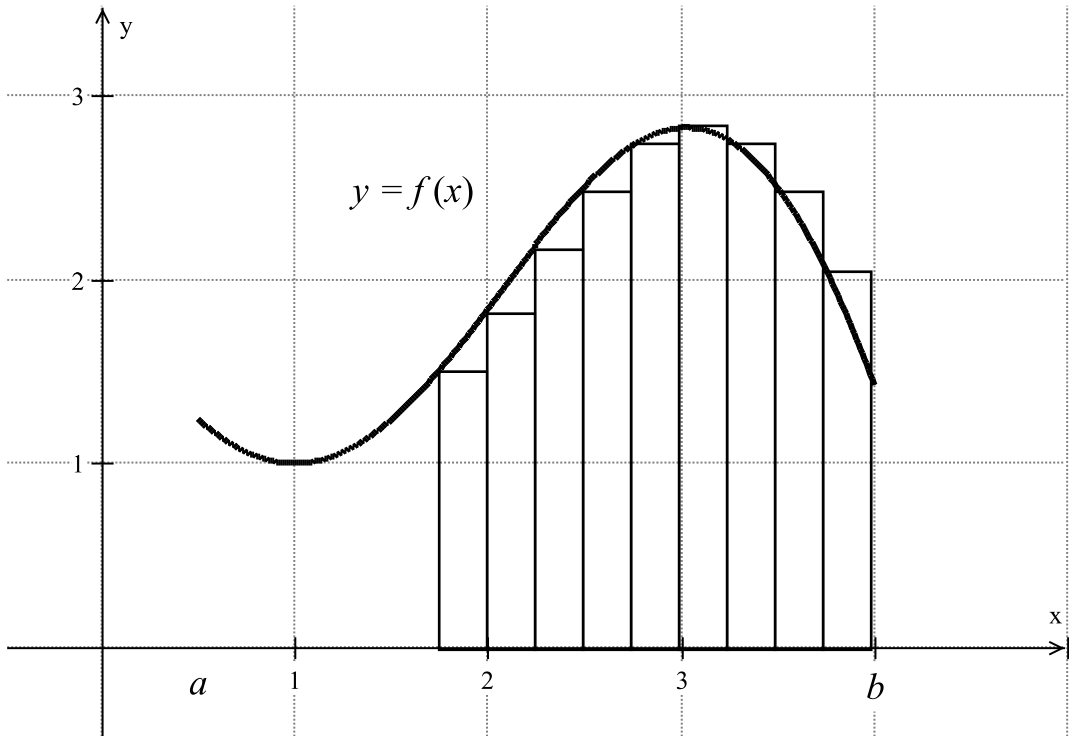
Chapter 4 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on the mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview of the last chapter, Anti-derivatives are known as Indefinite Integrals and this is because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an Analytic Geometry context of area. Though it is not necessary to know the theory behind this to be able to do it, the theory is a major subject of Integral Calculus, so we will explore it briefly in Section 3.0.

4.0 The Limit Definition of the Definite Integral

We know, from Geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider a figure where one side is the curve $y = f(x)$ and the other sides are the x-axis and the lines $x = a$ and $x = b$.



As we can see above, the area can be approximated by rectangles whose height is the y-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. The area of each rectangle would be $f(x) \cdot \Delta x$ and the total area of n rectangles would be $A = \sum_{i=1}^{n} f(x_i) \cdot \Delta x$. If we could make an infinite number of rectangles (which would be infinitely thin), we would have the exact area. The rectangles can be drawn several ways--with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But once they become infinitely thin, it will not matter how they were drawn--they will have virtually no width (represented by dx instead of the Δx) and a height equal to the y-value of the curve.

We can make an infinite number of rectangles mathematically by taking the Limit as n approaches infinity, or

$$\lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x.$$

Therefore,

$$\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x \int_a^b f(x)dx$$

b is the "upper bound" and a is the "lower bound." Mathematicians sometimes nuance this statement as

$$\boxed{\int_a^b f(x)dx = \lim_{n \to \infty} \sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x, \text{ where } \Delta x = \frac{b-a}{n}}$$

Objective
Understand the Limit Definition of the Definite Integral.

This has led to the following AP multiple-choice problems:

Ex 1 Which of the following limits is equal to $\int_2^5 x^2\,dx$?

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{1}{n}$

(B) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{k}{n}\right)^2 \frac{3}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{1}{n}$

(D) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(2 + \frac{3k}{n}\right)^2 \frac{3}{n}$

Since $\Delta x = \dfrac{b-a}{n} = \dfrac{5-2}{n} = \dfrac{3}{n}$, the answer must be (D).

Ex 2 Which of the following definite integrals are equal $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(-1 + \dfrac{4k}{n}\right)^2 \dfrac{4}{n}$?

I. $\int_{-1}^{3} x^2\,dx$ II. $\int_{0}^{4} (-1 + x)^2\,dx$ III. $\int_{0}^{1} 4(-1 + 4x)^2\,dx$

(A) I only (B) II only (C) III only

(D) I and II only (E) I, II, and III only

$\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(-1 + \dfrac{4k}{n}\right)^2 \dfrac{4}{n} = \int_{-1}^{3} x^2\,dx$ by a straight-forward application of the

Definition, with $\dfrac{4}{n} = \dfrac{b-(-1)}{n} \rightarrow b = 3$.

But if $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(-1 + \dfrac{4k}{n}\right)^2 \dfrac{4}{n}$ is viewed as $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(0 - 1 + \dfrac{4k}{n}\right)^2 \dfrac{4}{n}$--

meaning $a = 0$--then this limit is equal to $\int_{0}^{4} (-1 + x)^2\,dx$.

And if $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(-1 + \dfrac{4k}{n}\right)^2 \dfrac{4}{n}$ is viewed as $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} 4\left(0 - 1 + 4\dfrac{k}{n}\right)^2 \dfrac{1}{n}$, then the limit is equal to $\int_0^1 4(-1+4x)^2\,dx$. Therefore, the answer is (E), all three.

4.0 Multiple Choice Homework

1. For which of the following integrals is $\lim_{n \to \infty} \sum_{k=1}^{n} \sin\left(7 + \frac{4k}{n}\right)\frac{4}{n}$ the right hand Riemann sum approximation with n subintervals of equal length?

(A) $\int_{7}^{11} \sin(x)\,dx$ (B) $\int_{7}^{11} \sin(4x+7)\,dx$

(C) $\int_{7}^{11} \sin(4x)\,dx$ (D) $\int_{0}^{4} \sin(x+7)\,dx$

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2. For which of the following integrals is $\lim_{n \to \infty} \sum_{k=1}^{n} \ln\left(2 + \frac{5k}{n}\right)\frac{5}{n}$ the right hand Riemann sum approximation with n subintervals of equal length?

(A) $\int_{2}^{7} \ln(x)\,dx$ (B) $\int_{2}^{7} \ln(2+x)\,dx$

(C) $\int_{0}^{5} \ln(2+5x)\,dx$ (D) $\int_{2}^{7} \ln(2+5x)\,dx$

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3. Which of the following limits is equal to $\int_{3}^{9} \left(x^2 - 1\right)\,dx$?

(A) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\left(3 + \frac{6k}{n}\right)^2 - 1\right)\frac{6}{n}$ (B) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\left(2 + \frac{6k}{n}\right)^2\right)\frac{6}{n}$

(C) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(3 + \frac{6k}{n}\right)^2 \frac{6}{n}$ (D) $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\left(-1 + \frac{6k}{n}\right)^2 + 3\right)\frac{6}{n}$

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4. Which of the following limits is equal to $\int_0^7 \sqrt[3]{x}\, dx$?

(A) $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(\frac{7}{n}\right) \sqrt[3]{\frac{7k}{n}}$

(B) $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(\frac{7}{n}\right) \sqrt[3]{7 + \frac{k}{n}}$

(C) $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \left(7 + \frac{k}{n}\right)^{1/3} \left(\frac{1}{n}\right)$

(D) $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \sqrt[3]{\frac{49k}{n^2}}$

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5. Which of the following integrals is equal to $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} e^{\left(1 + \frac{k}{n}\right)} \frac{1}{n}$?

(A) $\int_0^1 e^x\, dx$

(B) $\int_1^2 e^x\, dx$

(C) $\int_1^2 e^{x+1}\, dx$

(D) $\int_0^2 e^{x+1}\, dx$

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6. Which of the following statements is false about $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \tan\left(3 + \frac{2k}{n}\right) \frac{2}{n}$
?

(A) $\int_3^5 \tan x\, dx$

(B) $\int_0^2 \tan(x+3)\, dx$

(C) $\int_0^1 2\tan(2x+3)\, dx$

(D) None of these

4.1 The Fundamental Theorem of Calculus

The most practical rule for Definite Integrals is:

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a,b]$, then

1) $\frac{d}{dx}\int_c^x f(t)\ dt = f(x)$ or $\frac{d}{dx}\int_c^u f(t)\ dt = f(u) \bullet D_u$

2) If $F'(x) = f(x)$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know--that an integral is an anti-derivative. The second part of the Fundamental Theorem says the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

This idea of the integral meaning the area may not make sense initially, mainly because we are used to Geometry, where area is always measured in square units. But that is only because the length and width are always measured in the same kind of units, so multiplying length and with must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the *x*-axis is time in seconds and the *y*-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet/sec--that is, feet. So the area under the curve equals the distance traveled in feet. In other words, the integral of velocity is distance.

Objectives

Evaluate Definite Integrals
Find the average value of a continuous function over a given interval
Differentiate integral expressions with the variable in the boundary

Let us first consider Part 2 of the Fundamental Theorem, since it has a very practical application. This part of the Fundamental Theorem gives us a method for evaluating definite integrals.

Ex 1 Evaluate $\int_2^8 (4x+3)\,dx$

$\int_2^8 (4x+3)\,dx = 2x^2 + 3x\Big|_2^8$ The antiderivative of $4x + 3$ is $2x^2 + 3x$. We use this notation when we apply the Fundamental Theorem

$= \left[(128+24)-(8+6)\right]$ Plug the upper limit of integration into the antiderivative and subtract it from the lower limit when plugged into the antiderivative

$= 138$ Check with Math 9.

Ex 2 Evaluate $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

$$\begin{aligned}\int_1^4 \frac{1}{\sqrt{x}}\,dx &= 2\sqrt{x}\Big|_1^4\\ &= \left[(4)-(2)\right]\\ &= 2\end{aligned}$$

Ex 3 Evaluate $\int_0^{\pi/2} \sin x\,dx$

$$\begin{aligned}\int_0^{\pi/2} \sin x\,dx &= -\cos x\Big|_0^{\pi/2}\\ &= -\left[0-1\right]\\ &= 1\end{aligned}$$

Ex 4 Evaluate $\int_1^2 \frac{4+u^2}{u^3}du$

$$\int_1^2 \frac{4+u^2}{u^3}du = \int_1^2 \left(4u^{-3}+u^{-1}\right)du$$
$$= -2u^{-2} + \ln|u| \Big|_1^2$$
$$= \left[\left(-2\cdot\frac{1}{2^2}+\ln 2\right)-\left(-2+\ln 1\right)\right]$$
$$= \frac{3}{2}+\ln 2$$

Ex 5 Evaluate $\int_{-5}^5 \frac{1}{x^3}dx$

This is a trick! We use the Fundamental Theorem on this integral because the curve is not continuous on $x \in [a,b]$. When $x = 0$, $\frac{1}{x^3}$ does not exist, so the Fundamental Theorem of Calculus does not apply.

Properties of Definite Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

Ex 6 If $\int_{-5}^{2} f(x)\ dx = -17$ and $\int_{5}^{2} f(x)\ dx = -4$, find $\int_{-5}^{5} f(x)dx$.

$$\int_{-5}^{5} f(x)\ dx = \int_{-5}^{2} f(x)\ dx + \int_{2}^{5} f(x)\ dx$$
$$= \int_{-5}^{2} f(x)\ dx - \int_{5}^{2} f(x)\ dx$$
$$= -17 - (-4)$$
$$= -13$$

One simple application of the definite integral is the Average Value Theorem. We all recall how to find the average of a finite set of numbers, namely, the total of the numbers divided by how many numbers there are. But what does it mean to take the average value of a continuous function? Let's say you drive from home to school – what was your average velocity (velocity is continuous)? What was the average temperature today (temperature is continuous)? What was your average height for the first 15 years of your life (height is continuous)? Since the integral is the sum of infinite number of function values, the formula below answers those questions.

