) dx$$

$$= 2\left[\frac{1}{7}x^7 - \frac{8}{5}x^5 + \frac{16}{3}x^3\right]_0^2$$

$$\approx 19.505$$

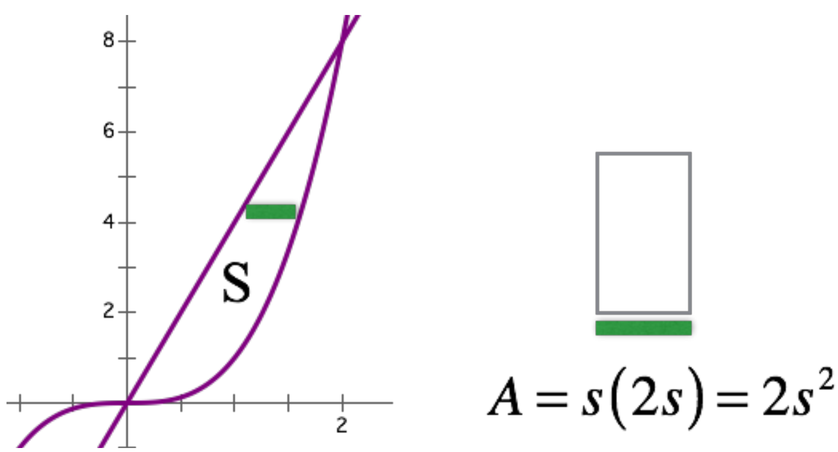
$$v = 2\int_0^2 \left(x^3 - 4x\right)^2 dx$$

$$= 2\int_0^2 \left(x^6 - 8x^4 - 16x^2\right) dx$$

$$= 2\left[\frac{1}{7}x^7 - \frac{8}{5}x^5 - \frac{16}{3}x^3\right]_0^2$$

$$= 19.505$$

Ex 4: The base of solid S is bounded by $y = x^3$ and $y = 4x$ in the first Quadrant. Find the volumes of the solid where region S is the base and the cross-sections perpendicular to the ***y*-axis** are rectangles that are twice as tall as they are wide.



This time, the rectangles are horizontal, so the equations need the x isolated:

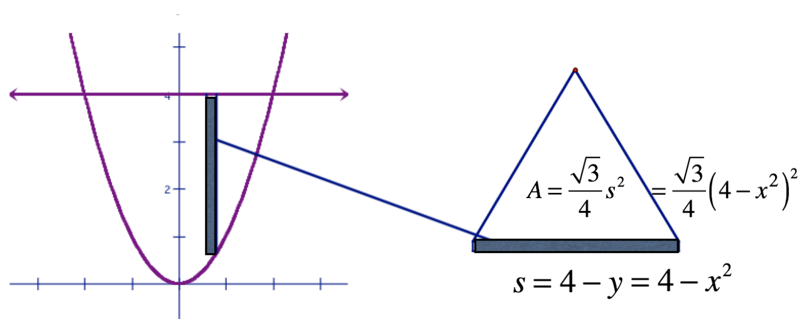
$$y = x^3 \rightarrow x = y^{1/3} \text{ and } y = 4x \rightarrow x = \frac{1}{4}y$$

The volume equation is $V = \int_c^d A(y)dy$, where the boundaries are the y-coordinates of the intersections of the curves—namely, $c = 0$ and $d = 8$. .

Now, $s = y^{1/3} - \frac{1}{4}y$ (right – left). Therefore,

$$\begin{aligned} V &= \int_c^d A(y)dy \\ &= 2\int_0^8 s^2 dy \\ &= 2\int_0^8 \left(y^{1/3} - \frac{1}{4}y\right)^2 dy \\ &\approx 4.876 \end{aligned}$$

Ex 5 The base of solid S is bounded by $y = x^2$ and $y = 4$. Cross-sections perpendicular to the x – axis are equilateral triangles. Find the volume of the solid.



$$v = \int_{-2}^{2} \frac{\sqrt{3}}{4}(4-x^2)^2 dx = \int_{0}^{4} y\sqrt{3}\, dy$$

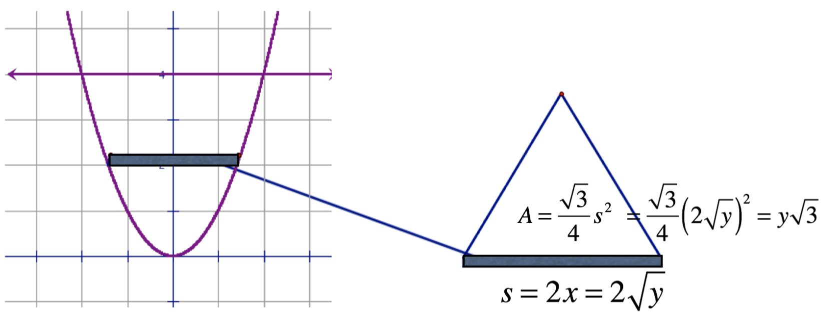
$$= \frac{\sqrt{3}}{4}\int_{-2}^{2}\left(16-8x^2+x^4\right)dx$$

$$= \frac{\sqrt{3}}{2}\int_{0}^{2}\left(16-8x^2+x^4\right)dx$$

$$= \frac{\sqrt{3}}{2}\left[16x-\frac{8}{3}x^3+\frac{1}{5}x^5\right]_0^2$$

$$= \frac{128\sqrt{3}}{15}$$

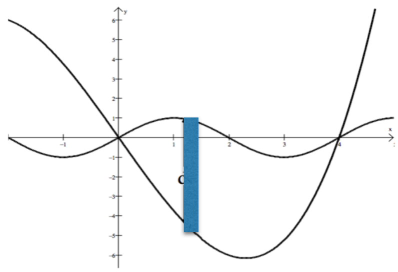
Ex 6 The base of solid *S* is bounded by $y = x^2$ and $y = 4$. Cross-sections perpendicular to the *y*– axis are equilateral triangles. Find the volume of the solid.

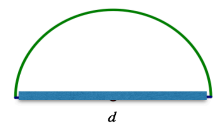


$$v = \int_{0}^{8} 2s^2 dy = 2\int_{0}^{8}\left(y^{1/3}-\frac{1}{4}y\right)^2 dy$$

$$\approx 4.876$$

Ex 7 The base of solid is the region S which is bounded by the graphs $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = \frac{1}{4}\left(x^3 - 16x\right)$. Cross-sections perpendicular to the x – axis are semicircles. What is the volume of the solid?





$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\frac{1}{2}d\right)^2$$

$V = \int_a^b \frac{1}{2}\pi r^2 dx$ Area formula for our cross section

$V = \int_0^4 \frac{1}{2}\pi r^2 dx$ Endpoints of the region of our solid

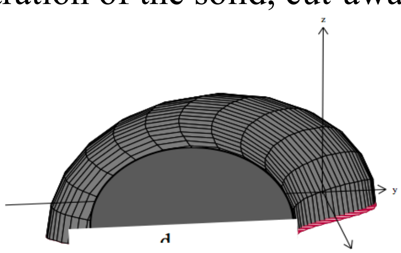
$$V = \frac{1}{2}\pi \int_0^4 \left(\frac{d}{2}\right)^2 dx$$

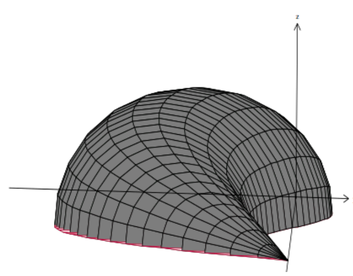
$V = \frac{1}{8}\pi \int_0^4 d^2 dx$ For our cross section $y = r$.

$V = \frac{\pi}{8}\int_0^4 \left(\sin\left(\frac{\pi}{2}x\right) - \frac{1}{4}\left(x^3 - 16x\right)\right)^2 dx$ We cannot integrate y with respect to x so we will substitute out for y.

$V = 30.208$

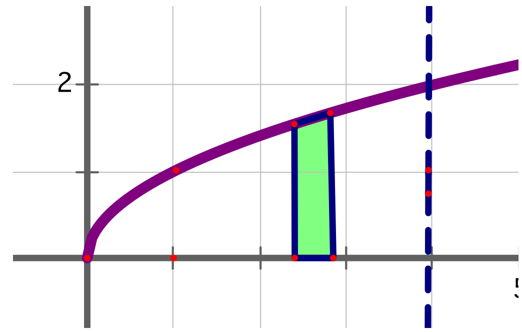
Here is an illustration of the solid, cut-away so that you can see the cross-section:



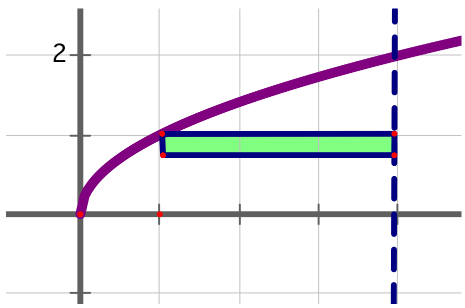


5.6 Free Response Homework

Find the volume of the solid described.

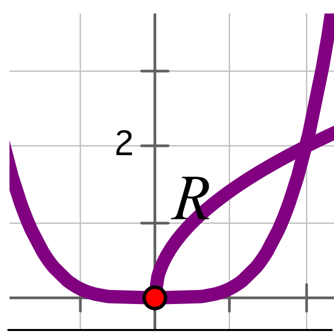


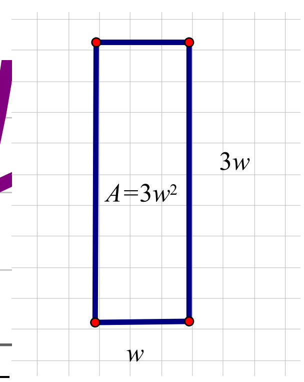
1. Consider the region R bounded by bounded by $y=\sqrt{x}$, the y-axis, and $x=4$. Find the volume of a solid with region R as its base and cross sections perpendicular to the x-axis which are squares.



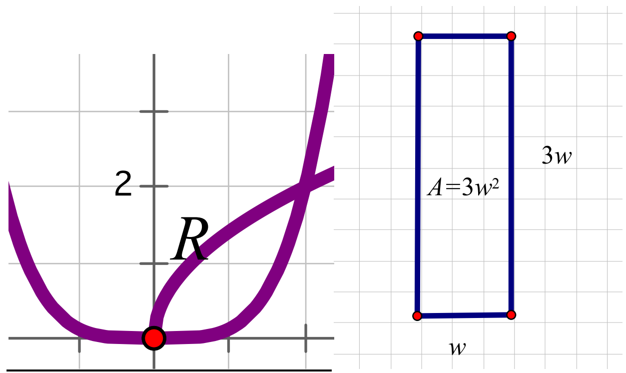
2. Consider the region R bounded by bounded by $y=\sqrt{x}$, the y-axis, and $x=4$. Find the volume of a solid with region R as its base and cross sections perpendicular to the y-axis which are squares.

3. Consider the region S bounded by bounded by $y=\sqrt{2x}$ and $y=\frac{1}{8}x^4$. Find the volume of a solid with region S as its base and cross sections perpendicular to the y-axis which are rectangles that are three times as tall as they are wide.



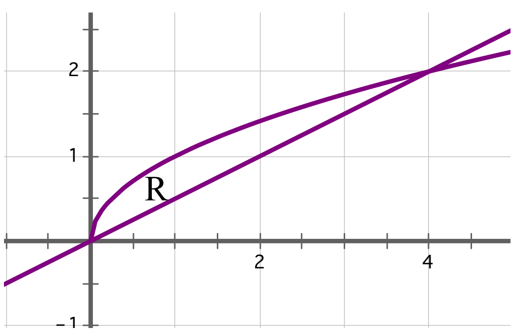


4. Consider the region S bounded by bounded by $y = \sqrt{2x}$ and $y = \frac{1}{8}x^4$. Find the volume of a solid with region S as its base and cross sections perpendicular to the x-axis which are rectangles that are three times as tall as they are wide.