Average Value Formula:

The average value of a function f on a closed interval $[a,b]$ is defined by

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it can start to make a little more sense.

$$\int_a^b f(x)\ dx = F(b) - F(a)$$ The Fundamental Theorem of Calculus

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{1}{b-a}\left[F(b) - F(a)\right] = \frac{F(b)-F(a)}{b-a}$$

Notice that this is just the average slope for $F(x)$ on $x \in [a,b]$. The average slope of $F(x)$ would be the average value of $F'(x)$. But since the definition in the Fundamental Theorem of Calculus says that $F'(x) = f(x)$, this is actually just the average value of $f(x)$.

Ex 7 Find the average value of $f(x) = x^2 + 1$ on $[0,5]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{avg} = \frac{1}{5-0}\int_0^5 \left(x^2+1\right)dx$$ Substitute in the function and interval

$$f_{avg} = {}^{28}\!/_{3}$$ Use Math 9 (or integrate analytically) to calculate answer

Ex 8 Find the average value of $h(\theta) = \sec\theta\tan\theta$ on $\left[0, \frac{\pi}{4}\right]$.

$$\begin{aligned} h_{avg} &= \frac{1}{b-a}\int_a^b h(\theta)\,d\theta \\ &= \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta \\ &= .524 \end{aligned}$$

Now let's take a look at the First Part of the Fundamental Theorem of Calculus:

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \text{ or } \frac{d}{dx}\int_a^u f(t)\,dt = f(u)\cdot\frac{du}{dx}$$

This part of the Fundamental Theorem of Calculus says that derivatives and integrals are inverses operations of each other. Confusion usually pops up with the seemingly “extra” variable *t* showing up. It is a dummy variable, somewhat like the parameter in parametric mode. It would disappear in the process of integrating.

Ex 9 Use Part I of the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t)=\int_2^t (4x+3)dx$.

$$f'(t)=\frac{d}{dt}\int_2^t (4x+3)dx=4t+3$$

Ex 10 Use Part I of the Fundamental Theorem of Calculus to find $f'(t)$ if $f(t)=\int_2^{3t^2} (4x+3)dx$.

$$f'(t)=\frac{d}{dt}\int_2^{3t^2} (4x+3)dx=\left(4\left(3t^2\right)+3\right)(6t)=72t^3+18t$$

In terms of the AP Test, the symbology is what is important and usually only comes into a process with L’Hopital’s Rule and limits.

Ex 11 $\lim\limits_{x\to 2}\dfrac{\int_2^x \ln t\,dt}{x^2-4}=$

$f(x)=\lim\limits_{x\to 2}\int_2^x \ln t\,dt=\int_2^2 \ln t\,dt=0$ and $g(x)=\lim\limits_{x\to 2}\left(x^2-4\right)=2^2-4=0$, so L’Hopital’s Rule applies:

$$\lim_{x\to 2}\frac{\int_2^x \ln t\,dt}{x^2-4}\overset{L'H}{=}\lim_{x\to 2}\frac{\frac{d}{dx}\int_2^x \ln t\,dt}{\frac{d}{dx}\left(x^2-4\right)}=\lim_{x\to 2}\frac{\ln x}{2x}=\frac{\ln 2}{4}$$

4.1 Free Response Homework

Use Part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

1. $\int_{-1}^{3} x^5 dx$

2. $\int_{2}^{7} (5x-1)\,dx$

3. $\int_{-5}^{5} \frac{2}{x^3} dx$

4. $\int_{-3}^{-1} \left(\frac{x^7 - 4x^3 - e}{x} \right) dx$

5. $\int_{1}^{2} \frac{3}{t^4} dt$

6. $\int_{\pi/4}^{3\pi/4} \csc y \cot y\, dy$

7. $\int_{0}^{\pi/4} \sec^2 y\, dy$

8. $\int_{1}^{9} \frac{3}{2z} dz$

9. $\int_{0}^{e^2-1} \frac{1}{x+1} dx$

10. $\int_{\pi}^{\frac{5\pi}{4}} \sin y\, dy$

11. $\int_{3}^{5} (x^2+5x+6)\,dx$

12. $\int_{3}^{e^2+2} \frac{1}{x-2} dx$

13. $\int_{\pi}^{\frac{3\pi}{4}} \cos y\, dy$.

14. $\int_{5}^{e^3+4} \frac{1}{x-4} dx$

15. $\int_{1}^{2} \left(\frac{x^2 - 4x + 7}{x} \right) dx$

16. $\int_{1}^{16} \left(\frac{2x^2 - 1}{\sqrt[4]{x}} \right) dx$

Use the following values for problems 17 – 27 to evaluate the given integrals

$\int_{-2}^{5} f(x)dx = -2$	$\int_{1}^{5} f(x)dx = 3$
$\int_{-2}^{1} g(x)dx = 4$	$\int_{5}^{1} g(x)dx = 9$
$\int_{1}^{5} h(x)dx = 7$	$\int_{5}^{-2} h(x)dx = -6$

17. $\int_{-2}^{1} f(x)dx =$

18. $\int_{-2}^{5} g(x)dx =$

19. $\int_{-2}^{1} h(x)dx =$

20. $\int_{1}^{5} [f(x) - g(x)]dx =$

21. $\int_{-2}^{5} [g(x) + h(x)]dx =$

22. $\int_{-2}^{1} [h(x) - f(x)]dx =$

23. $\int_{-2}^{5} [h(x) + f(x)]dx =$

24. $\int_{1}^{5} [2f(x) + 3h(x)]dx =$

25. $\int_{-2}^{1} [2f(x) - 3g(x)]dx =$

26. $\int_{-2}^{5} \left[\frac{1}{2}g(x) + 4h(x)\right]dx =$

27. $\int_{1}^{5} \left[\frac{1}{3}h(x) + 2f(x)\right]dx =$

28. $\int_{0}^{2} f(x)\, dx$ where $f(x) = \begin{cases} x^4, & \text{if } 0 \le x < 1 \\ x^5, & \text{if } 1 \le x \le 2 \end{cases}$

Find the Average Value of each of the following functions.

29. $F(x)=(x-3)^2$ on $x\in[3,7]$

30. $H(x)=\sqrt{x}$ on $x\in[0,3]$

31. $F(x)=\sec^2 x$ on $x\in\left[0,\frac{\pi}{4}\right]$

32. $F(x)=\frac{1}{x}$ on $x\in[1,3]$

33. $f(t)=t^2-\sqrt{t}+5$ on $t\in[1,4]$

34. $f(t)=t^2-\sqrt{t}+5$ on $t\in[4,9]$

35. If a cookie taken out of a 450˚F oven cools in a 60˚F room, then according to Newton's Law of Cooling, the temperature of the cookie *t* minutes after it has been taken out of the oven is given by

$$T(t)=60+390e^{-.205t}.$$

What is the average value of the cookie during its first 10 minutes out of the oven?

36. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the given equation

$$L(t)=10-3\cos\left(\frac{\pi t}{182}\right),$$

where *t* the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

37. During one summer in the Sunset, the temperature is modeled by the function $T(t)=50+15\sin\frac{\pi}{12}t$, where T is measured in F° and t is measured in hours after 7 am. What is the average temperature in the Sunset during the six-hour Chemistry class that runs from 9 am to 3 pm?

Use Part I of the Fundamental Theorem of Calculus to find the derivative of the function.

39. $g(y)=\int_2^y t^2 \sin t\,dt$

39. $g(x)=\int_0^x \sqrt{1+2t}\,dt$

40. $F(x)=\int_x^2 \cos(t^2)\,dt$

41. $h(x)=\int_2^{1/x} \arctan t\,dt$

42. $y=\int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt$

43. If $F(x)=\int_1^x f(t)\,dt$, where $f(t)=\int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du$, find $F''(2)$.

44. $\frac{d}{dx}\left[\int_e^{x^2} \ln\left(t^2+1\right)dt\right]$

45. If $h(x)=\int_\pi^{\sqrt{x}} e^{5t}\,dt$, find $h'(x)$

46. $\frac{d}{dx}\int_{10}^{x^2} t\ln(t)\,dt$

47. $h(m)=\int_5^{\cos m} t^2 \cos^{-1}(t)\,dt$, find $h'(m)$

48. $h(y)=\int_5^{\ln y} \frac{e^t}{t^4}\,dt$, find $h'(y)$

49. $\frac{d}{dx}\int_{e^x}^{5} \left(t^3+t+1\right)dt$

<u>4.1 Multiple Choice Homework</u>

1. If the function $y=x^3$ has an average value of 9 on $x\in[0, k]$, then $k=$

a) 3 b) $\sqrt{3}$ c) $\sqrt[3]{18}$ d) $\sqrt[4]{36}$ e) $\sqrt[3]{36}$

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2. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1, 2]$

a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

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3. If the average of the function $f(x) = |x - a|$ on $[-1, 1]$ is $\frac{5}{4}$, what is/are the values of a?

a) ± 1 b) $\pm\frac{1}{2}$ c) $\pm\frac{1}{4}$ d) 0 e) None of these

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4. What is the average rate of change of the function $f(x) = x^4 - 5x$ on the closed interval [0, 3] ?

a) 8.5 b) 8.7 c) 22 d) 33 e) 66

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5. The average value of $y = e^x \cos x$ on $x \in \left[0, \frac{\pi}{2}\right]$ is

a) 0 b) 1.213 c) 1.905 d) 2.425 e) 3.810

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6. $\lim_{x \to 2} \dfrac{x^2 - 4}{\int_2^x \cos(\pi t)\, dt} =$

a) 0 b) 1 c) 2 d) 4 e) DNE

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7. $\lim_{x\to\pi} \frac{\int_{\pi}^{x} \left(\cos^2 t\right) dt}{\sin 2x} =$

a) -1 b) $-\frac{1}{2}$ c) 0 d) $\frac{1}{2}$ e) 1

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8. $\lim_{x\to 2} \frac{\int_{-2}^{x} t^3 dt}{x^2 - 4}$

a) 0 b) 2 c) 4 d) 6 e) DNE

---

9. $\lim_{x\to 1} \frac{\int_{1}^{x} e^{t^2} dt}{x^2 - 1}$

a) 0 b) 1 c) $\frac{e}{2}$ d) e e) DNE

---

10. If $\int_{-5}^{2} f(x)\, dx = -17$ and $\int_{5}^{2} f(x)\, dx = -4$, then $\int_{-5}^{5} f(x)\, dx =$

a) -21 b) -13 c) 0 d) 13 e) 21

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11. Let f and g be continuous functions such that $\int_{0}^{6} f(x)\, dx = 9$, $\int_{3}^{6} f(x)\, dx = 5$, and $\int_{3}^{0} g(x)\, dx = -7$. What is the value of $\int_{0}^{3} \left(\frac{1}{2} f(x) - 3g(x) \right) dx =$

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

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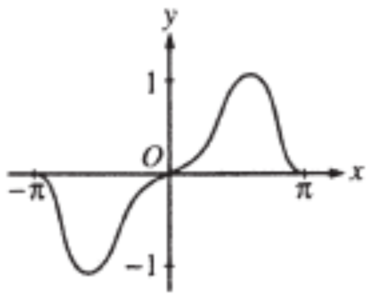
12. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $x \in [-\pi, \pi]$?

(A)



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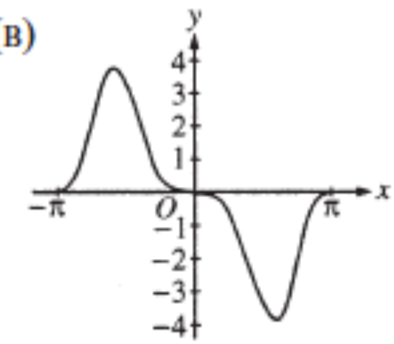

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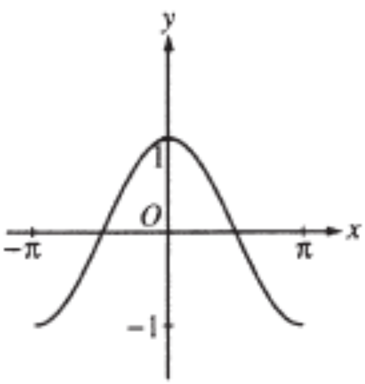

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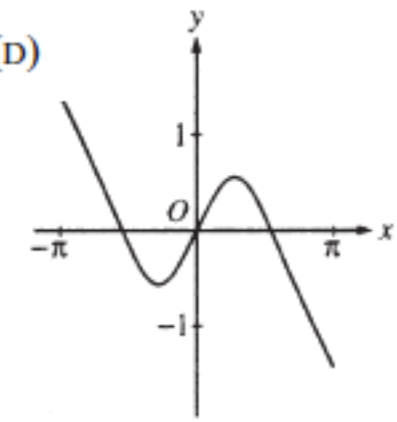
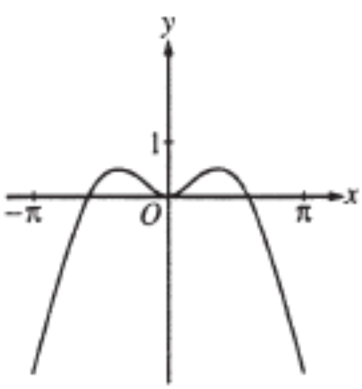


(E)



4.2 Definite Integrals and The Substitution Rule

Let's revisit u – subs with definite integrals and pick up a couple of more properties for the definite integral.