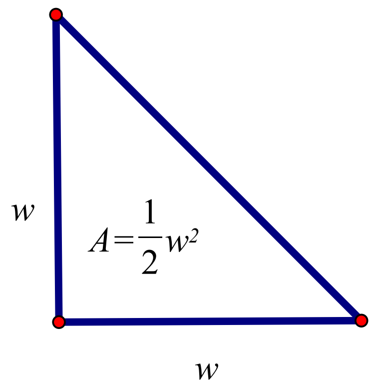
Find the volume of the solid described.

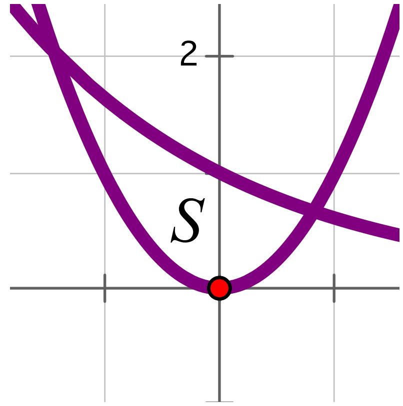
5. Consider region R bounded by the curves $y = \sqrt{x}$ and $y = \frac{1}{2}x$. Find the volume of a solid with region R as its base and cross sections which are squares perpendicular to the x-axis.



6. Consider the region R bounded by $y = \sqrt{x}$ and $y = \frac{1}{2}x$. Find the volume of a solid with region R as its base and cross sections which are squares perpendicular to the y-axis.

7. Let *T* be the region shown above bounded by the graphs of $y = e^{-0.5x}$ and $y = x^2$. Find the volume of a solid with region *T* as its base and cross sections perpendicular to the *x*-axis which are isosceles-right triangles with a leg on region *S*. [Note, a calculator is needed to find the boundaries.]





8. The base is a circle of radius 4 and the cross-sections perpendicular to the *x*-axis are squares.

9. The base is the ellipse $9x^2 + 4y^2 = 36$ and the cross-sections perpendicular to the *x*-axis are isosceles right triangles with the hypotenuse in the base.

10. The base is the region bounded by $y = x^2$ and $y = 1$, and the cross-sections perpendicular to the *y*-axis are squares.

11. The base is the region $y = x^2$ and $y = 1$, and the cross-sections perpendicular to the *x*-axis are squares.

12. The base is the region bounded by $y = x^2$ and $y = 1$, and the cross-sections perpendicular to the *y*-axis are equilateral triangles.

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid that has the described region as its base and the given cross-sections.

13. $y = \sqrt{x}$, $y = e^{-2x}$, $x = 1$; the cross-sections are semi-circles.

14. $y = \ln\left(x^2 + 1\right)$ *and* $y = \cos x$; the cross-sections are squares.

15. The region bounded by $y = \frac{1}{x}$, $x = 1$ and $x = 4$ where the cross-sections are squares.

16. The cross-sectional areas of a felled tree cut into 10-foot sections are given by in table below. Use the Midpoint Riemann Sum with n=5 to estimate the volume of the tree.

x	0	10	20	30	40	50	60	70	80	90	100
Area	.68	.65	.64	.61	.58	.59	.53	.55	.52	.50	.48

17. The end table above is made of nine stacked wooden cylinders. The table below shows the radius and the area of the top surface of each cylinder at each height above the floor.

height	0	3	6	9	12	15	18	21	24	27
diameter	n/a	12	10	8	6	4	6	8	10	12
area	n/a	36π	25π	16π	9π	4π	9π	16π	25π	36π

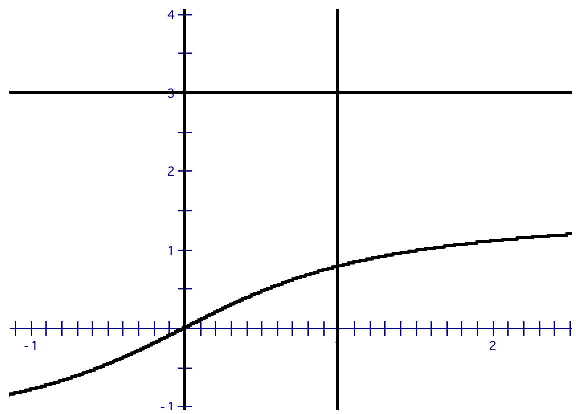
a) Use a right-hand Riemann sum to determine the volume of wood used to make the table.

b). Suppose there were more cylinders of a smaller height and that the radius of each cylinder is modeled by $r(h) = 4 + 2\cos\left(\frac{\pi}{27}h\right)$, where h is the height above the floor. Given that $V_{cyl} = \pi r^2 h$, what would the volume be if there were an infinite number of disks?

5.6 Multiple Choice Homework

1. Let R be the region in Quadrant I bounded by the graph of $y=e^{-x^2/2}$, the line $x=3$. The region R is the base of a solid for which each cross-section perpendicular to the x-axis is a square. What is the volume of the solid?

a) 0.886 b) 0.906 c) 1.078 d) 1.245 e) 2.784



2. The base of a solid is the region bounded by $y=\tan^{-1}x$ $y=0$, $y=3$, and $x=1$. If each cross-section of the solid perpendicular to the x-axis is a square, the volume of the solid is

a) 2.561 b) 6.612 c) 8.757

d) 20.773 e) 27.504

---

3. The base of a solid is a region in the first quadrant bounded by the x-axis, the y-axis, and the line $x + 3y = 5$. If cross sections of the solid are squares perpendicular to the y-axis, what is the volume of the solid?

a) 4.167 b) 4.629 c) 8.727

d) 12.500 e) 13.889

---

4. The base of a solid is a region bounded by $y = \tan^2 x$ and $y = 4 - x^2$. If each cross-section perpendicular to the *x*-axis is a rectangle with a base in the region and height equal to twice the base, the volume of the solid is

a) 21.033 b) 33.038 c) 40.524 d) 42.066 e) 81.048

---

5. The base of a solid is the region bounded by $y = \sin x$ and $y = \cos x$ for $0 \le x \le \frac{\pi}{4}$. If each cross-section of the solid perpendicular to the *x*-axis is a semicircle, the volume of the solid is

a) 0.039 b) 0.112 c) 0.224

d) 0.448 e) 0.897

---

6. The base of a solid S is the region enclosed by the graph of $y = \sqrt{\ln x}$, the line $x = e$, and the *x*-axis. If the cross sections of *S* perpendicular to the *x*-axis are rectangles with a height twice as big as the base, the volume of *S* is

a) 0.5 b) 0.667 c) 1.140 d) 2 e) 6.362

---

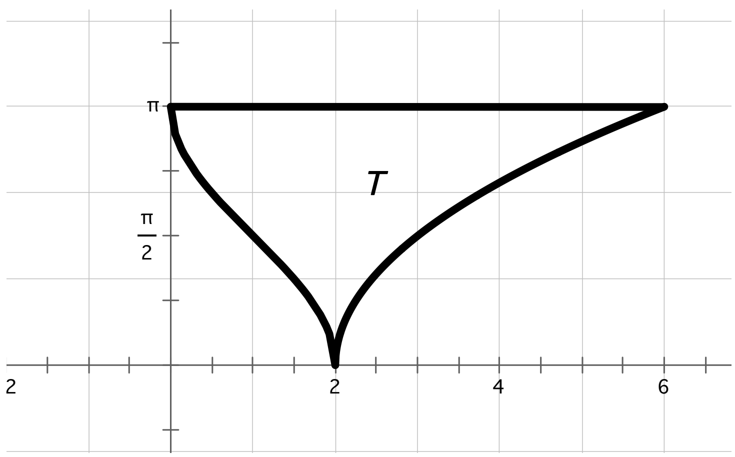
7. The base of a solid is the region enclosed by $x^2 + 4y^2 = 4$. If each cross-section of the solid perpendicular to the *x*-axis is a square, the volume of the solid is

a) $\frac{8}{3}$ b) $\frac{8\pi}{3}$ c) $\frac{16}{3}$ d) $\frac{32}{3}$ e) $\frac{32\pi}{3}$

---

8. The base of a solid is the region enclosed by $y = \cos x$ and the x-axis for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. If each cross-section of the solid perpendicular to the x-axis is a square, the volume of the solid is

a) $\frac{\pi}{4}$ b) $\frac{\pi^2}{4}$ c) $\frac{\pi}{2}$ d) $\frac{\pi^2}{2}$ e) 2



9. Let ***T*** be the region above bounded by

$$y = \frac{\pi}{2} - \sin^{-1}(x-1), \quad y = \frac{\pi}{2}\sqrt{x-2}, \quad \text{and} \; y = \pi$$

Let ***T*** be the base of a solid whose cross-sections are isosceles right triangles with one leg in ***T*** and the leg is perpendicular to the x-axis. The volume of the solid is

a) 6.225 b) 7.330 c) 39.111

d) 105.585 e) 108.895

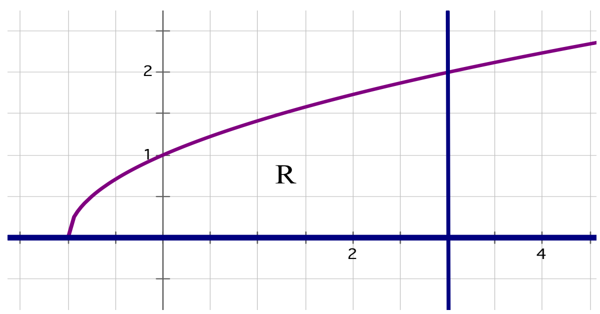
5.7 AP-Style Volume FRQs

There are basically two kinds of questions on the AP Exam.

- Primarily algebraic. These questions from sections 6.1 through 6.6 will be mixed together in a single problem. The trick is to recognize that the subsections are actually separate questions and treat them accordingly.
- Volume in Context. These questions are word problems that often include related rates questions as part d). They also can be table problems like Example 4.

Objectives:
Solve AP-style area and volume problems.

Ex 1 Let R be the region bounded by the *x*-axis, $y=\sqrt{x+1}$, and the line $x=3$ as shown below.



(a) Find the volume of the solid formed if region R is rotated about the *x*-axis.

(b) Let the base of a solid be the region R. If the cross-sections of the solid perpendicular to the *y*-axis are rectangles that are twice as tall as they are wide, find the volume of the solid.

(c) Set up, but **do not solve**, an expression for the volume of a solid formed if Region R is revolved about the line $x=4$.

(a) Find the volume of the solid formed if region R is rotated about the x-axis.

$$Volume = \pi \int_{-1}^{3} \left[(x+1)^{1/2} \right]^2 dx = \pi \int_{-1}^{3} (x+1)\, dx$$

$$= \pi \left[\frac{x^2}{2} + x \right]_{-1}^{3} = \pi \left[\left(\frac{9}{2} + 2 \right) - (0) \right] = \frac{13\pi}{2}$$

(b) Let the base of a solid be the region R. If the cross-sections of the solid perpendicular to the x-axis are rectangles that are twice as tall as they are wide, find the volume of the solid.

Cross-section Method with $A(x) = bh = (x+1)^{1/2} \left(2(x+1)^{1/2} \right)$

$$Volume = \int_{-1}^{3} (x+1)^{1/2} \left(2(x+1)^{1/2} \right) dx = \int_{-1}^{3} (2x+2)\, dx =$$

$$\left[x^2 + 2x \right]_{-1}^{3} = 15 - (-1) = 16$$

(c) Set up, but **do not solve**, an expression for the volume of a solid formed if Region R is revolved about the line $x = 4$.