Objectives
Evaluate definite integrals using the Fundamental Theorem of Calculus. Evaluate definite integrals applying the Substitution Rule, when appropriate. Use proper notation when evaluating these integrals.

Ex 1 Evaluate $\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt$.

Let $u = t^3+1$ $\quad u(0)=1$

$du = 3t^2 dt$ $\quad u(2)=9$

$$\begin{aligned}\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt &= \frac{1}{3}\int_0^2 \left(3t^2\sqrt{t^3+1}\right)dt \\ &= \frac{1}{3}\int_1^9 \left(\sqrt{u}\right)du \\ &= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^9 \\ &= \frac{2}{9}\left[9^{3/2}-1^{3/2}\right] \\ &= \frac{52}{9}\end{aligned}$$

Ex 2   Evaluate $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx.$

$$\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx \qquad = \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \cos u \; du$$

Let $u = x^2$ $u(0) = 0$

$du = 2xdx$ $u(\sqrt{\pi}) = \pi$

$$= \left.\frac{1}{2}\sin u\right|_0^{\pi}$$

$=0$

Ex 3 $\int_1^2 \frac{e^{1/x}}{x^2}dx$

$$\int_1^2 \frac{e^{1/x}}{x^2}dx \qquad = -\int_1^{1/2} e^u du$$

Let$u = 1/x$

$du = -1/x^2 dx$

$u(1) = 1$

$u(2) = 1/2$

$$= -\left. e^u \right|_1^{1/2}$$

$= -e^{1/2} + e$

Ex 4 $\int_{2}^{0} x\sqrt{x^2+a^2}\,dx \qquad a>0$

$$\int_{2}^{0} x\sqrt{x^2+a^2}\,dx = -\int_{0}^{2} x\sqrt{x^2+a^2}\,dx$$

Let $u = x^2 + a^2$

$du = 2xdx$

$$= -\int_{0}^{2} x\sqrt{x^2+a^2}\,dx$$

$$= -\frac{1}{2}\int_{a^2}^{4+a^2} \sqrt{u}\,du$$

$=$

$$= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{a^2}^{4+a^2}$$

$$= -\frac{1}{3}\left[\left(4+a^2\right)^{3/2} - a^3\right]$$

Ex 5 Find the average value of $f(x) = x\sin x^2$ on $[0,5]$.

$$f_{avg} = \frac{1}{b-a}\int_{a}^{b} f(x)\,dx$$

$$f_{avg} = \frac{1}{5-0}\int_{0}^{5} \left(x\sin x^2\right)dx$$

Let $u = x^2 \qquad u(0) = 0$

$du = 2xdx \qquad u(5) = 25$

$$f_{avg} = \frac{1}{5}\left[\frac{1}{2}\int_{0}^{5} \left(2x\sin x^2\right)dx\right]$$

$$f_{avg} = \frac{1}{5}\left[\frac{1}{2}\int_{0}^{5} \left(2x\sin x^2\right)dx\right] = \frac{1}{10}\int_{0}^{25} (\sin u)\,du$$

Note that the average value set-up occurs before the substitution, therefore, the coefficient is not affected by the substitution **of the boundaries**.

$$f_{avg} = -\frac{1}{10}\left[\cos u\right]_{0}^{25} = -8.797\,E\,-4 \;=\; 0$$

4.2 Free Response Homework

Evaluate the definite integral, if it exists.

1. $\int_0^1 x^2(1+2x^3)^5 dx$

2. $\int_0^{\pi} \sec^2(t/4)dt$

3. $\int_1^2 \frac{e^{1/x}}{x^2} dx$

4. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta} d\theta$

5. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

6. $\int_{-1}^{2} \frac{dx}{2x+5}$

7. $\int_0^{\pi} \frac{\sin x}{2-\cos x} dx$

8. $\int_2^4 \frac{dx}{xLnx}$

9. $\int_0^{Ln2} \frac{e^x}{1+e^{2x}} dx$

10. $\int_{\pi/6}^{\pi/2} \cos^5 x \sin x dx$

11. $\int_{\pi}^{2\pi} \cos\frac{1}{2}\theta \, d\theta$

12. $\int_0^2 f(x)\, dx$where $f(x)=\begin{cases} x^4, & if\ 0 \le x < 1 \\ x^5, & if\ 1 \le x \le 2 \end{cases}$

13. $\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$

14. $\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y} dy$

15. $\int_0^{\pi} \frac{\cos x}{2+\sin x} dx$

16. $\int_0^{\pi} \frac{\sin y}{2+\cos y} dy$

17. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec(m^2)\tan(m^2)\,dm$

18. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\tan^3 x\,dx$

19. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

20. $\displaystyle\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$

21. $\displaystyle\int_{\sqrt{3}}^{\sqrt{4}} ye^{y^2-3}\,dy$

22. $\displaystyle\int_2^{e^3+1} \frac{(\ln(x-1))^4}{x-1}\,dx$

Find the Average Value of each of the following functions.

23. $f(x) = \cos x\sin^4 x$ on $x \in [0,\ \pi]$

24. $g(x) = xe^{-x^2}$ on $x \in [1,\ 5]$

25. $G(x) = \dfrac{x}{(1+x^2)^3}$ on $x \in [0,\ 2]$

26. $h(x) = \dfrac{x}{(1+x^2)^2}$ on $x \in [0,\ 4]$

4.2 Multiple Choice Homework

1. $\displaystyle\int_1^4 \frac{dx}{(1+\sqrt{x})^2\sqrt{x}}$

a) $\dfrac{6}{5}$ b) $\dfrac{1}{3}$ c) $\dfrac{2}{3}$ d) $\dfrac{4}{9}$ e) $\dfrac{3}{2}$

---

2. If $\int_1^4 h(x)dx = 6$, then $\int_1^4 h(5-x)dx =$

a) -6 b) -1 c) 0 d) 3 e) 6

---

3. $\int_e^{e^2} \frac{1}{x\ln x} dx =$

a) $\ln(\ln(2))$ b) $\frac{2}{e^2}$ c) $\ln 2$ d) $\frac{1-2e}{2e^2}$ e) DNE

---

4. The average value of $y = e^{6x}$ on $x \in [0, 4]$ is

a) $\frac{e^{24}-1}{4}$ b) $\frac{e^{24}-1}{6}$ c) $\frac{e^{24}}{24}$ d) $\frac{e^{24}}{6}$ e) $\frac{e^{24}-1}{24}$

---

5. $\int \left(2-\sin\frac{t}{5}\right)^2 \cos\frac{t}{5}\, dt$

a) $-\frac{5}{3}\left(2-\sin\frac{t}{5}\right)^3 + c$ b) $\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3 + c$

c) $\frac{1}{3}\left(2-\sin\frac{t}{5}\right)^3 + c$ d) $5\left(2-\sin\frac{t}{5}\right)^3 + c$

e) $-\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3 + c$

---

6. The average value of $g(x) = (2x+3)^2$ on $x \in [-3, -1]$ is

a) $\frac{7}{3}$ b) -4 c) 5 d) $\frac{14}{3}$ e) 3

---

7. The average value of $g(x) = e^{7x}$ on $x \in [0, 2]$ is

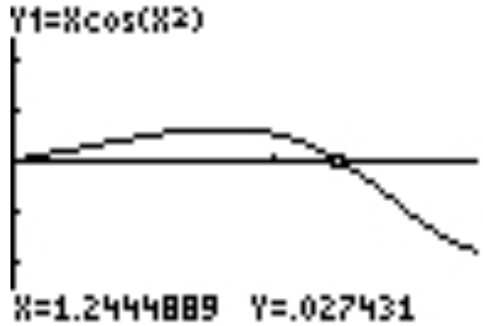
a) $\frac{1}{14}e^{14}$ b) $\frac{1}{7}\left(e^{14}-1\right)$ c) $\frac{1}{14}\left(e^{14}-1\right)$

d) $\frac{1}{2}\left(e^{14}-1\right)$ e) $\frac{1}{7}e^{14}$

4.3 Context for Definite Integrals: Area, Displacement, and Net Change

Since we originally defined the definite integrals in terms of "area under a curve," we need to consider what this idea of "area" really means in relation to the definite integral.

Let's say that we have a function, $y = \left(x\cos\left(x^2\right)\right)$ on $x \in \left[0, \sqrt{\pi}\right]$. The graph looks like this:



In the last section, we saw that $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)\, dx = 0$. But we can see there is area under the curve, so how can the integral equal the area and equal 0? Remember that the integral was created from rectangles with width *dx* and height *f(x)*. So the area below the x-axis would be negative, because the *f(x)*-values are negative.

Ex 1 What is the area under $y = \left(x\cos\left(x^2\right)\right)$ on $x \in \left[0, \sqrt{\pi}\right]$?

We already know that $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)\, dx = 0$, so this integral cannot represent the area. As with example 1b, we really are looking for the positive number that represents the area (total distance), not the difference between the positive and negative "areas" (displacement). The commonly accepted context for area is a positive value. So,

$$
\begin{aligned}
Area &= \int_0^{\sqrt{\pi}} \left| x\cos\left(x^2\right) \right| \, dx \\
&= \int_0^{1.244} x\cos\left(x^2\right) \, dx - \int_{1.244}^{\sqrt{\pi}} x\cos\left(x^2\right) \, dx \\
&= 1
\end{aligned}
$$

We could have used our calculator to find this answer:

When we use the phrase "area under the curve, we really mean the area between the curve and the *x*-axis. CONTEXT IS EVERYTHING. The area under the curve is only equal to the definite integral when the curve is completely above the *x*-axis. When the curve goes below the x-axis, the definite integral is negative, but the area, by definition, is positive.

Objectives:

Relate definite integrals to area under a curve.
Understand the difference between displacement and total distance.
Extend that idea to understanding the difference between the two concepts in other contexts.

Remember:

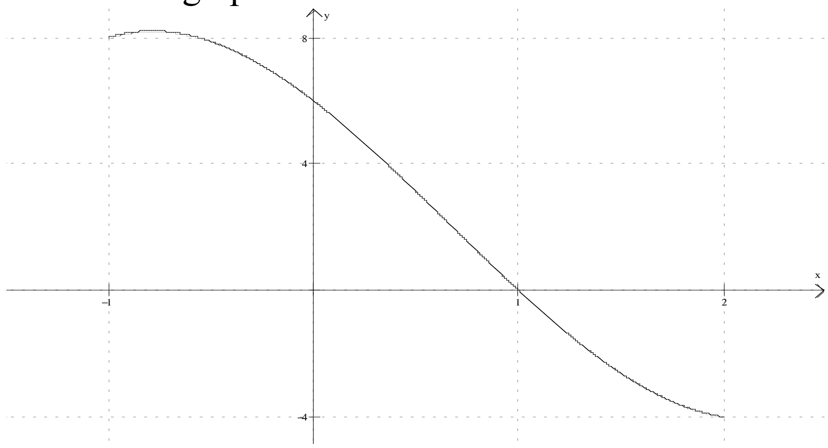
1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

Ex 2 Find the area under $y = x^3 - 2x^2 - 5x + 6$ on $x \in [-1, 2]$.

A quick look at the graph reveals that the curve crosses the x-axis at x=1.



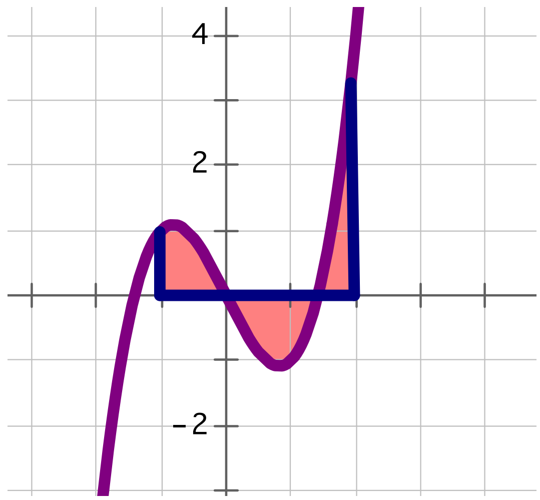
If we integrate y on $x \in [-1, 2]$, we will get the **difference** between the areas, not the sum. To get the total area, we need to set up two integrals:

$$
\begin{aligned}
Area &= \int_{-1}^{1} \left(x^3 - 2x^2 - 5x + 6\right) dx + \left(-\int_{1}^{2} \left(x^3 - 2x^2 - 5x + 6\right) dx\right) \\
&= \left.\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-1}^{1} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{1}^{2} \\
&= \frac{32}{3} + \frac{29}{12} \\
&= \frac{157}{12}
\end{aligned}
$$

<u>Steps to Finding Total Area:</u>

1. Draw the function.
2. Find the zeros between $x = a$ and $x = b$.
3. Set up separate integrals representing the areas above and below the x-axis.
4. Change the sign on those integrals which represent the negative values (i.e., those where the curve is below the x-axis).
5. Solve the integral expression.

Ex 3   Find the area under $y = x^3 - 2x$ on $x \in [-1, 2]$.



Note that there are three regions here, therefore there will be three integrals:

$$
\begin{aligned}
Area &= \int_{-1}^{0} \left(x^3 - 2x\right)\,dx + \left(-\int_{0}^{\sqrt{2}} \left(x^3 - 2x\right)\,dx\right) + \int_{\sqrt{2}}^{2} \left(x^3 - 2x\right)\,dx \\
&= \left.\frac{x^4}{4} - x^2\right]_{-1}^{0} - \left[\frac{x^4}{4} - x^2\right]_{0}^{\sqrt{2}} + \left[\frac{x^4}{4} - x^2\right]_{\sqrt{2}}^{2} \\
&= \frac{3}{4} - (-1) + 1 \\
&= \frac{11}{4}
\end{aligned}
$$

F could be a function that describes anything – volume, weight, time, height, temperature. F' represents its rate of change. The left side of that equation accumulates the rate of change of *F* from *a* to *b* and the right side of the equation says that accumulation is difference in *F* from *a* to *b*.

Imagine if you were leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your calculus homework (gasp!). You head back home, pick up your assignment, and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? You are six miles from where you started but you have traveled 12 miles. These are the two different ideas behind displacement and total distance.

Vocabulary:

Displacement – How far apart the starting position and ending position are. (It can be positive or negative.)
Total Distance – how far you travel in total. (This can only be positive.)

$$\textit{Displacement} = \int_a^b v\,dt \qquad \textit{Total Distance} = \int_a^b |v|\,dt$$

$$\textit{Position at } t = a \quad = x(a) + \int_a^b v\,dt$$

Ex 4 A particle moves along a line so that its velocity at any time t is $v(t) = t^2 + t - 6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during the time period $1 \le t \le 4$.

$$\begin{aligned}(a) \quad \int_a^b v\,dt &= \int_1^4 \left(t^2 + t - 6\right)dt \\ &= \left.\frac{t^3}{3} + \frac{t^2}{2} - 6t\right|_1^4 \\ &= -4.5\end{aligned}$$

(b) $\int_a^b |v|\,dt = \int_1^4 \left(t^2 + t - 6\right) dt$

$$= -\int_1^2 \left(t^2 + t - 6\right) dt + \int_2^4 \left(t^2 + t - 6\right) dt$$

$$= -\frac{t^3}{3} - \frac{t^2}{2} + 6t\bigg|_1^2 + \frac{t^3}{3} + \frac{t^2}{2} - 6t\bigg|_2^4$$

$$= 10\frac{1}{6}$$

Note that we used the properties of integrals to split the integral into two integrals that represent the separate positive and negative distanced and then made the negative one into a positive value by putting a – in front. We split the integral at $t = 2$because that would be where $v(t) = 0$.

Ex 5 AB 1997 # 1

<u>Steps to solving Integrals in Context</u>

1. Graph the function or find the sign pattern for the velocity.
2. Identify the zeros (if any) between $x = a$ and $x = b$.
3. Set up separate integrals to represent the sections above the *x*-axis and below the *x*-axis.
4. Change the signs on the integrals representing the regions below the *x*-axis .
5. Perform the integration and solve.

4.3 Free Response Homework

Find the area under the curve of the given equation on the given interval.

1. $y = x^3 - 2x^2 - 3x$ on $x \in [-2, 2]$

2. $y = x^3 - 4x^2 + 4x$ on $x \in [-1, 2]$

3. $y = x^3 - 2x^2 - x + 2$ on $x \in [-3, 3]$

4. $y = \frac{\pi}{2}\cos x\left(\sin\left(\pi + \pi \sin x\right)\right)$ on $x \in \left[-\frac{\pi}{2}, \pi\right]$

5. $y = \frac{-x}{x^2 + 4}$ on $x \in [-2, 2]$

6. $y = \frac{4 - x^2}{x^2 + 4}$ on $x \in [-3, 3]$

7. $y = \frac{\sin\sqrt{x}}{\sqrt{x}}$ on $x \in \left[.01, \pi^2\right]$

8. $y = x\sqrt{18 - 2x^2}$ on $x \in [-2, 1]$

9. $y = 3\sin x \sqrt{1 - \cos x}$ on $x \in \left[-\frac{\pi}{2}, \frac{\pi}{3}\right]$

10. $y = x^2 e^{x^3}$ on $x \in [0, 1.5]$

11. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \le t \le 3$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

12. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \le t \le 6$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

13. Find the area under the curve $f(x)=e^{-x^2}-x$ on $x\in[-1,2]$ (do not use absolute values in your setup, break it into multiple integrals).

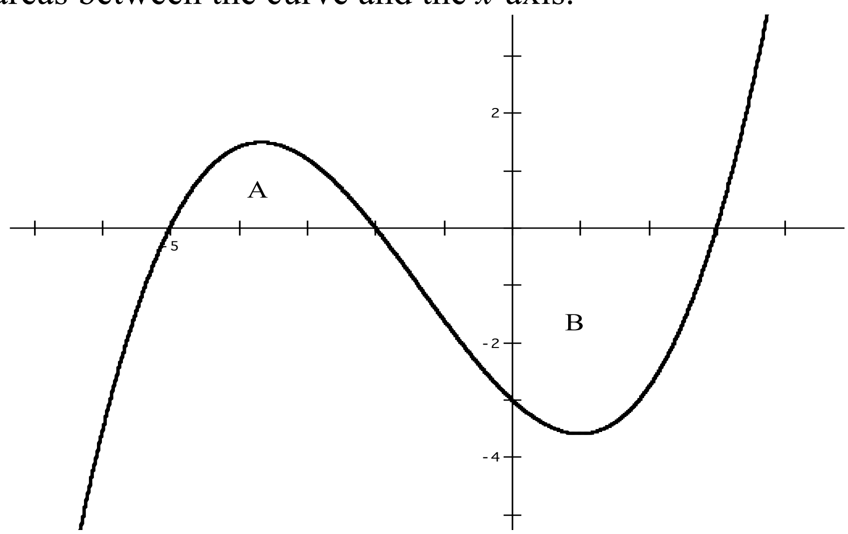
14. Find the area under the curve $f(x)=e^{-x^2}-2x$ on $x\in[-1,2]$ (do not use absolute values in your setup, break it into multiple integrals).

15. Find the area under the curve $f(x)=\dfrac{x}{x^2+1}+\cos(x)$ on $x\in[0,\pi]$

16. Find the area under the curve $g(x)=-1-x\sin x$ on $x\in[0,2\pi]$

<u>4.3 Multiple Choice Homework</u>

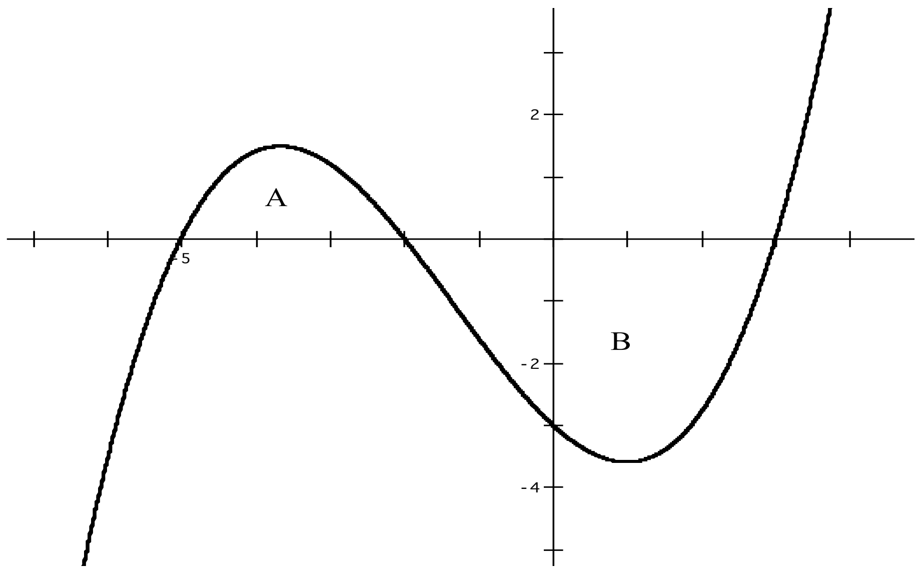
1. The graph of $y=f(x)$ is shown below. A and B are positive numbers that represent he areas between the curve and the x-axis.



In terms of A and B, $\int_{-5}^{3} f(x)\,dx+\int_{-2}^{3} f(x)\,dx=$

a) A b) $A-B$ c) $2A-B$ d) $A+B$ e) $A-2B$

2. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x-axis.

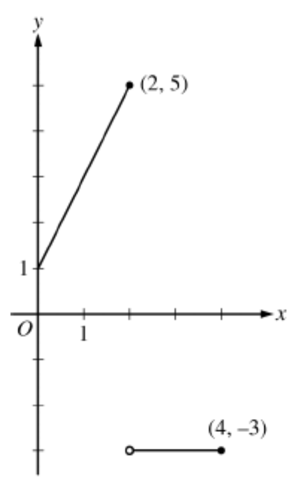


In terms of A and B, $2\int_{-5}^{3} f(x)\,dx - \int_{-2}^{3} f(x)\,dx =$

a) A b) $A - B$ c) $2A - B$ d) $A + B$ e) $A + 2B$

---

3. The graph of $f(x)$ on $0 \le x \le 4$ is shown.



What is the value of $\int_0^4 f(x)dx$?

a) -1 b) 0 c) 2 d) 6 e) 12

4. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. The total distance traveled by the particle from 0 to 3 is

a) 0.667 b) 0.704 c) 1.540 d) 2.667 e) 2.901

5. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2}\sin(2t)$ meters per second. What is the total distance, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$ seconds?

a) 0.835 b) 1.625 c) 2.055 d) 2.261 e) 7.025

6. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. The total displacement of the particle from t=0 to t=3 is

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

7. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2}\sin(2t)$ meters per second. What is the total displacement, in meters, of the particle during the time interval $0 \leq t \leq 2$ seconds?

a) 0.835 b) 1.625 c) 1.661 d) 2.261

4.4 Graphical Analysis II

An important part of calculus is being able to read information from graphs. Earlier, we looked at graphs of first derivatives and answered questions concerning the traits of the "original" function. This section will answer the same questions, even with similar graphs – but this time the "original" function in will be defined as an integral in the form of the FTC. **This is a major AP topic.**

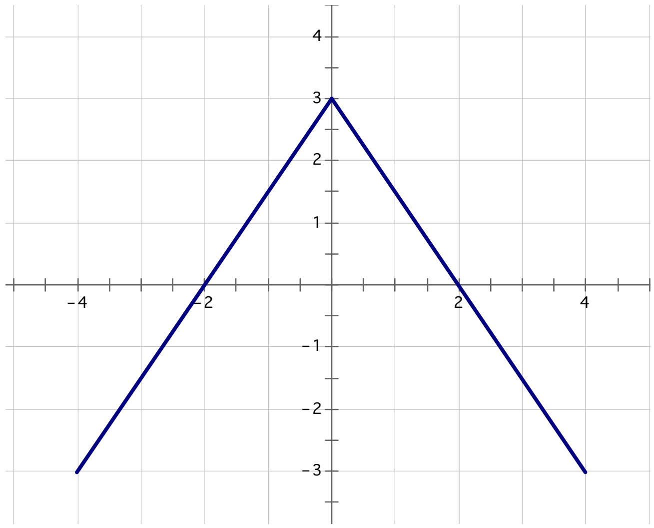
Objectives
Use the graph of a function to answer questions concerning maximums, minimums, and intervals of increasing and decreasing
Use the graph of a function to answer questions concerning points of inflection and intervals of concavity.
Use the graph of a function to answer questions concerning the area under a curve.