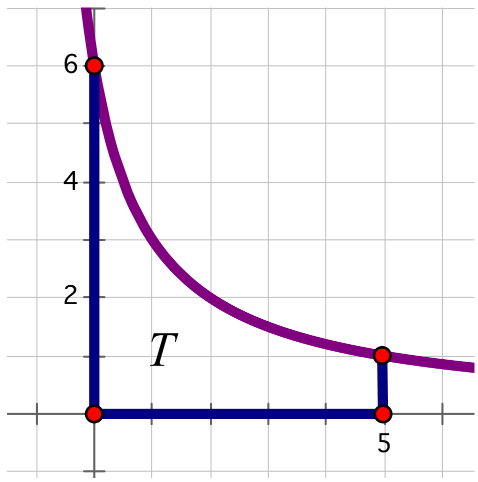
Washer Method with horizontal rectangles:

$$y = \sqrt{x+1} \rightarrow x = y^2 - 1$$

$$R = 4 - (y^2 - 1) \qquad\qquad r = 4 - 3 = 1$$

$$V = \pi \int_{0}^{2} \left(\left[4 - (y^2 - 1) \right]^2 - [1]^2 \right) dy$$

Ex 2 Let T be the region bounded by $y=\frac{6}{x+1}$, $y=0$, $x=0$, and $x=5$.



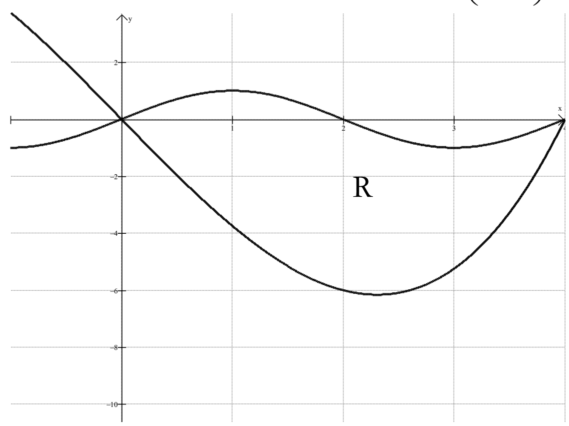
a) Find the area of region T. Show the anti-differentiation step.
b) Find the perimeter of region T. Show the set-up.
c) Find the volume of the solid generated when T is rotated about the x – axis. Show the anti-differentiation steps.

a) $$A=\int_0^5 \frac{6}{x+1}dx=6\int_1^6 \frac{1}{u}du=6\ln u\Big|_1^6=6\left[\ln 6-\ln 1\right]=6\ln 6$$

b) $$P=6+5+1+\int_0^5 \sqrt{1+\left(-12\left(x+1\right)^{-2}\right)^2}\,dx=24.007$$

c) $$V=\pi\int_0^5\left(\frac{6}{x+1}\right)^2 dx=36\pi\int_0^5\left(x+1\right)^2 dx=36\pi\int_1^6 u^2du=36\pi\left[\frac{u^3}{3}\right]_1^6=$$
$$=12\pi\left[6^3-1\right]=2580\pi$$

Ex 3 Let R be the region bounded by the graphs $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = \frac{1}{4}\left(x^3 - 16x\right)$



(a) Find the area of the region R.

(b) Find the volume of the solid when the region R is rotated around the line $y = -10$.

(c) Find the volume of the solid where the region R is the base and the cross-sections perpendicular to the x-axis are rectangles which are four times as tall as they are wide.

(a) Find the area of the region R.

$$Area_R = \int_0^4 \left[\left(\sin\frac{\pi}{2}x\right) - \left(\frac{1}{4}\left(x^3 - 16x\right)\right)\right] dx = 16$$

(b) Find the volume of the solid when the region R is rotated around the line $y = -10$.

$$V = \pi \int_0^4 \left(\sin\frac{\pi}{2}x - (-10)\right)^2 - \left(\frac{1}{4}\left(x^3 - 16x\right) - (-10)\right)^2 dx = 243.981$$

(c) Find the volume of the solid where the region R is the base and the cross-sections perpendicular to the x-axis are rectangles which are four times as tall as they are wide.

$$V=\int_0^4\left(\sin\frac{\pi}{2}x-\frac{1}{4}\left(x^3-16x\right)\right)\cdot 4\left(\sin\frac{\pi}{2}x-\frac{1}{4}\left(x^3-16x\right)\right)dx=307.691$$

Ex 4 A tree stands perpendicular to the ground. The tree's trunk has horizontal cross sections that are circles. The radius r of the tree trunk is a differentiable function of the height h of the tree measured from the ground. Both r and h are measured in feet. Selected values of r and h are given in the table below.

h height from the ground (feet)	0	1	3	5	8
$r(h)$ radius of the tree trunk (feet)	3	2.5	2	2.5	1.5

---

a) Use a Right Riemann sum with subintervals indicated by the table to approximate the value of $\frac{1}{8}\int_0^8 r(h)\,dh$. Using correct units, explain the meaning of this value in the context of this problem.

b) Must there be a height within the first eight feet of tree trunk where $\frac{dr}{dh}=0$? Explain.

c) Write an expression involving one or more integrals to calculate the volume of the tree trunk from $h=0$ to $h=8$. Use a Left Riemann sum to approximate the value of this expression.

d) For heights above 8 feet, the radius is given by $g(h) = \frac{1}{h^2} + \frac{95}{64}$. A squirrel climbs up the tree at a rate of $\frac{dh}{dt} = 3 \text{ ft}/\text{sec}$. How quickly is the radius of the tree changing when the squirrel is 9 feet above the ground?

---

a) Use a Right Riemann sum with subintervals indicated by the table to approximate the value of $\frac{1}{8}\int_0^8 r(h)\,dh$. Using correct units, explain the meaning of this value in the context of this problem.

$$\frac{1}{8}\int_0^8 r(h)\,dh \approx \frac{1}{8}\left[2.5 + 2(2) + 2(2.5) + 3(1.5)\right] = 2$$

Two feet is the approximate average radius of the tree over the first 8 feet of the tree.

---

b) Must there be a height within the first eight feet of tree trunk where $\frac{dr}{dh} = 0$? Explain.

Yes. The function is continuous and differentiable, therefore, the mean Value Theorem applies. $r(1) = r(5)$, therefore, the $\frac{dr}{dh} = \frac{r(5) - r(1)}{5 - 1} = 0$

---

c) Write an expression involving one or more integrals to calculate the volume of the tree trunk from $h = 0$ to $h = 8$. Use a Left Riemann sum to approximate the value of this expression.

$$V = \pi\int_0^8 \left[r(h)\right]^2 dh \approx \pi\left[3^2 + 2(2.5)^2 + 2(2)^2 + 3(2.5)^2\right] = 42\pi \ ft^3$$

---

d) For heights above 8 feet, the radius is given by $g(h) = \frac{1}{h^2} + \frac{95}{64}$. A squirrel climbs up the tree at a rate of $\frac{dh}{dt} = 3 \text{ ft}/\text{sec}$. How quickly is the radius of the tree changing when the squirrel is 9 feet above the ground?

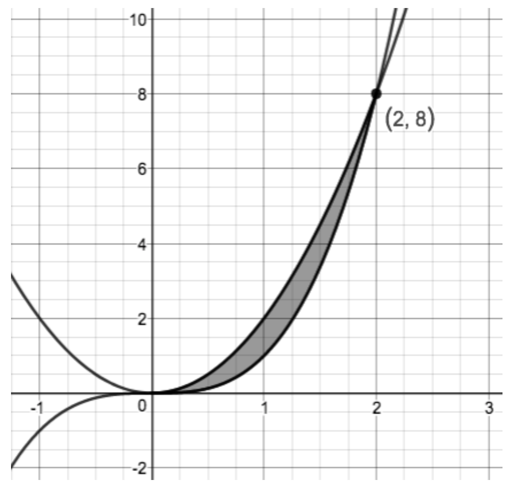
$$r = g(h) = \frac{1}{h^2} + \frac{95}{64} = h^{-2} + \frac{95}{64}$$

$$\frac{dr}{dt} = g'(h) = -2h^{-3}\frac{dh}{dt}$$

$$g'(9) = -2(9)^{-3}(3) = -\frac{2}{243} \, ft/\text{sec}$$

6.7 Free Response Homework

1. Let R be the region bounded by $y = 2x^2$ and $y = x^3$, shaded in the picture below. The curves intersect at the origin and at the point $(2,8)$.



a. Find the area of R.

b. Set up, but do not evaluate, an expression involving one or more integrals that would find the volume of the solid whose base is R and whose cross-sections parallel to the y-axis are semicircles.

c. Find the volume of the solid formed by revolving R around the y-axis.

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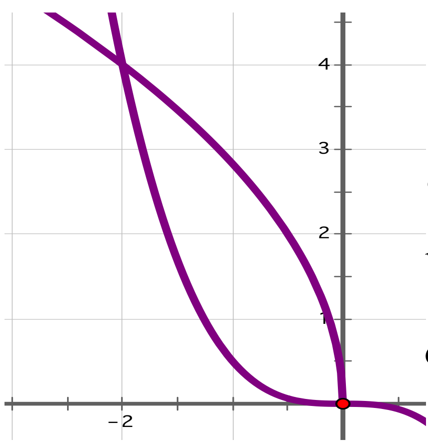
2. Let T be the region bounded by $y = -\frac{1}{2}x^3$ and $y = \sqrt{-8x}$ in Quadrant II.

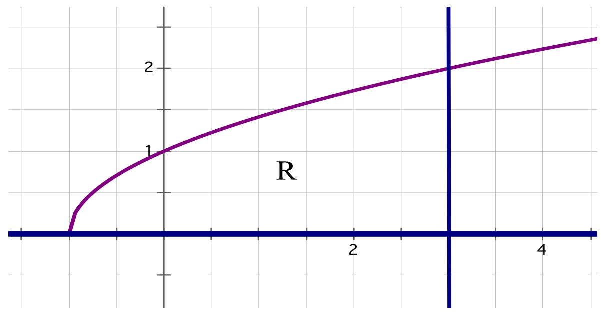


a) Find the volume of the solid generated when T is rotated about the x – axis. Show the anti-differentiation steps.

b) Find the volume of the solid generated when T is rotated about the y – axis. Show the anti-differentiation steps.

c) Set up, but do not evaluate, an expression involving one or more integrals that would find the volume of the solid whose base is T and whose cross-sections perpendicular to the y-axis are squares.

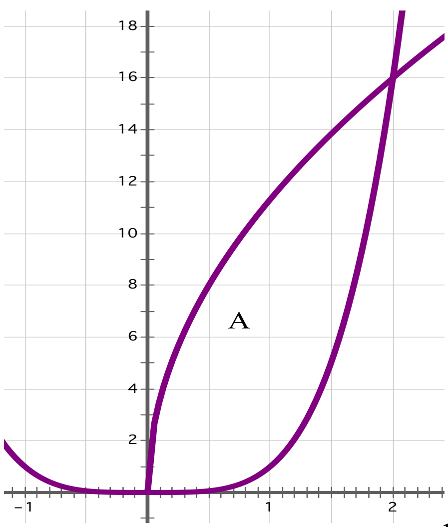
3. Let R be the region bounded by the x-axis, $y=\sqrt{x+1}$, and the line $x=3$ as shown below.



a. Find the area of region R.

b. Find the volume of the solid formed by revolving region R about the line $y=3$.

c. Set up, but **do not solve**, an expression for the volume of a solid formed if Region R is revolved about the line $x=4$.



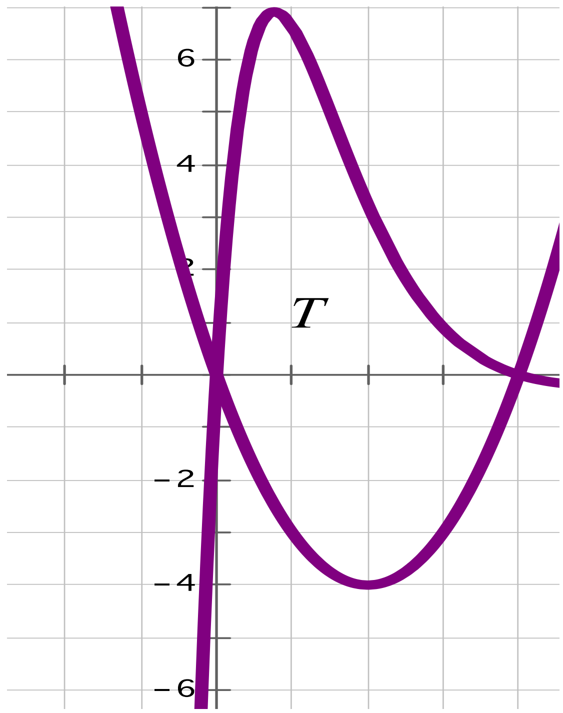
4. Given the functions below are $y=x^4$ and $y=8\sqrt{2x}$. Let A be the region bounded by the two curves and the x-axis.

a. Find the area of region A.

b. Find the volume of the solid when region A is rotated around the x-axis.

c. Set up, but **do not solve**, an expression for the volume of a solid formed if Region A is revolved about the line $x=-2$.