As we saw before, the graphs of a function and its derivatives are related. We summarized that relationship in a table that we will now expand.

G(x)= $\int_0^x F(t)\,dt$	The *y*-values of G(x)	Increasing EXTREME Decreasing	Concave up POI Concave down
F(x)	Area under *F(x)*	Positive ZERO Negative	Increasing EXTREME Decreasing
F'(x)			Positive ZERO Negative

NB. The AP Test will always require the statement that $g(x)=\int_0^x f(t)\,dt$ **means that** $g'(x)=f(x)$**, and the arguments about extremes, concavity and increasing and decreasing be made from the point of view of the derivatives.**

Ex 1 The graph of f is shown below. f consists of three line segments.



Let $g(x)=\int_0^x f(t)\,dt$.

a) Find $g(-2)$, $g'(-2)$, and $g''(-2)$.

b) For what values on $x\in(-4,4)$ is $g(x)$ increasing? Explain your reasoning.

c) For what values on $x\in(-4,4)$ is $g(x)$ concave up? Explain your reasoning.

d) Sketch a graph of $g(x)$ on $x\in[-4,4]$.

a) $g(-2)=-3$. $g(-2)$ would equal the area of the triangle from -2 to 0, but the integral goes from 0 to -2, therefore the integral equals the negative of the area.

$g'(-2)=f(-2)=0$.

$g''(-2)=f'(-2)=\frac{3}{2}$. This is the slope of $f(x)$ at x=-2.

b) $g(x)$is increasing when $g'(x)$ is positive. $g'(x)=f(x)$. Therefore, $g(x)$is increasing on $x\in[-2,2]$

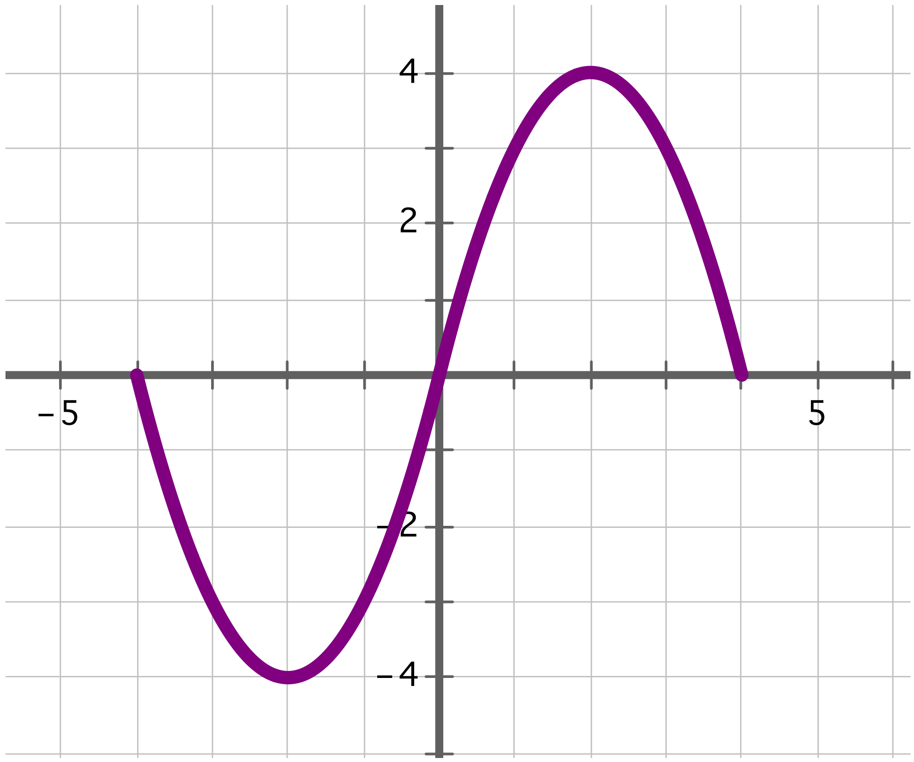
c) $g(x)$ concave up when $f(x)$ is increasing. Therefore, $g(x)$is concave up on $x\in(-4,0)$

d) Because of the equal triangle areas, $g(-4)=g(0)=g(4)=0$

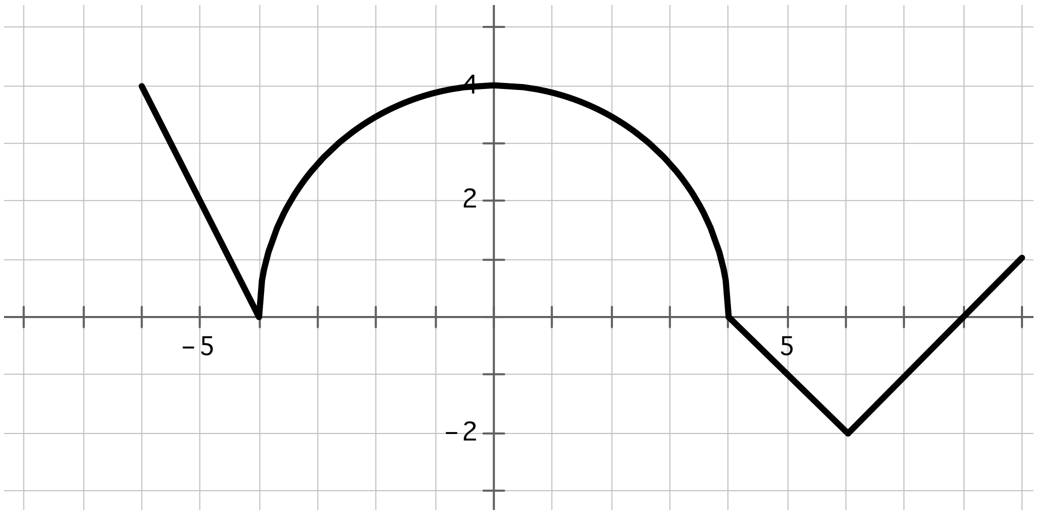
$g(x)$ has a relative max at $x=2$ and a relative min at $x=-2$.

$g(x)$is increasing on $x \in [-2, 2]$ and decreasing on $x \in [-4, -2] \cup [2, 4]$
$g(x)$ is concave up on $x \in (-4, 0)$ and concave down on $x \in (0, 4)$.

The graph looks something like this:



Ex 3 Let $h(x)=\int_0^x g(t)\,dt$. The graph of g is shown below.



(a) Find $h(4)$.

(b) Find $h(0)$, $h'(0)$, and $h''(0)$.

(c) What is the instantaneous rate of change of $h(x)$ at $x = 1$?

(d) Find the absolute minimum value of h on $[-6, 8]$.

(e) Find the x – coordinate of all points of inflection of h on $(-6, 8)$.

(f) Let $F(x)=\int_x^0 g(t)\,dt$. When is F increasing?

(a) $h(4)=\int_0^4 g(t)\,dt$ would equal the area from 0 to 4. This is one quarter of the circle with radius 4. So, $h(4)=\int_0^4 g(t)\,dt = 4\pi$.

(b) $h(0)=\int_0^0 g(t)\,dt = 0$

$$h'(0)=\frac{d}{dx}\int_0^x g(t)\,dt = g(0)=4$$

$$h''(0)=g'(0)=0$$

(c) The instantaneous rate of change of *h(x)* at $x = 1$ would be *g(1)*. To find *g(1),* we would need the equation of the circle. Since I t has its center at the origin and radius = 4, the equation is $x^2 + y^2 = 16$ or $y = \sqrt{16 - x^2}$.

$$g(1) = \sqrt{16 - 1^2} = \sqrt{15}$$

(d) Since $g(x) = h'(x)$, the critical values of h would be the zeros and endpoints of g, namely, x = -6, -6, 4, 9. The graph of *g(x)* yields a sign pattern:

$$\begin{array}{lccccccccc} g & & 0 & + & 0 & + & 0 & - & 0 & \\ & \longleftarrow & & & & & & & & \longrightarrow \\ x & & -6 & & -4 & & 2 & & 8 & \end{array}$$

From this, we can see that $x = -4$ and 2 cannot be minimums. $x = 2$ is a maximum because the sign of $g(x) = h'(x)$ switches from + to - and $x = -4$ is not an extreme because the sign does not change. Now all we need are *h(-6)* and *h(-9)*. Whichever is smaller is the absolute minimum.

$$h(-6) = \int_0^{-6} g(t)\,dt = -\int_{-6}^{0} g(t)\,dt = -(4 + 4\pi)$$

$$h(9) = \int_0^9 g(t)\,dt = 4\pi - 4$$

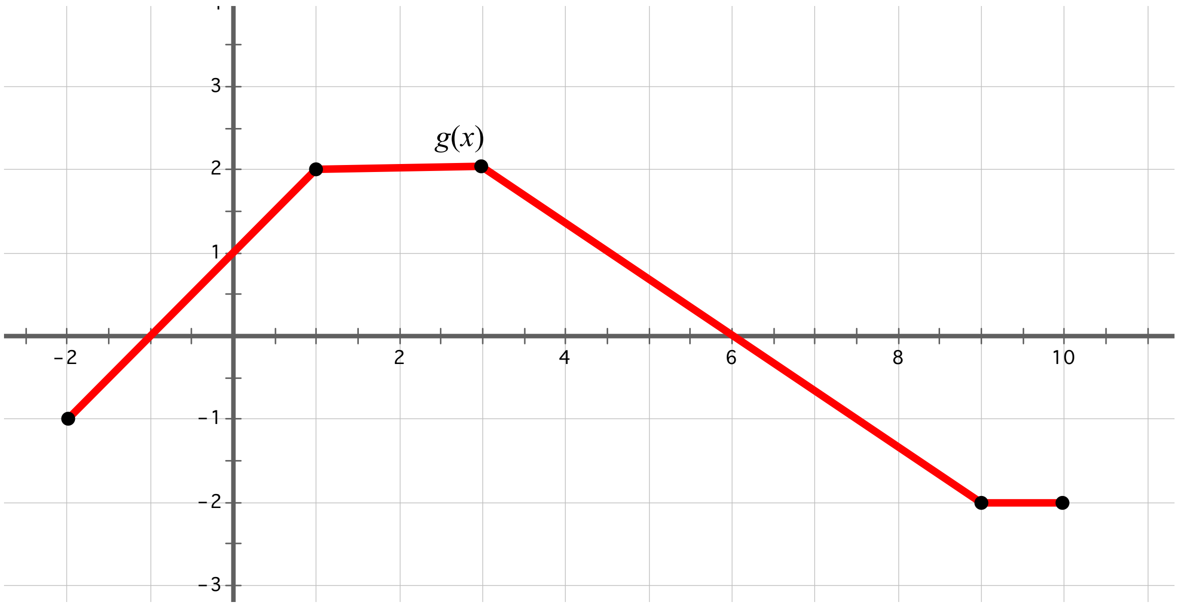
h(-6) is the smaller value so the absolute minimum is $-(4 + 4\pi)$

(e) $g'(x) = h''(x)$ so the points of inflection on h would be where the slope of g changes sign, namely at $x = -4$, 0 and 6.

(f) $F(x) = \int_x^0 g(t)\,dt \to F'(x) = -g(x)$. So F is increasing when g is negative, namely on $x \in [4, 8]$.