5. Region T is bounded by the curves $f(x) = -6(x^2 - 4x)e^{-x}$ and $g(x) = x^2 - 4x$



(a) Find the area of region T. Show the set-up.
(b) Find the volume of the solid generated when R is revolved about the line $y = -4$.
(c) Let the base of the solid be the region R. Find the volume of the solid where the cross-sections perpendicular to the x-axis are squares.

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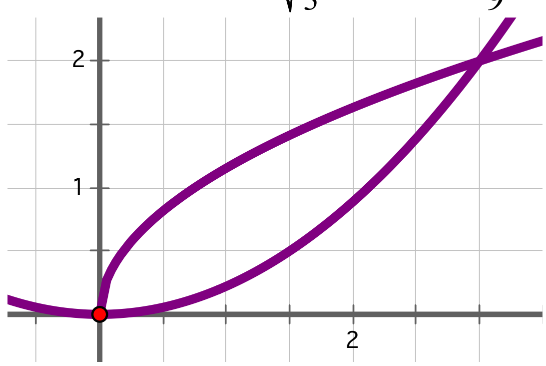
6. Let R be the region bounded by $y = \sqrt{\frac{4}{3}x}$ and $y = \frac{2}{9}x^2$.



a) Find the area of region R. Show the anti-differentiation steps.

b) Find the volume of the solid generated when R is rotated about the y – axis. Show the setup, but use your calculator to find the answer.
c) Find the volume of the solid generated when R is rotated about the x – axis. Show the anti-differentiation steps.

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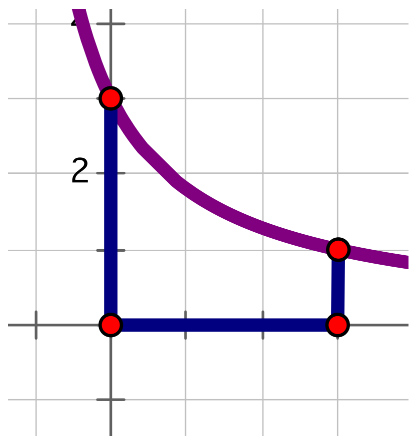
7. Let T be the region bounded by $y=\frac{9}{2x+3}$, $y=0$, $x=0$, and $x=3$.



a) Find the area of region T. Show the anti-differentiation step.
b) Find the perimeter of region T. Show the set-up.
c) Find the volume of the solid generated when T is rotated about the x – axis. Show the anti-differentiation steps.

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8. Let R be the region bounded by the curves $f(x)=\ln x$, $g(x)=e^{-x}$ and $x = 4$.

a. Find the area of the region bounded by three curves in the first quadrant.
b. Find the volume of the solid generated by rotating the region around the line $y = 2$.
c. Find the volume of the solid generated by rotating the region around the line $y = -1$.

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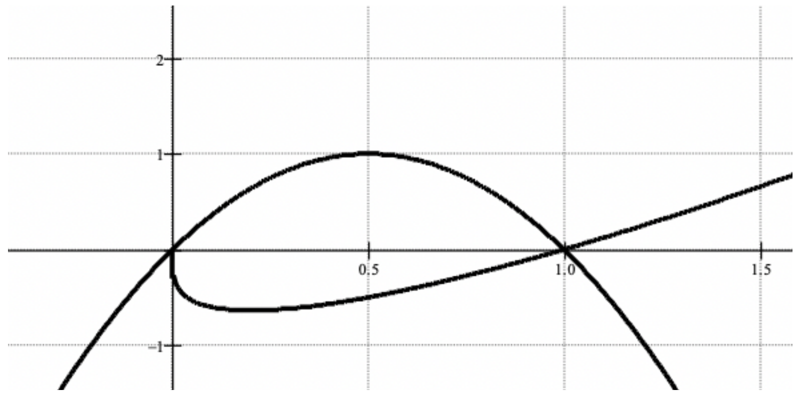
9. Let f and g be the functions given by $f(x)=4x(1-x)$ and $g(x)=\sqrt[4]{2x}(x-1)$



a. Find the area of the region bounded by the two curves.

b. Find the volume of the solid generated by rotating the region around the line $y = 3$.

c. Find the volume of the solid generated by rotating the region around the line $y = -2$.

d. Set up an integral expression for the curve $g(x)=\sqrt[4]{2x}(x-1)$ and the curve $h(x)=x(1-x)$ which is not pictured above that is rotated around the line $y = k$, where $k > 2$, in which the volume equals 10. Do not solve this equation.

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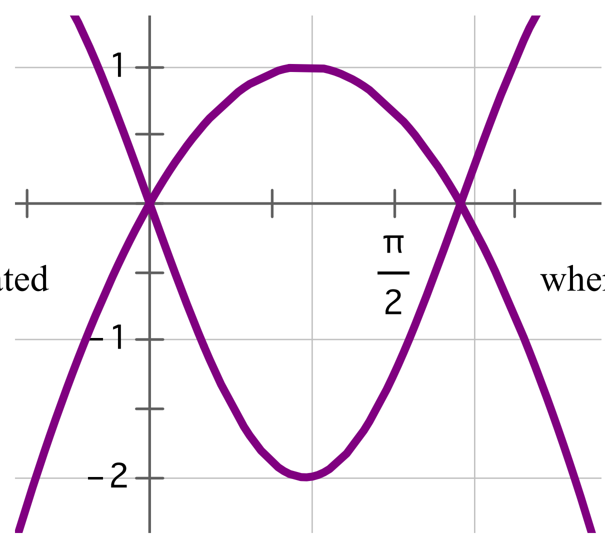
10. Let S be the region shown above bounded above by the graph of $y=-2\sin\left(\frac{\pi}{2}x\right)$ and below the graph of $y=2x-x^2$.

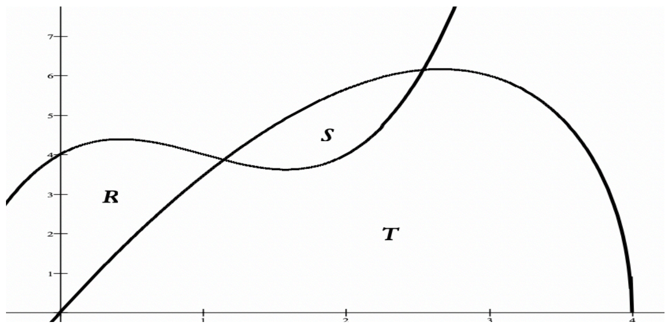


a) Find the area of region S.

b) Find the volume of the solid generated when S is revolved about the line $y=2$.

c) Let the base of the solid be the region S. Find the volume of the solid where the cross-sections perpendicular to the x-axis are rectangles that are three times as tall as they are wide.

11. Given the curves $f(x)=x^3-3x^2+2x+4$, $g(x)=2x\sqrt{4-x}$ and in the first quadrant.



a. Find the area of the regions *R*, *S*, and *T*.

b. Find the volume of the solid generated by rotating the curve $g(x)=2x\sqrt{4-x}$ around the line $y=8$ on the interval $x\in[1,3]$.

c. Find the volume of the solid generated by rotating the region *S* around the line $y=-1$.

d. Find the volume of the solid generated if *R* forms the base of a solid whose cross sections are squares perpendicular to the *x*-axis

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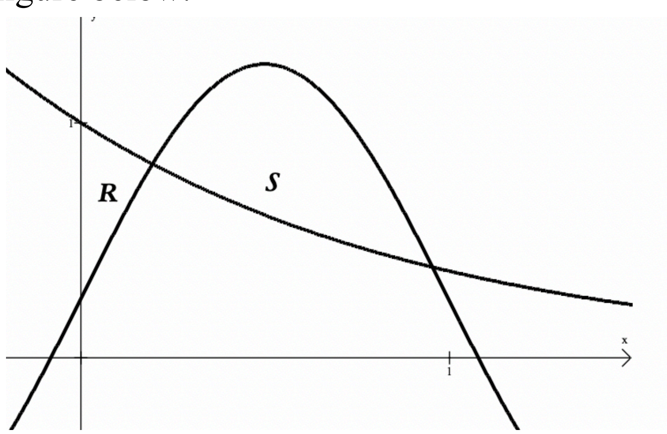
12. Let *f* and *g* be the functions given by $f(x)=\frac{1}{4}+\sin(\pi x)$ and $g(x)=e^{-x}$. Let R be the region in the first quadrant enclosed by the y – axis and the graphs of f and g, and let *S* be the region in the first quadrant enclosed by the graphs of f and g, as shown in the figure below.



a. Find the area of *R*

b. Find the area of S

c. Find the volume of the solid generated when S is revolved around the horizontal line $y = -2$

d. The region R forms the base of a solid whose cross-sections are rectangles with bases perpendicular to the x-axis and heights equal to half the length of the base. Find the volume of the solid.

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13. Dr Quattrin has a 30-foot-long, irregularly shaped swimming pool at his house. He and Mrs. Quattrin measure the width and depth of the pool at regular intervals starting at the south end. The data is presented below.

x in feet	0	5	10	15	20	25	30
$w(x)$ in feet	0	14.6	13.5	14.5	17.4	11.6	0
$h(x)$ in feet	0	3	4	6	7.4	8	0

(a) Use a trapezoidal sum to approximate the surface area of the pool.

(b) Assume the cross-sections are rectangular. Use a midpoint Reiman sum to approximate the volume of the pool.

(c) Approximate $w'(15)$. Use your approximation to write the equation of the line tanget to $w(x)$ at $x = 15$.

(d) Suppose the depth of the pool is modeled by

$$f(x) = -.0001565x^4 + 0.007969x^3 - 0.12954x^2 + 1.06724x - 0.931776.$$

Mrs. Quattrin swims the length of the pool in a leisurely 15 seconds. How quickly is the depth of the water changing when $t = 8\ sec$?

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14. A heart attack that is caused by a 100 percent blockage of the left anterior descending (LAD) artery is referred to as a "Widow maker" because it is often fatal if not treated quickly. The LAD has circular cross-sections, and the table below shows the diameter $k(x)$ of a particular heart patient's LAD, where x is the distance from where the LAD branches off of the main left artery.

x mm	0	4	9	16	19	23	27
$k(x)$ in feet	3	2.8	2.8	3.1	2.7	2.9	3.1

$k(x)$ is a twice-differeiable function.

a) Write an integral expression which would determine the average diameter of this patient's LAD.

b) Using a right-hand Reiman sum, determine the volume of blood in this LAD.

c) Explain why there is at least one x-value where $k(x) = 2.83$.

d) Explain where there are at least two x-values where $k'(x) = 0$.

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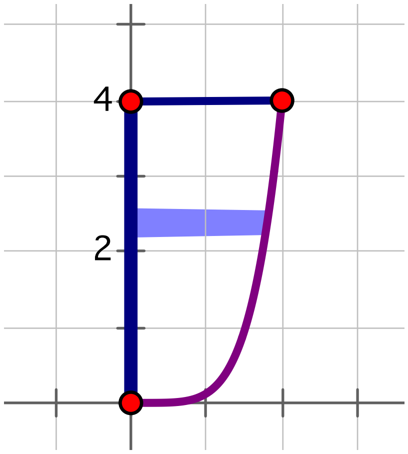
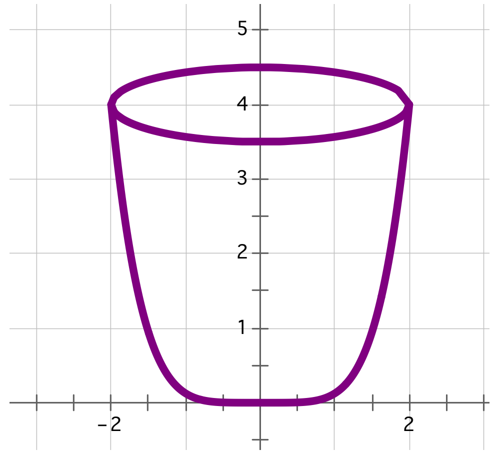
15. The inner surface of an ancient crucible, used to smelt iron ore, is found to have an inner surface that conforms to the solid formed by the region bounded by $y = \frac{1}{8}x^5$, $x = 0$, and $y = 4$ being revolved about the y-axis. The width and height are measured in feet.





a) Find the volume of the solid.

b) Find an equation for the volume $V(h)$ of molten metal in the crucible, where h is the depth of the liquid.

c) Assume the same size and shape vessel is used to store water, and that the water evaporates at a constant rate such that $\frac{dh}{dt} = -\frac{1}{5}\ ft/hr$. How fast is the volume changing when $h = 3$.

d) Show that the rate of change of the volume of the container is directly proportional to the exposed surface area of the water.

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16. The oldest dam and conduit system in the United States is the one built to irrigate the lands of the Old Mission San Diego about 1801. The tile-lined aqueduct runs from the diverter dam on the San Diego River, through Mission Gorge, to the Mission 4 miles away. The cross-section shown below can be modeled by $y = \frac{1}{1.21}x^2 - 1$, where x and y are measured in feet.

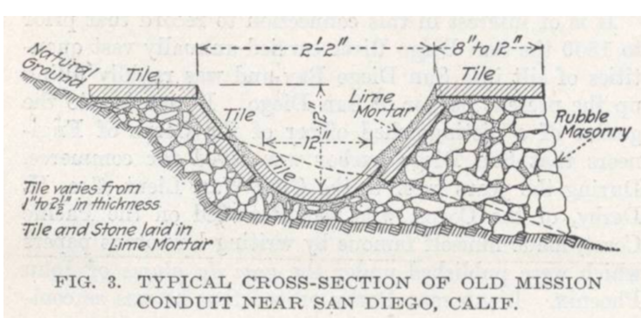
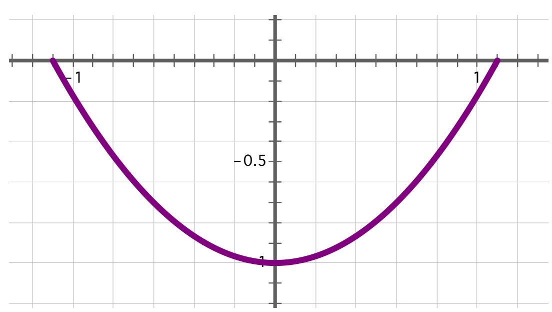


FIG. 3. TYPICAL CROSS-SECTION OF OLD MISSION CONDUIT NEAR SAN DIEGO, CALIF.



a) What would the area of the cross-section be if the water were 0.8 feet deep $(y = -0.2)$?

b) What is the volume of water which the aqueduct could hold before overflowing?

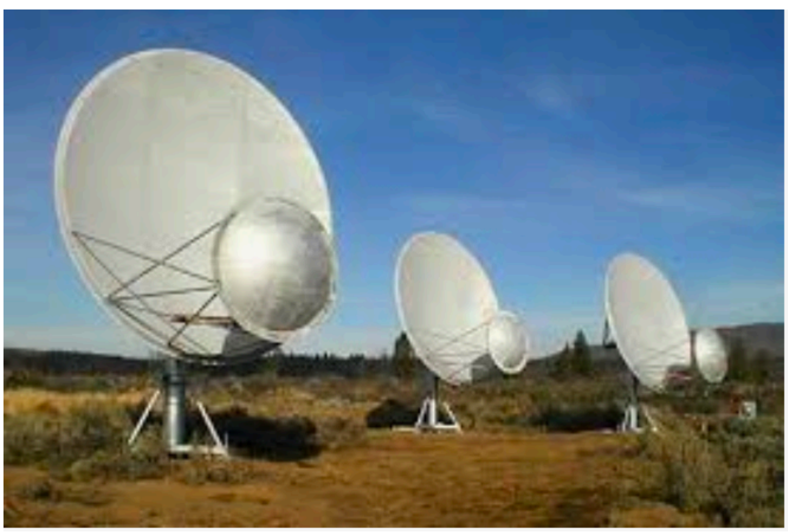
c) The volumetric flow $f(t)$ at a location along the aqueduct is the product of the cross-sectional area and the velocity of the water at that location. Suppose the velocity of the flow of water in the aqueduct was modeled by

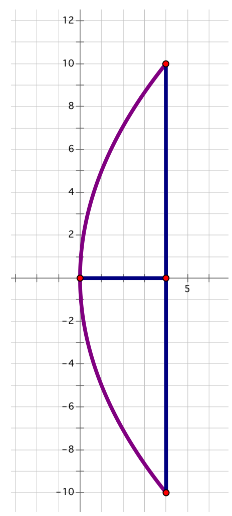
$$v(t) = 3 + 2\cos\sqrt{t+10}.$$

Find the average volumetric flow during the first hour the dam gate is open if the water were 0.8 feet deep.

d) If the average volumetric flow exceeds $6.2\ \text{ft}^3/\text{min}$ during any 20-minute period, the dam gates need to be closed. $f(t)$ has a maximum at $t = 30$. Do the gates need to be closed? Show the computations that lead to your answer.

The ATA-42 Problem

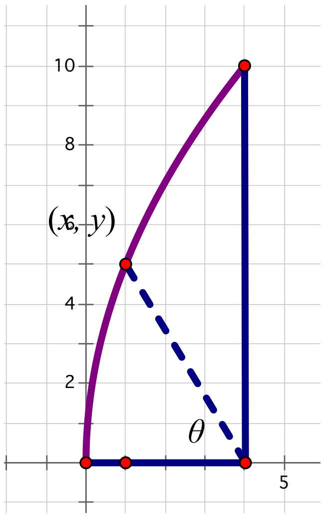




17. The SETI Institute (search for extra-terrestrial intelligence) began using the Allen Telescope Array (ATA-42) in 2016 to search for intelligence among radio waves. ATA-42 is a set of 42 radio receivers with parabolic dishes. Each dish conforms to the shape formed by revolving the equation $y = 5\sqrt{x}$ revolved about the *x*-axis from $x = 0$ to $x = 4$ ft.

a) Find the volume of the bowl of one reflector.

b) Find the arc length of the bowl's profile.

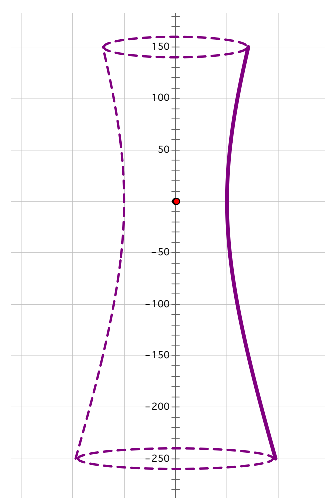


c) Consider a drop of morning dew sliding down the reflector of the telescope. What is the relationship between the angle θ and the position (x, y) on the reflector?

d) If $\frac{dy}{dt} = -.1\ {}^{in}/_{min}$ and $\frac{dx}{dt} = -.03\ {}^{in}/_{min}$, how fast is θ changing when the position is $(1, 5)$?

The Cooling Tower Problem

18. Many power plants cool their reactions with convective air flow through a hyperboliod tower. The shape increases air flow while minimizing construction material. Consider the shape of a tower formed by revolving the hyperbola $f(y) = 50\sqrt{\frac{1}{22500}y^2 + 1}$ on $y \in [-250, 150]$ about the y-axis, where y is measured in feet from the narrowest part of the tower.



a. Find the volume of the interior of the tower. Show the antiderivative. Indicate the units.

b. Assume that the inner wall is of the shape formed by revolving $f(y)$ about the y-axis and the outer wall is of the the shape formed by revolving

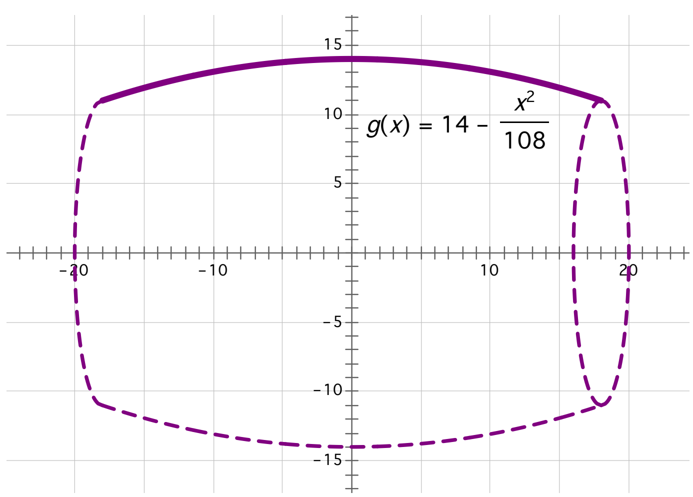
$g(y) = 50.583\sqrt{\frac{1}{22500}y^2 + 1}$ about the y-axis. Find the volume of material needed to make the tower.

c. Concrete costs \$90 per cubic yard (27 cubic feet). Find the concrete cost to build the tower. Indicate units.

d. The temperature S in the tower varies according to the function $S(h) = 70e^{-0.001(150-h)}$, where h is the height, in feet, from the top of the tower. An object is dropped into the tower from the top. It falls at a rate of $\frac{dh}{dt} = -32t \; {}^{ft}\!/_{sec}$ and the height the obect has fallen is $h = -16t^2 \; ft$. How fast is the temperature S changing when the object has been falling for 3 seconds? Indicate the units.

<u>Wine Barrel Problem</u>

19. Johannes Kepler provided fundamental advances before Newton or Leibnitz ever developed the Calculus. He did this when he remarried after his first wife's death and, at the wedding party, became incensed because he believed the wine merchant was cheating him on the price by volume of the wine. (He thought the way of measuring was inaccurate, but it turned out thatKepler was wrong.)



A 53-gallon wine barrel has a lid diameter of 22 inches, a bilge diameter (the diameter of the widest part of the barrel) of 28, and a height of 36 inches. If the barrel were lying on its side, the same shape could be created by rotating the parabola $y = 14 - \frac{1}{108}x^2$ on $x \in [-18, 18]$ about the x-axis.

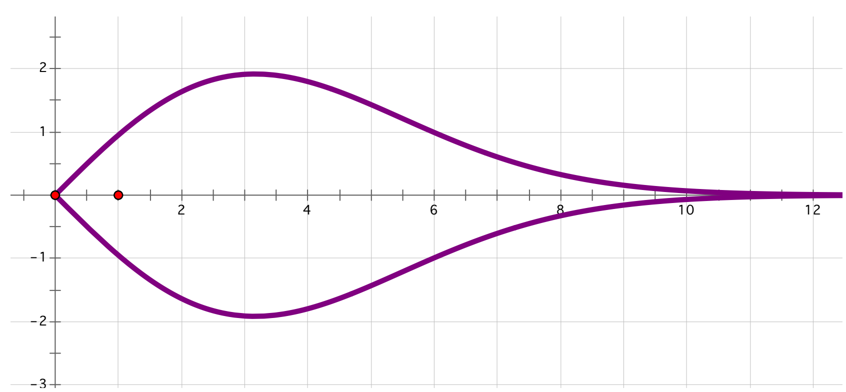
a. Find the volume, in cubic inches of the shape formed by the rotation. Show the antiderivative steps.

b. The surface area of a solid of rotation is found by the equation $A = 2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$. Find the surface area of the sides of wine barrel, including both end lids. Show the set up and indicate the correct units.

c. Assume that the volume of wine in the barrel can be approximated by a cylinder with a diameter of 25" $(V_{\text{cyl}} = \pi r^2 h)$. The barrel is placed upright and wine is scooped out in pitchers to serve to the wedding guests. If the wine is removed from the barrel at $4800\, in^3/hr$, how fast is the level of the wine in the barrel is dropping? Indicate units.