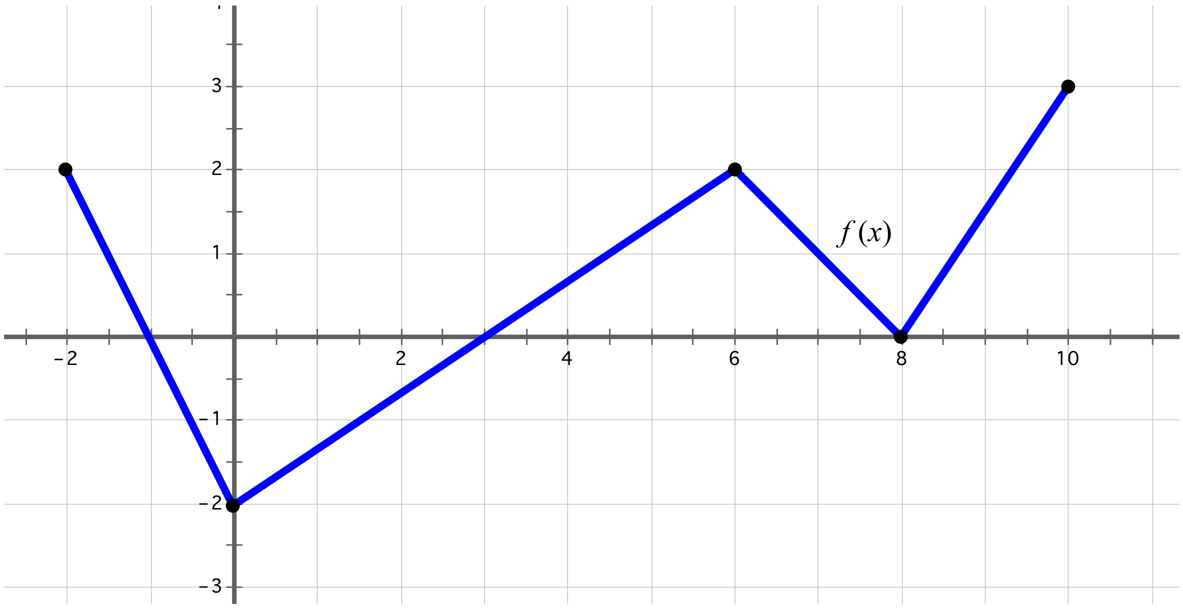
4.4 Free Response Homework

1. Let $f(x)=\int_3^x g(t)dt$ for $-2 \le t \le 10$, where the graph of the differentiable function f is shown below.



a) Find $f(0)$, $f'(0)$, and $f''(0)$.
b) Find the average rate of change of $f(x)$ on $-2 \le t \le 10$?
c) At what x-values is $f(x)$ decreasing and concave up? Justify your answer.
d) Find the x-coordinate of the absolute minimum of $f(x)$. Justify your answer.

2. Let $g(x) = \int_{-2}^{x} f(t)dt$ for $-2 \le t \le 10$, where the graph of the differentiable function f is shown below.



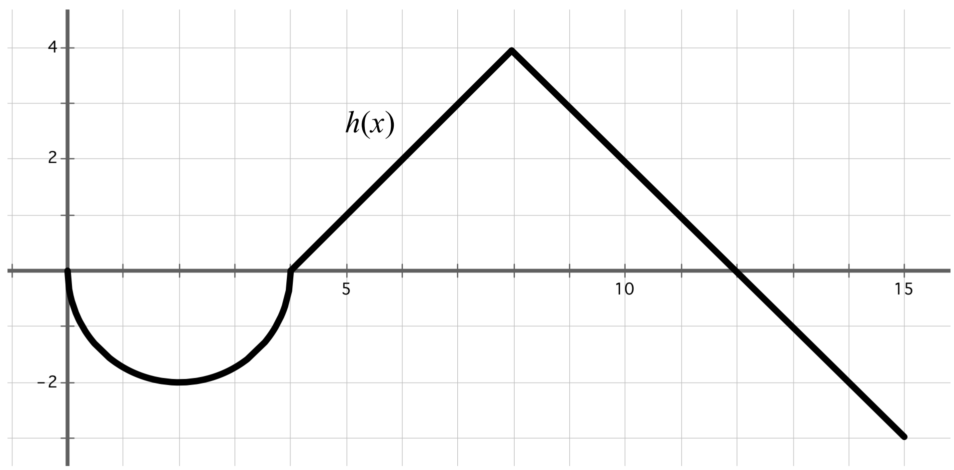
a) Find $g(2)$, $g'(2)$, and $g''(2)$.

b) Find the average rate of change of $g(x)$ on $-2 \le t \le 10$?

c) At what x-values is $g(x)$ increasing and concave down? Justify your answer.

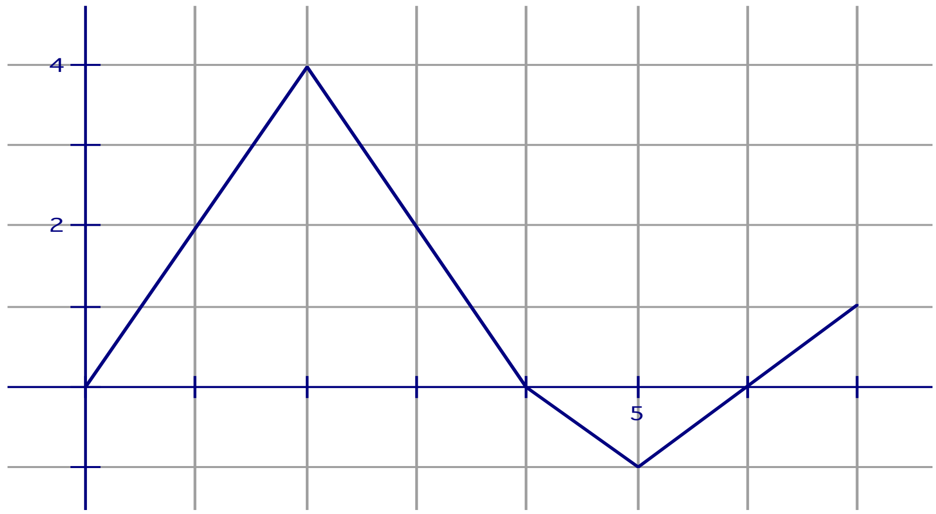
d) Find the x-coordinate of the absolute minimum of $g(x)$. Justify your

3. Let $k(x) = \int_0^x h(t)\,dt$ for $0 \le t \le 15$, where the graph of the differentiable function *f* is shown below.



a) Find $k(4)$, $k'(4)$, and $k''(4)$.
b) Find the average rate of change of $k(x)$ on $0 \le t \le 15$?
c) At what *x*-values is $k(x)$ decreasing and concave down? Justify your answer.
d) Find the *x*-coordinate of the absolute minimum of $k(x)$. Justify your

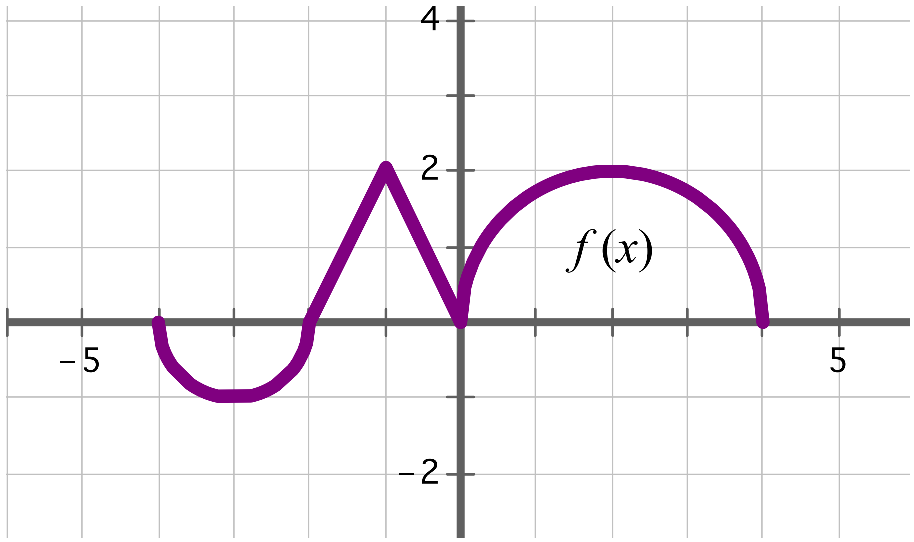
4. Let $g(x) = \int_2^x f(t)dt$ for $0 \le t \le 7$, where the graph of the differentiable function f is shown below.



a) Find $g(3)$, $g'(3)$, and $g''(3)$.
b) Find the average rate of change of $g(x)$ on $0 \le c \le 3$?
c) For how many values of c, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Justify your answer.

d) Find the x-coordinate of each point of inflection of the graph of g on the interval $0 < t < 7$. Justify your answer.

5. Let $h(x)=\int_0^t f(t)\,dt$ on $x\in[-4,\ 4]$. Let the graph of be comprised of two semicircles and a line segment as shown below.

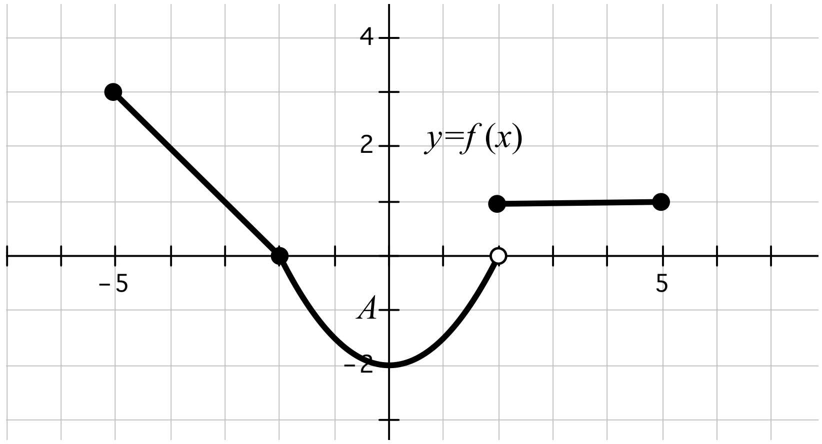


(a) Find $h(2)$, $h'(2)$, and $h''(2)$.

(b) Find the average rate of change of $h(x)$ on $x\in[0,\ 2]$.

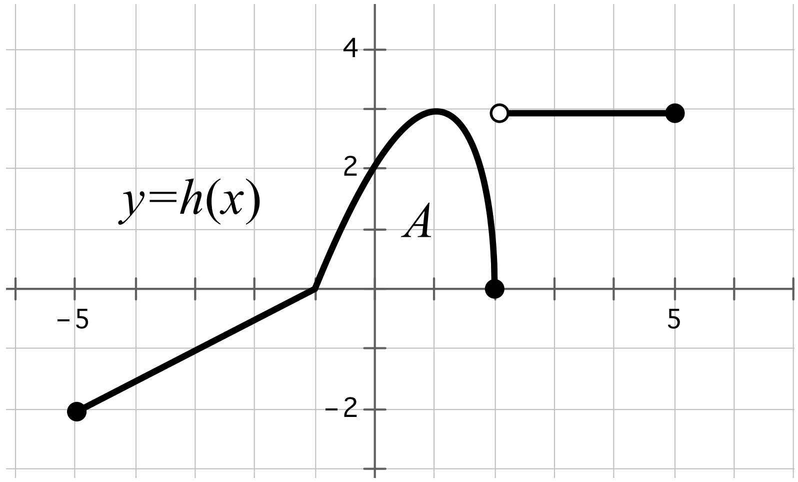
(c) At what x-values is $h(x)$ decreasing and concave up? Justify your answer.

(d) What is the absolute maximum value of $h(x)$ on the interval $x\in[-4,\ 4]$?



6. The graph above, $f(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a parabola. Let $g(x) = 4 + \int_{-2}^{x} f(t)\, dt$, and let Area A equal 5.7.

(a) Find $g(-5)$ and $g'(-5)$.
(b) Find $g(2)$.
(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute maximum? Explain.
(d) On what interval(s) is $g(x)$ both decreasing and concave down? Explain why.



7. The graph above, $h(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a radial function. Let $g(x) = 4 + \int_{-1}^{x} h(t)\,dt$. The area A is 5.

(a) Find $g(-5)$ and $g'(-5)$.
(b) Find $g(2)$.
(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute minimum? Explain.
(d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why.