Alien Sea Creature Problem

20. Xenobiologists visiting another planet find sea creatures whose silhouette (outline) resembles the region bouned by $f(x) = xe^{-.05x^2}$ and $g(x) = -xe^{-.05x^2}$ from $x = 0$ to $x = 12$



a) Find the area of the silhouette. Show the antiderivative.

b) If the creature's body shape was created by revolving $f(x)$ about the x-axis, find the volume of said creature.

c) Dissection reveals that the creature has a cross section perpendicular to the x-axis which is a regular hexagon with the diameter s in the silhouette region. Given that the area of a regular hexagon is $\mathrm{A}=\frac{3\sqrt{3}}{2}r^2$, find the volume of the creature. Show the setup.

d) If the live creature swam by at $2\,{}^{in}\!/_{sec}$, how fast would the diameter be changing when $x=3$?

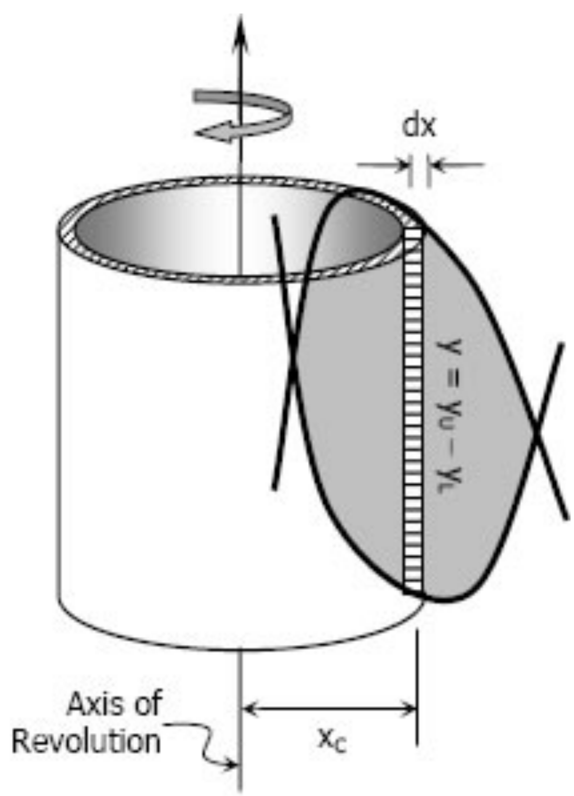
21. AP Packet: AB 2013 #5, AB 2014 #2, AB 2015 #2, AB 2019 #5

6.8 Volume by the Shell Method

There is another method we can use to find the volume of a solid of rotation. With the Disc/Washer method we drew our sample pieces perpendicular to the axis of rotation. Now we will investigate calculating volumes when our sample pieces are drawn parallel to the axis of rotation (**parallel** and **shell** rhyme – that's how you can remember they go together).

Objectives:
Find the volume of a solid rotated when a region is rotated about a given line.

Take the following function, let's call it $f(x)$, from a to b and rotate it about the y – axis.



$$V = 2\pi\int_{x_1}^{x_2} x_C(y_U - y_L)\,dx$$

How do we begin to find the volume of this solid?

Imagine taking a tiny strip of the function and rotating this little piece around the y – axis. What would that piece look like once it was rotated?
What is the surface area formula for a cylindrical shell? $2\pi rl$

Now would the surface area of just this one piece give you the volume of the entire solid?
How would we add up all of the surface areas of all these little pieces?

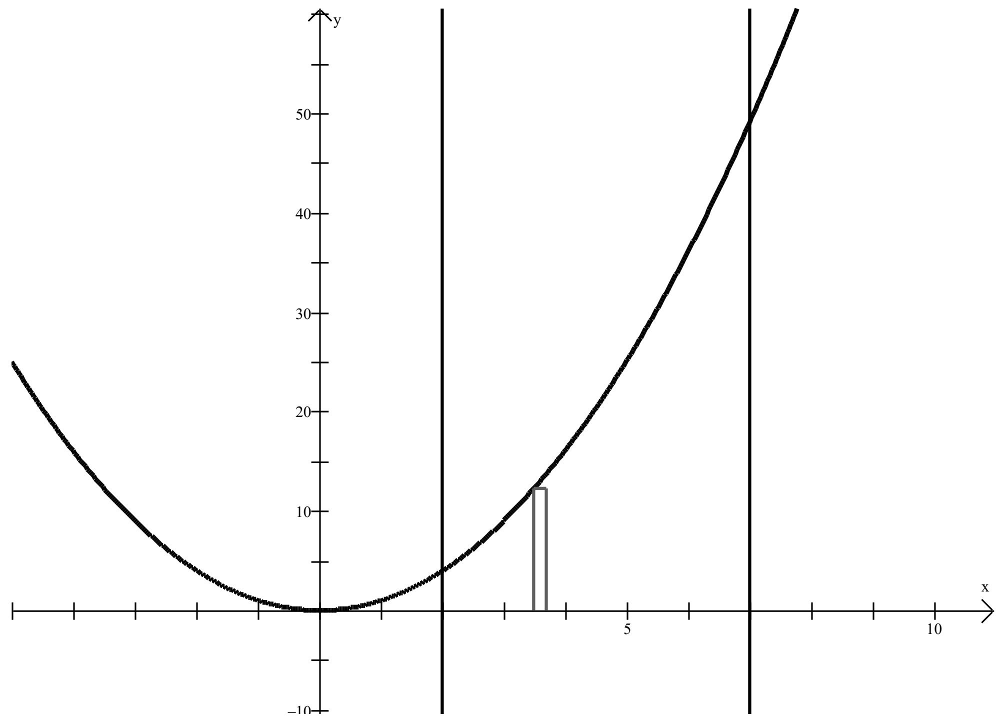
We arrive at our volume formula.

Volume by Shell Method: The volume of the solid generated when the region bounded by function $f(x)$, from $x = a$ and $x = b$, where $f(x) \geq 0$ [or $g(y)$, from $y = c$ and $y = d$, where $g(y) \geq 0$], is rotated about the y – axis is given by

$$V = 2\pi \int_a^b x \cdot f(x)\,dx \text{ or } V = 2\pi \int_c^d y \cdot g(y)\,dy$$

** Everything rhymes with the shell method – Shell/Parallel/ $2\pi rl$ so the disc method must be of the form Disc/Perpendicular/ πr^2.

Ex 1 Let R be the region bounded by the equations $y = x^2$, $y = 0$, $x = 1$, $x = 3$.
Find the volume of the solid generated when R is rotated about the y – axis.



$V = 2\pi\int_a^b x \cdot y\,dx$ Our pieces are parallel to the y – axis, so our integrand will contain dx

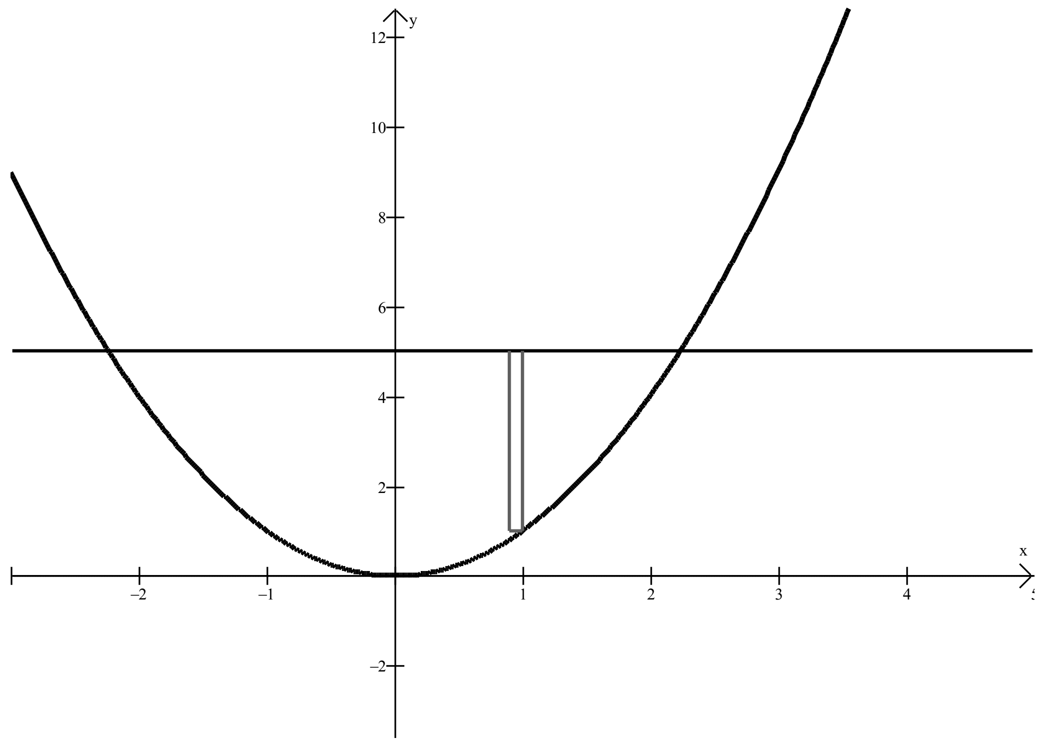
$V = 2\pi\int_a^b x \cdot y\,dx$ We cannot integrate y with respect to x so we will substitute out for y

$V = 2\pi\int_a^b x \cdot x^2 dx$ The expression for y is x^2

$V = 2\pi\int_1^3 x \cdot x^2 dx$ Our region extends from $x = 1$ to $x = 3$

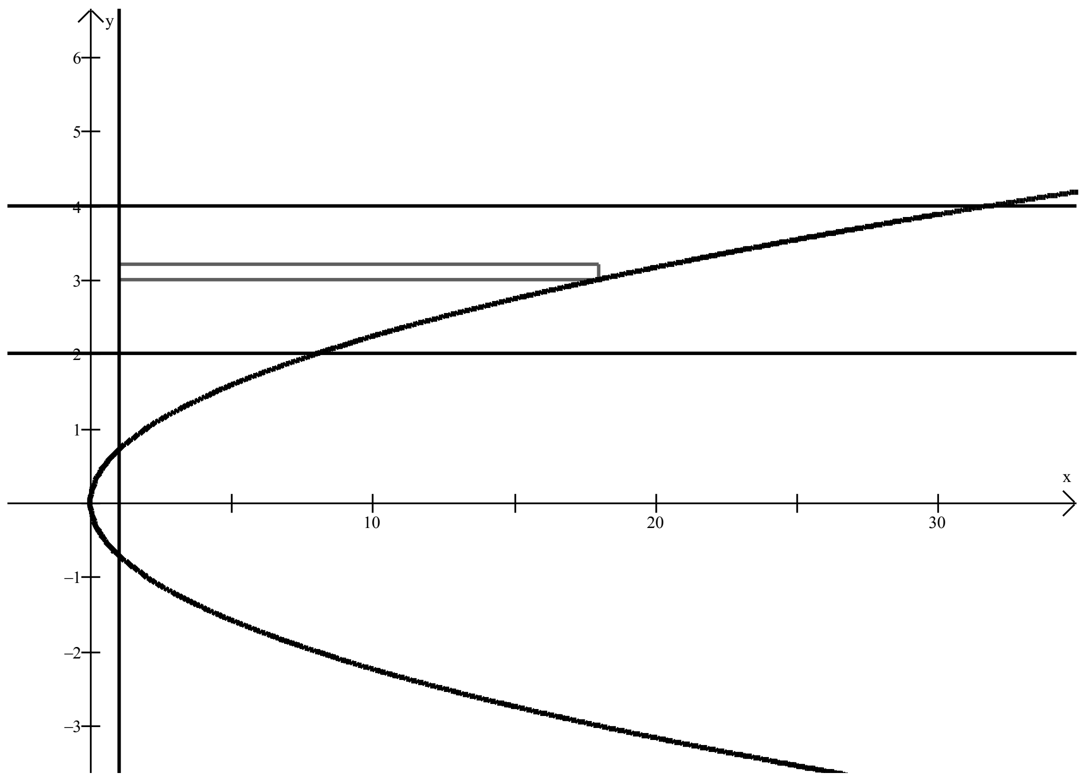
$V = 40\pi$

Ex 2 Let R be the region bounded by $y=x^2, y=5$, and $x=0$. Find the volume of the solid generated when R is rotated about the $y-$axis.



$V=2\pi\int_a^b x\cdot(5-y)\,dx$	Our pieces are parallel to the $y-$axis, so our integrand will contain dx. We cannot integrate y with respect to x so we will substitute out for y.
$V=2\pi\int_a^b x\cdot(5-x^2)\,dx$	The expression for y is x^2.
$V=2\pi\int_0^{\sqrt{5}} x\cdot(5-x^2)\,dx$	Our region extends from $x=0$ to $x=\sqrt{5}$
$V=50\pi/4$	

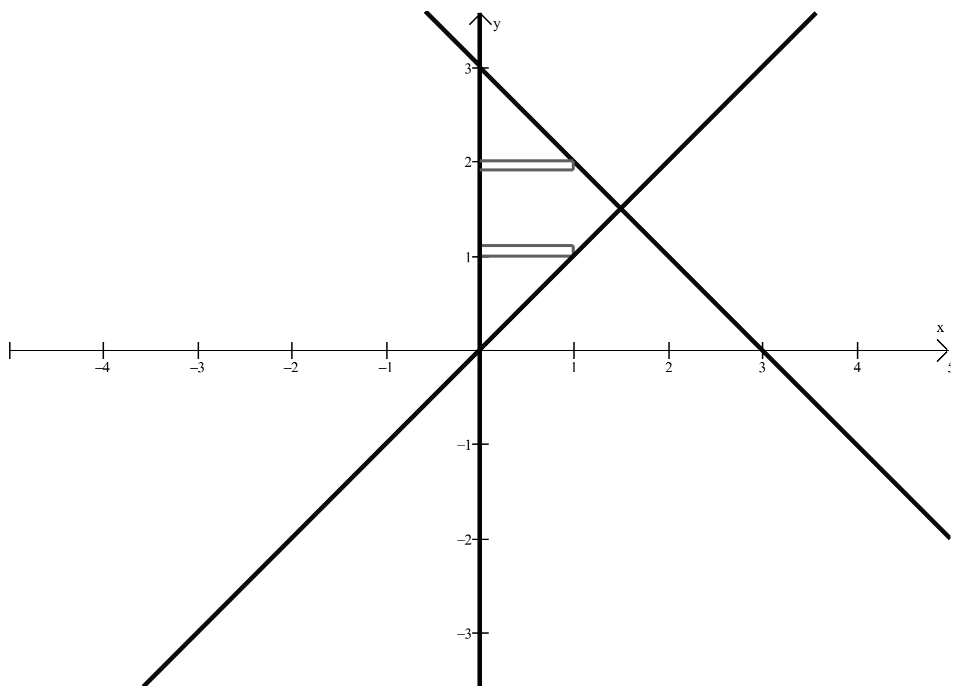
Ex 3 Let R be the region bounded by $x=2y^2, x=1,\ y=2$ and $y=4$. Find the volume of the solid generated when R is rotated about the $x-$axis.



$$V=2\pi\int_c^d y\cdot x\,dy$$

$$V=2\pi\int_2^4 y\cdot 2y^2 dy$$
$$V=4\pi\int_2^4 y^3 dy$$
$$V=240\pi$$

Ex 4 Let R be the region bounded by $y=x, x+y=3$, and $x=0$. Find the volume of the solid generated when R is rotated about the x – axis.



$$V = 2\pi\int_c^d y\cdot x\,dy$$

Our pieces are parallel to the x – axis, so our integrand will contain dy. But we will need two integrals as our region of rotation is bounded by different curves. The dividing line is $y = \frac{3}{2}$

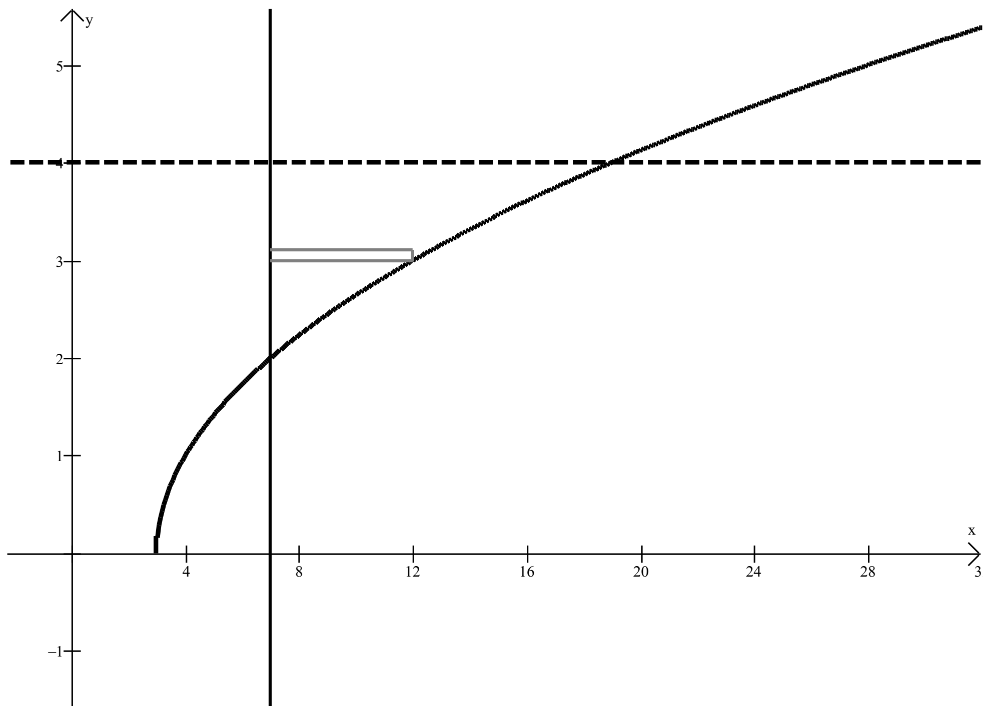
$$V = 2\pi\int_0^{3/2} y\cdot x_1 dy + 2\pi\int_{3/2}^{3} y\cdot x_2 dy$$

$$V = 2\pi\int_0^{3/2} y\cdot y\,dy + 2\pi\int_{3/2}^{3} y\cdot(-y+3)\,dy$$

$$V = \frac{27\pi}{4}$$

The Shell Method can be applied to rotating regions about lines other than the axes, just as the Disk and Washer Methods were.

Ex 5 Let *R* be the region bounded by the equations $y=\sqrt{x-3}$, the x – axis, and $x=7$. Find the volume of the solid generated when *R* is rotated about the line $y=4$.



$V=2\pi\int_c^d y\cdot x dy$ Our pieces are parallel to the x – axis, so our integrand will contain dy

$V=2\pi\int_c^d (4-y)\cdot(7-x)dy$ We cannot integrate x with respect to y so we will substitute out for x

$V=2\pi\int_c^d (4-y)\cdot\left(7-\left(y^2+3\right)\right)dy$ The expression for x is y^2+3

$V=2\pi\int_0^2 (4-y)\cdot\left(7-\left(y^2+3\right)\right)dy$ Our region extends from $y=0$ to $y=2$

$V=108.909$

The Washer Method yields the same result.

$$V = \pi \int_a^b \left(R^2 - r^2\right) dx$$

$$V = \pi \int_a^b \left(\left(y_1\right)^2 - \left(4 - y_2\right)^2\right) dx$$

$$V = \pi \int_3^7 \left[\left(4\right)^2 - \left(4 - \sqrt{x-3}\right)^2\right] dx$$

$$V = 108.909$$

6.8 Free Response Homework

Find the volume of the solid formed by rotating the described region about the given line.

1. $y = \sec x, y = 1, x = 0, x = \frac{\pi}{6}$; about the y-axis.

2. $y = x^3, x = 0, y = 8$; about the y-axis.

3. $y = \frac{1}{x^{2/3}}, x = 1, x = 8, y = 0$; about the y-axis.

4. $y = e^{-x^2}, y = 0, x = -2, x = 2$; about the x-axis.

5. $y = x(x-1)^2$ and the x-axis; about the y-axis.

6. $y = \sin x^2$ and the x-axis on $x \in \left[0, \sqrt{\pi}\right]$; about the y-axis.

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid formed by rotating the described region about the given line.

7. $y = \sqrt{x}, y = e^{-2x}, x = 1$; about the line $x = 1$.

8. $y = \ln\left(x^2 + 1\right), y = \cos x$; about the line $x = 2$.

9. $y = x^2, y = 2^x$; about the line $x = -1$.

Area and Volume Practice Chapter Test

CALCULATOR ALLOWED

1. The area of the region enclosed by $y = x^2 - 4$ and $y = x - 4$ is given by

a) $\int_0^1 (x - x^2)\,dx$ b) $\int_0^1 (x^2 - x)\,dx$ c) $\int_0^2 (x - x^2)\,dx$

d) $\int_0^2 (x^2 - x)\,dx$ e) $\int_0^4 (x^2 - x)\,dx$

---

2. Which of the following integrals gives the length of the graph $y = \tan x$ between $x=a$ to $x=b$ if $0 < a < b < \frac{\pi}{2}$?

a) $\int_a^b \sqrt{x^2 + \tan^2 x}\,dx$ b) $\int_a^b \sqrt{x + \tan x}\,dx$

c) $\int_a^b \sqrt{1 + \sec^2 x}\,dx$ d) $\int_a^b \sqrt{1 + \tan^2 x}\,dx$

e) $\int_a^b \sqrt{1 + \sec^4 x}\,dx$

---

3. Let R be the region in the first quadrant bounded by $y = e^{x/2}$, $y = 1$ and $x = \ln 3$. What is the volume of the solid generated when R is rotated about the x-axis?

a) 2.80 b) 2.83 c) 2.86 d) 2.89 e) 2.92

---

4. A region is bounded by $y = \frac{1}{\sqrt{x}}$, the x-axis, the line $x = m$, and the line $x = 2m$, where $m > 0$. A solid is formed by revolving the region about the x-axis. The volume of the solid

a) is independent of m.

b) increases as m increases.

c) decreases as m increases.

d) increases until $m = \frac{1}{2}$, then decreases.

e) is none of the above

---

5. Let R be the region in the first quadrant bounded by $y = \sin^{-1} x$, the y-axis, and $y = \frac{\pi}{2}$. Which of the following integrals gives the volume of the solid generated when R is rotated about the y-axis?