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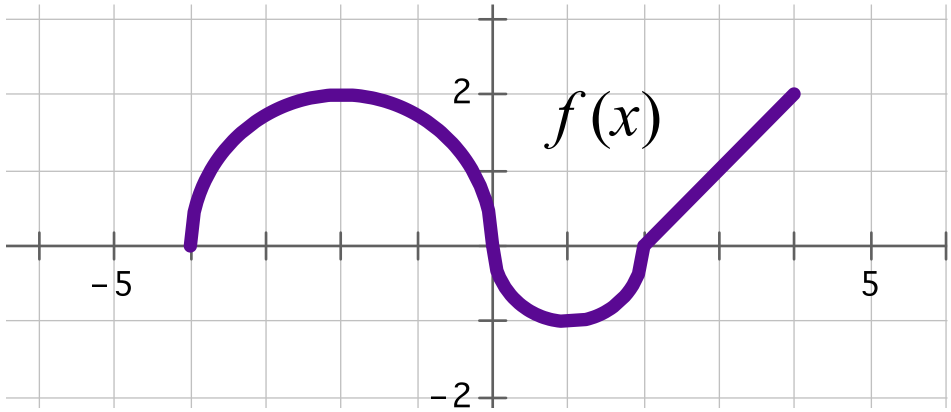
8. Let $h(x)=\int_0^t f(t)\,dt$ on $x\in[-4,\ 4]$. Let the graph of be comprised of two semicircles and a line segment as shown below.



(a) Find $h(2)$, $h'(2)$, and $h''(2)$.

(b) Find the average rate of change of $h(x)$ on $x\in[0,\ 2]$.

(c) At what x-values is $h(x)$ decreasing and concave up? Justify your answer.

(d) What is the absolute maximum value of $h(x)$on the interval $x\in[-4,\ 4]$?

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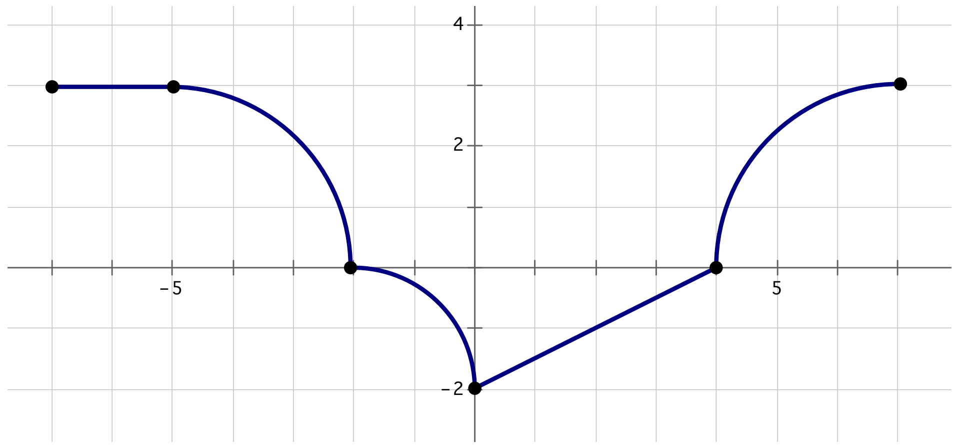
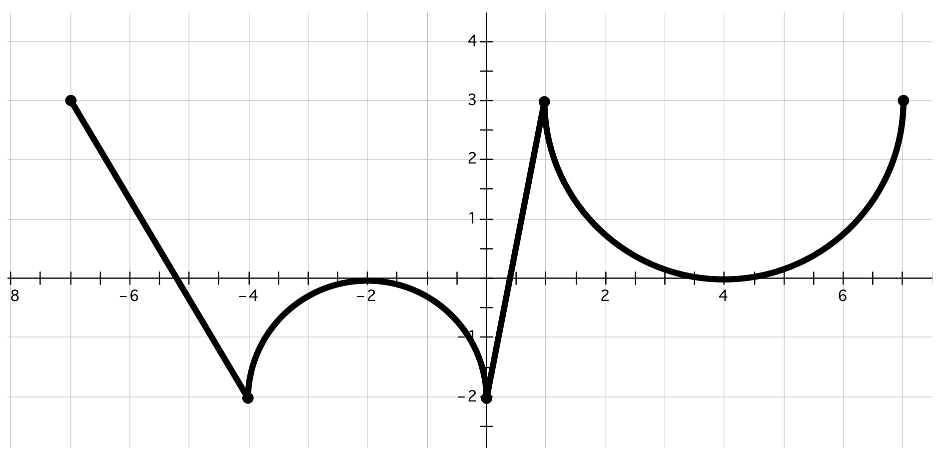
9. Let $g(x)=\int_{-2}^{x} f(t)\,dt$ for $-7 \le t \le 7$, where the graph of the differentiable function *f* is shown below.



a) Find $g(4)$, $g'(2)$, and $g''(4)$.
b) Find the average rate of change of $g(x)$ on $-7 \le t \le 0$?
c) At what *x*-values is $g(x)$ decreasing and concave up? Justify your answer.
d) Find the *x*-coordinate of the absolute minimum of $g(x)$. Justify your answer.

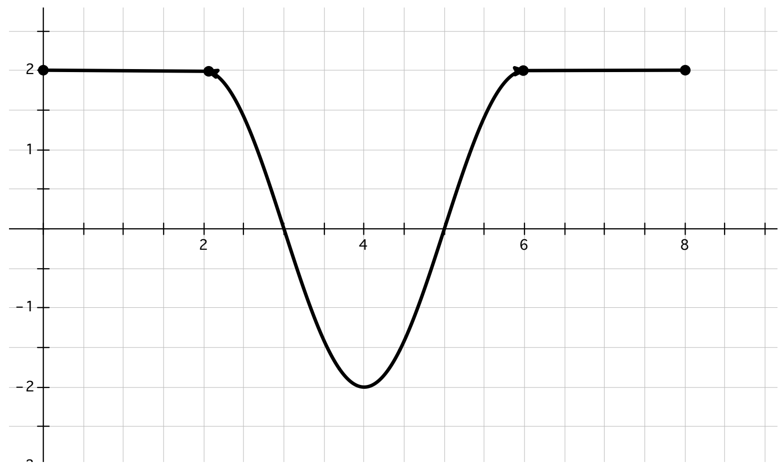
10. Let $g(x)=\int_{-4}^{x} f(t)\,dt$ for $-7 \le t \le 7$, where the graph of the differentiable function *f* is shown below.



a) Find $g(0)$, $g'(0)$, and $g''(0)$.
b) Find the equation of the line tangent to $g(x)$ at $x=0$.
c) At what *x*-values is $g(x)$ decreasing and concave up? Justify your answer.

d) Find the x-coordinate of the absolute maximum of $g(x)$. Justify your answer.

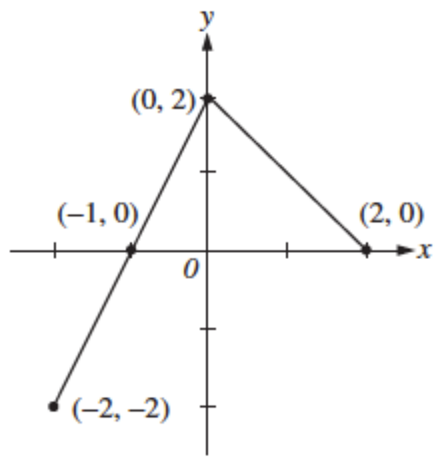
11. Let $g(x) = 10 + \int_0^x f(t)dt$ for $0 \le t \le 8$ and let $f(t)$ be the differentiable function (shown below) comprised of two horizontal line segments and one cycle of the cosine wave $y = 2\cos\left[\frac{\pi}{2}(t-2)\right]$.



a) Find $g(4)$, $g'(4)$, and $g''(4)$.
b) Find the average rate of change of $g(x)$ on $0 \le t \le 8$?
c) Find the average value of $f(x)$ on $0 \le t \le 8$.
d) Find the absolute minimum of $g(x)$. Justify your answer.

12. Handout of AP Questions: BC 2002B #4, BC 1999 #5, BC 2003 #5, BC 2002 #4, BC 2009B #5

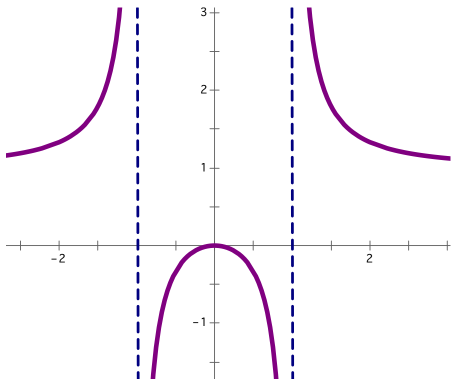
4.4 Multiple Choice Homework



Graph of f

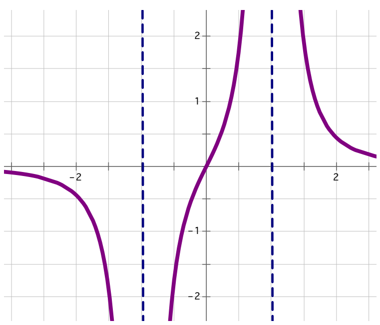
1. The graph of the function f shown above consists of two line segments. If g is the function defined by $g(x) = \int_0^x f(t)\,dt$, then $g(-1) =$

a) -2 b) -1 c) 0 d) 1 e) 2

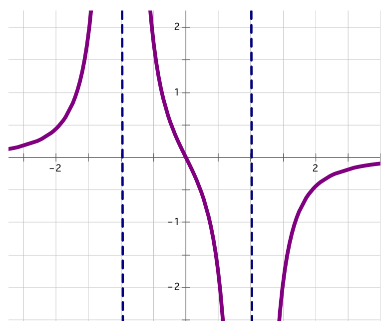


2. Let the graph above be $g(x)=\int_0^x f(t)\,dt$. Which of the following could be the graph of f ?

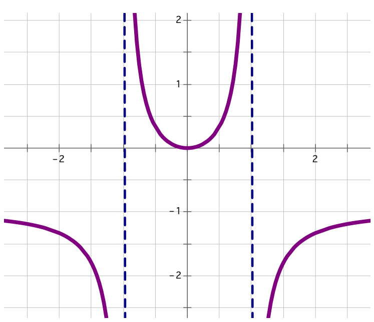
a)



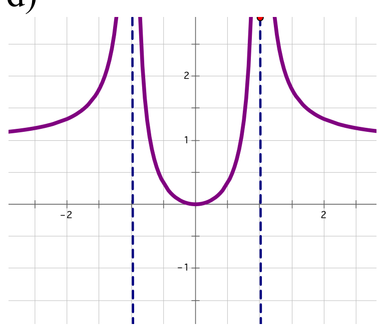
b)



c)



d)



3. Let g be the function given by $g(x)=\int_1^x 100(t^2-3t+2)\,dt$. Which of the following statements about g must be true?

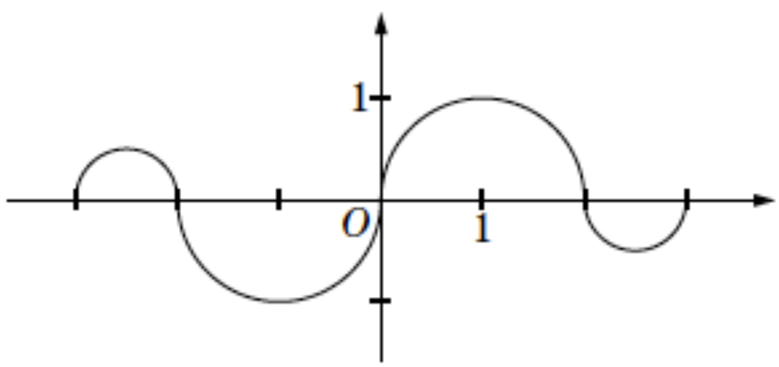
I. g is increasing on $x\in(1,\ 2)$.

II. g is increasing on $x\in(2,\ 3)$.

III. $g(3)<0$

a) I only b) II only c) III only

d) II and III only e) I, II, and III



Graph of f

4. The graph of the function f above consists of four semicircles. If $g(x)=\int_0^x f(t)\,dt$, where is $g(x)$ nonnegative?

a) $x\in[-3,\ 3]$ only b) $x\in[-3,\ -2]\cup[0,\ 2]$ only

c) $x\in[0,\ 3]$ only d) $x\in[0,\ 2]$ only

e) $x\in[-3,\ -2]\cup[0,\ 3]$ only

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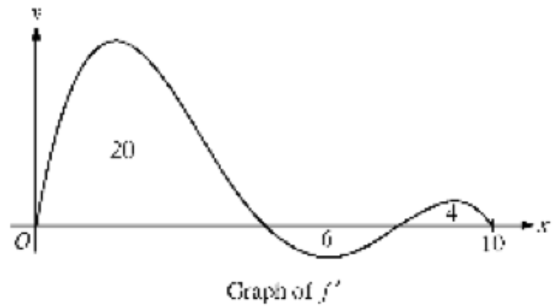
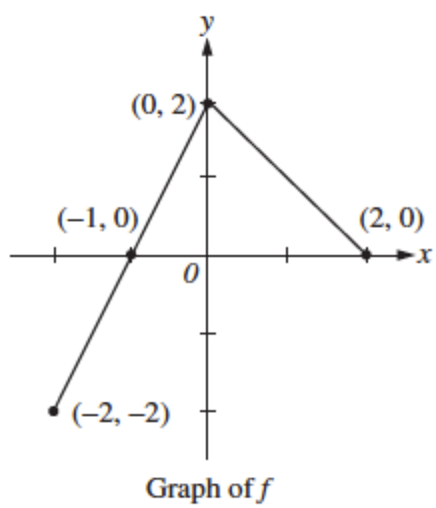
5. The graph of f', the derivative of f, is shown below, for $0 \le x \le 10$.