a) $\pi \int_0^{\pi/2} y^2 \, dy$

b) $\pi \int_0^1 \left(\sin^{-1} x\right)^2 dx$

c) $\pi \int_0^{\pi/2} \left(\sin^{-1} x\right)^2 dx$

d) $\pi \int_0^{\pi/2} (\sin y)^2 \, dy$

e) $\pi \int_0^1 (\sin y)^2 \, dy$

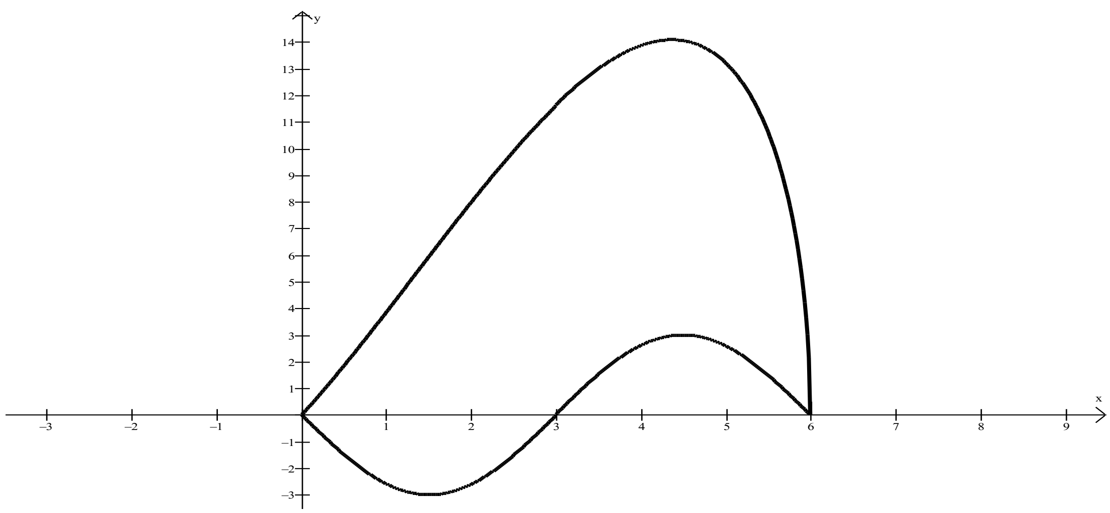
---

6. The base of a solid is the region enclosed by $y = \cos x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. If each cross-section of the solid perpendicular to the x-axis is a square, the volume of the solid is

a) $\frac{\pi}{4}$ b) $\frac{\pi^2}{4}$ c) $\frac{\pi}{2}$ d) $\frac{\pi^2}{2}$ e) 2

---

7. Let R be the region bounded by the graphs $f(x) = x\sqrt{12 + 4x - x^2}$ and $g(x) = -3\sin\left(\frac{\pi}{3}x\right)$ pictured below.



a) Find the area of R.

b) Find the volume of the figure if R is rotated around the line $y = -4$.

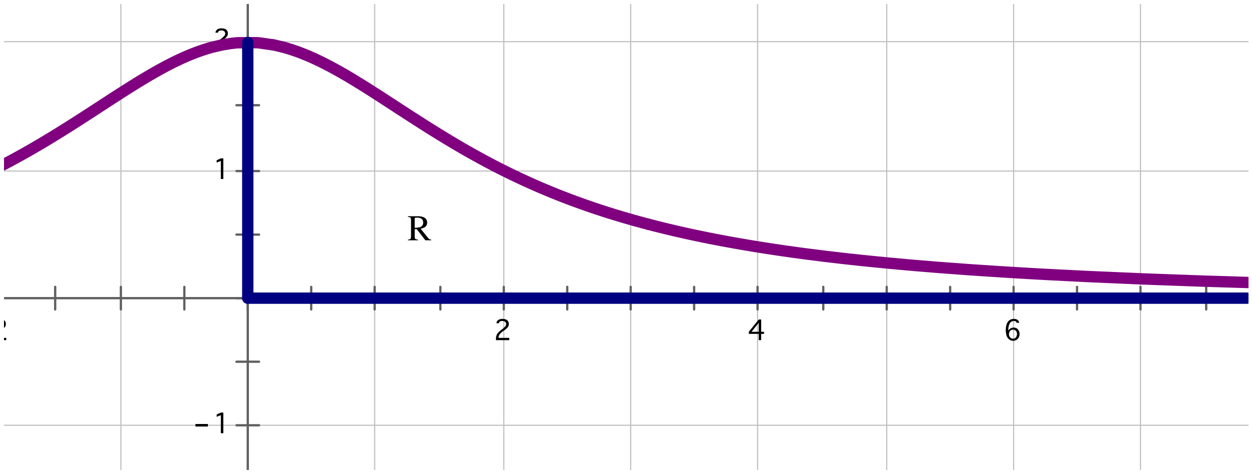
c) The region, R, is the base of a solid whose cross-sections are squares perpendicular to the x-axis. Find the volume of this solid.

---

8. Let S be the region in Quadrant I bounded by the graphs of $y = 2e^{-3x}$ and $x = 2$.

a. Find the area of region S. Show the anti-derivatives.

b. Find the perimeter of region S in #2.



9. The picture above is the graph of $f(x)=\dfrac{8}{x^2+4}$.

---

a. Find the area of the region in Quadrant I between $f(x)=\dfrac{8}{x^2+4}$, the line $x=2$, and the x-axis.

b. Set up, but do not solve, the formula for the volume of the solid formed by revolving region R above about the *x*-axis.

c. Set up, but do not solve, the formula for the volume of the solid formed by revolving region R above about **the *y*-axis**.

5.1 Free Response Answers

1.	19.5	2.	2.810	3.	60.252	4.	.5
5.	$\frac{32}{3}$	6.	$\frac{4}{3}$	7.	5.684	8.	$\frac{8}{3}$
9.	1.369	10.	9	11.	.350	12.	1.168
13.	2.106						

5.1 Multiple Choice Answers

a.	B	2.	A	3.	B	4.	B	5.	B	6.	B

5.2 Free Response Answers

1.	6.103	2.	99	3.	10.667	4.	.881
5.	1.106	6.	2.003	7.	34.553	8.	2
9.	18.268	10.	6.411	11.	6.158	12.	5.196

5.2 Multiple Choice Answers

1.	A	2.	B	3.	D	4.	D	5.	E	6.	E
7.	E										

5.3 Free Response Answers

1.	$\frac{\pi}{5}$	2.	10.036	3.	$\frac{\pi}{2}$	4.	$\frac{32\pi}{3}$
5.	2π	6.	38.024	7.	$\frac{\pi^2}{2}$	8.	2π

9.	$\frac{3\pi}{10}$	10.	1.207	11.	8π	12.	$\frac{592\pi}{15}$
13.	$\frac{768\pi}{7}$	14.	27.018	15.	94.612	16.	$2\pi\sqrt{3}-\frac{2\pi^2}{3}$

5.3 Multiple Choice Answers

1.	E	2.	B	3.	C	4.	B	5.	E	6.	A
7.	B	8.	D	9.	D						

5.4 Free Response Answers

1.	$\frac{\pi}{5}$	2.	$\frac{128\pi}{7}$	3.	8π	4.	$\frac{16\pi}{3}$
5.	2.986	6.	64π	7.	144π	8.	2.937
9.	$\frac{64\pi}{15}$	10.	$\frac{3\pi}{4}$	11.	$\frac{\pi}{2}$	12.	0.445
13.	$\frac{3\pi}{10}$	14.	52.12	15.	$\frac{128\pi}{5}$	16.	9.913

5.4 Multiple Choice Answers

1.	D	2.	E	3.	E	4.	C	5.	D
6.	B	7.	E	8.	D				

5.5 Free Response Answers

1.	$\frac{7424\pi}{15}$			2.	$\frac{5504\pi}{15}$		
3.	8.997	4.	7.632	5.	$\frac{10\pi}{21}$	6.	1.361
7.	$\frac{104\pi}{15}$	8.	$\frac{16\pi}{15}$	9.	$\frac{32\pi}{15}$	10.	$\frac{48\pi}{5}$
11.	3.447	12.	27.561	13.	374.525	14.	22.222

5.5 Multiple Choice Answers

1. C 2. D 3. E 4. D 5. C

5.6 Free Response Answers

1.	$\frac{64}{3}$	2.	$\frac{416}{15}$	3.	6.031	4.	3.242
5.	0.533	6.	6 1.067	7.	0.687	8.	341.333
9.	24	10.	2	11.	$\frac{16}{15}$	12.	$\frac{\sqrt{3}}{2}$
13.	.085	14.	916	15.	0.75		
16.	58	17a.	582π	17b.	13186.456		

5.6 Multiple Choice Answers

1. A 2. B 3. E 4. D 5. B 6. D

7. A 8. C 9. A

5.7 Free Response Answers

1a. $\frac{4}{3}$ 1b. $V=\frac{\pi}{8}\int_0^2\left(\frac{2x^2-x^3}{2}\right)^2dx$ 1c. $\frac{16\pi}{5}$

2a. $\frac{88\pi}{7}$ 2b. 63.827 2c. $V=\int_0^4\left((-2y)^{1/3}-\frac{1}{8}y^2\right)^2dy$

3a. $\frac{16}{3}$ 3b. 16 3c. $V=\pi\int_0^2\left(\left[4-\left(y^2-1\right)\right]^2-[4-3]^2\right)dy$

4a. $Area=\frac{124}{15}$ 4b. 625.526

4c. $Volume=\pi\int_0^{16}\left(\left(y^{1/4}\right)^2-\left(\frac{y^2}{128}\right)^2\right)dy=\pi\int_0^{16}\left(y^{1/2}-\frac{y^4}{16384}\right)dy$

5a. 23.326 5b. 676.640 5c. 170.182

6a. 2 6b. 16.965 6c. 11.310

7a. $A=\frac{9}{2}\ln 3$ 7b. 10.747 7c. 28.274

8a. 27.180 8b. 599.644 8c. 107.233 8d. 66.842

9a. 1.089 9b. 6.630 9c. 17.665

9d. $V=\int_0^1\left[[k-x(1-x)]^2-\left[k-\sqrt[4]{2x}(x-1)\right]^2\right]dx$

10a. $A_R=6.185;\ A_S=2.193$ 10b. $V_S=23.814$

10c. $V=55.345$ 10d. $V_S=8.607$

12a. $A_R=0.071$ 12b. $A_S=0.328$ 12c. $V_S=5.837$

13d. $V_S = .007$

13a. 359ft^2 13b. 2236ft^3

13c. $w'(15) \approx .39$. Tangent Line: $w - 15.5 = .39(x - 15)$

13d. $0.9929 \,{}^{\text{ft}}\!/_{\text{sec}}$

14a. $\frac{1}{27}\int_0^{27} k(x)\ dx$ 14b. 55.5425mm^3 14c. IVT 14d. MVT

15a. $\frac{20\pi}{7}\text{ft}^3$ 15b. $\frac{5\pi}{56}(8h)^{7/5}$ 15c. $-7.800 \,{}^{\text{ft}^2}\!/_{\text{hr}}$

15d. $\frac{dV}{dt} = \pi(8h)^{2/5}\frac{dh}{dt} = \pi\left[(8h)^{1/5}\right]^2\frac{dh}{dt} = A\frac{dh}{dt}$

16a. 1.442ft^2 16b. $24{,}780.8\text{ft}^3$ 16c. $5.109 \,{}^{\text{ft}^3}\!/_{\text{min}}$ 16d. Yes.

17a. $V = 200\pi\ in^3$ 17b. $L = 21.965 in$

17c. $\tan\theta = \frac{y}{4 - x}$ 17d. $-0.013 \,{}^{rad}\!/_{\text{min}}$

18a. $5352343.038\ ft^3$ 18b. $V = 125{,}544.038\ ft^3$

18c. $\$418{,}480.13$ 18d. $\left.\frac{dS}{dt}\right| = 6.680\ {}^\circ F.$

19a. $V = 19{,}203.928\text{in}^3$ 19b. $A = 3751.100\ in^2$ 19c. $\frac{dh}{dt} = -9.778 \,{}^{\text{in}}\!/_{\text{hr}}$

20a. -19.985 20b. 44.021 20c. 36.405 20d. 0.255

21. AP Central

5.8 Free Response Answers

1.	.064	2.	60.319	3.	70.689	4.	1.969
5.	.209	6.	2π	7.	.554	8.	14.676
9.	55.428						

Practice Test Answers

1. A 2. E 3. B 4. A 5. D 6. C

7a. 54.295 b. 1012.159 c. 713.902

8a. 0.665 b. 5.479

9a. π

9b. $V=\pi\int_0^2\left(\frac{8}{x^2+4}\right)^2 dx$

9c. $V=\pi\int_0^2\left(\sqrt{\frac{8}{y}-4}\right)^2 dx$