The area of the region between the graph of f' and the x-axis are 20, 6 and 4, respectively. If $f(0)=2$, what is the maximum value of f on the closed interval $0 \le x \le 10$?

a) 16 b) 20 c) 22 d) 30 e) 32



6. The graph of the function *f* shown above consists of two-line segments. If *g* is the function defined by $g(x)=\int_{-1}^{x} f(t)\,dt$, then $g(-2)=$

a) -2 b) -1 c) 0 d) 1 e) 2

4.5 Accumulation of Rates

As we saw with the Riemann sums, in $\int R(t)\,dt$, $R(t)\cdot dt$ is the area of a rectangle (height times base). The $\int$ is the sum of the areas. The concept in these "Accumulation of Rates" problems is that, since a definite integral is a sum of values, then **an integral of a rate of change equals the total change.**

Objective
Analyze the relationship between rates of change and integrals.

Beginning in 2002, AP shifted emphasis on understanding of the accumulation aspect of the Fundamental Theorem to a new kind of rate problem. Previously, accumulation of rates problems were mostly in context of velocity and distance, though the 1996 Cola Consumption problem hinted at the direction the test would take. The "Amusement Park Problem" of 2002 caught many students and teachers off guard, though. Almost every year since then, the test has included this "Accumulation of Rate" kind of problem. Here is an example similar to the 2002 Amusement Park Problem:

Ex 1 The rate at which people enter a park is given by the function $E(t)=\dfrac{15600}{t^2-24t+160}$, and the rate at which they are leaving is given by $L(t)=\dfrac{9890}{t^2-38t+370}-76$. Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8\le t\le 24$. At t = 8 there are no people in the park.

a) How many people have entered the park at 4 pm (t = 16)? Round your answer to the nearest whole number.

b) The price of admission is \$36 until 4 pm (t = 16). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

c) Let $H(t)=\int_8^t E(x)-L(x)dx$ for $8 \le t \le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

d) At what time *t*, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

Notice that several questions are about the meaning of the derivative or integral in context of the word problem and require the use of proper units.

Functions	**Motion**	**Accumulation of Rates**	**Formula**	**Units**
$f(x)$	Position	Total Change	$\int_a^b R(t)dt$	Amount
$f'(x)$	Velocity	Rate	$R(t)$	Amount per time
$f''(x)$	Acceleration	Rate increasing or decreasing	$R'(t)$	Amount per time per time

It is important to remember several key phrases:

Key Phrases for decoding Accumulation of Rates questions:

Total change: $\int_a^b R(t)\,dt$ or $\int_a^t [\text{incoming rate} - \text{outgoing rate}]\,dx$

Total rate of change: $\text{incoming rate} - \text{outgoing rate}$

Total Amount: $Total(t) = \text{initial value} + \int_a^t [\text{incoming rate} - \text{outgoing rate}]\,dx$

Instantaneous rate of change: $\frac{dx}{dt}$ or $R(t)$

Average rate of change: $\frac{f(b) - f(a)}{b - a}$

Average value of $f(x)$: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(\text{Rate of Change})$ is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(\text{Rate of Change})$ is positive

*Note that it can be a function that is increasing or decreasing, or it could be a derivative that is increasing or decreasing, etc.

Consider the Units!!!

If we consider the units involved in integrating a rate, this becomes more apparent. If $R(t)$ is measured in miles per hour,

$$\int_a^b R(t)\,dt = \int_a^b \frac{miles}{hour}(hours) = sum\ of\ miles = \text{total miles.}$$

So,

$$\int_a^b R(t)\,dt = \int_a^b \frac{units}{time}(time) = sum\ of\ units = \text{total units.}$$

Amount increasing or decreasing would be $\frac{d}{dt}(units) = \frac{units}{time}$

Rate would be given in $\frac{units}{time}$

$\frac{f(b)-f(a)}{b-a}$ would be in $\frac{units}{time}$

$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$ would be $\frac{1}{time}\int_a^b \frac{units}{time}(time) = \frac{units}{time}$

Amount increasing or decreasing would be $(units)\frac{1}{time} = \frac{units}{time}$

Amount increasing or decreasing at an increasing or decreasing rate would be $(units)\frac{1}{time}\cdot\frac{1}{time} = \frac{units}{time^2}$

Rate increasing or decreasing would be $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{time}\left(\frac{units}{time}\right) = \frac{units}{time^2}$

Now let us try to actually do Ex 1:

Ex 1   The rate at which people enter a park is given by the function $E(t)=\dfrac{15600}{t^2-24t+160}$, and the rate at which they are leaving is given by $L(t)=\dfrac{9890}{t^2-38t+370}-76$. Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8 \le t \le 24$. At t = 8 there are no people in the park.

a) How many people have entered the park at 4 pm (t = 16)? Round your answer to the nearest whole number.

Since $E(t)$ is a rate in people per hour, the number of people who have entered the park will be an integral from $t=8$ to $t=16$.

$$\text{Total entered} = \int_8^{16} E(t)\,dt = 6126.105 \approx 6126 \text{ people}$$

Note that this is the basic accumulation of rate definition.

b) The price of admission is \$36 until 4 pm ($t=16$). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

Since there are different entry fees for different times of day, we need to determine how many people paid each fee:

$$\text{Total entered before 4pm } = \int_8^{16} E(t)\,dt = 6126 \text{ people}$$

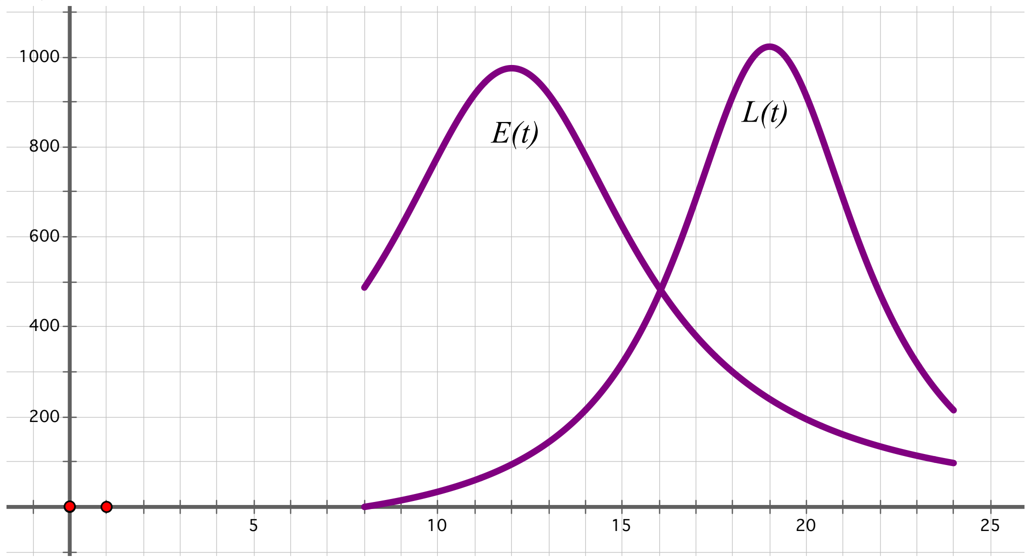
$$\text{Total entered after 4 pm } = \int_{16}^{24} E(t)\,dt = 1808 \text{ people}$$

The total revenue that the park gets from admissions means multiplying the number of people by the admission charge:

Total revenue = \$36(6126) + \$20(1808) = \$256,696

c) Let $H(t)=\int_8^t E(x)-L(x)\,dx$ for $8\le t\le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

While the graph is not necessary for solving this problem, sometimes it helps to visualize the situation. Below are the graphs of $E(x)$ and $L(x)$ on $8\le t\le 24$:



Note how each function increases and then decreases. Since $H(t)=\int_8^t E(x)-L(x)\,dx$, we can use the Fundamental Theorem of Calculus to determine the derivative.

$$\begin{aligned} H'(t) &= \frac{d}{dt}\int_8^t E(x)-L(x)\,dx \\ &= E(t)-L(t) \\ &= E(16)-L(16) \\ &= 14 \end{aligned}$$

But we still need to interpret the meaning of the numbers.

$H(16) = 5023$ people – since we know that integrating a rate gives total change, and $H(t)$ was defined as an integral of the difference of two rates, $H(16)$ is how many people entered the park minus how many people left the park. In other words, $H(16)$ is how many people are **in the park** at 4 pm.

$H'(16) = 14$ people per hour. Since the original equations were rates and $H'(t) = E(t) - L(t)$, $H'(16)$is the rate of change of the number of people in the park at $t = 16$. In other words, the number of people in the park is increasing at 14 people per hour at 4 pm.

d) At what time t, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

To find the maximum number of people, we have to set the derivative $H'(t)$ equal to zero. **We must also check the endpoints**, which are also critical values (we could just do a sign pattern instead to verify that the zero is the maximum).

$$H'(t) = E(t) - L(t) = 0$$

Using a graphing calculator, we find the zero is at $t = 16.046$. Or, we can find the point of intersection of $E(x)$ and $L(x)$. The critical values are $t = 0$, 16.046, and 24.

When $t = 0$, $H = 0$ (this was given at the beginning of the problem)
When $t = 16.046$, $H(16.046) = \int_8^{16.046} [E(x) - L(x)]dx = 5023$
When $t = 24$, $H(24) = \int_8^{24} [E(x) - L(x)]dx = 1453$

Because 5023 is the highest of these three H values, the maximum number of people in the park occurs at $t = 16.046$.

Ex 2 A certain industrial chemical reaction produces synthetic oil at a rate of $S(t)=\frac{15t}{1+3t}$. At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of $R(t)=2+5\sin\left(\frac{4\pi}{25}t\right)$. Both functions have units of gallons per hour, and the reaction runs from $t = 0$ to $t = 6$. At time of $t = 0$, the reaction vessel contains 2500 gallons of oil.

a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measure.

$$\text{Amount removed} = \int_0^6 \left[2+5\sin\left(\frac{4\pi}{25}t\right)\right]dt = 31.816 \text{ gallons}$$

b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time t.

$$P(t) = 2500 + \int_0^t \left[S(x) - R(x)\right] dx$$

Note: Since the variable t is the upper boundary, as "dumby variable"—namely, x—needs to be used in the integand.

c) Find the rate at which the total amount of oil is changing at $t = 4$.

$$S(4) - R(4) = -1.909 \text{ gal/hr.}$$

d) For the interval indicated above, at what time t is the amount of oil in the reaction vessel at a minimum? What is the minimum value? Justify your answers.

Critical values occur when $S(t) = R(t)$ and at the endpoints. Our graphing calculator shows the time when $S(t) = R(t)$ is $t = 5.117$.

t	$P(t)$
0	2500
5.117	2492.367
6	2493.277

The minimum is 2492.367 gallons and occurs at $t = 5.117$.

SUMMARY

- Realize that these problems are actually three or four different problems with a common source rather than a single problem.
- Let the units dictate the mathematical set-up.
- Slow down and read critically
- These are CALCULATOR problems.

4.5 Free Response Homework

1. In 1881, the silver mines in Tombstone, Arizona, struck the local aquifer at 520 feet and began to flood. The owners of the Grand Central Mine bought the Cornish engines from the Comstock Mines to pump the water out. On a given day, the water was seeping into the mine at a rate of $S(t)=96+\dfrac{204\ln(t+1)}{t+.1}\ {}^{gal}\!/_{hr}$, and the pumps could drain the water at a rate of $D(t)=414+375\sin\left(\dfrac{x^2}{72}\right){}^{gal}\!/_{hr}$. When the pumps start, there are 10,000 gallons of water in the mine.

   (a) How many gallons of water were pumped out of the mine during the time interval $0\le t\le 24$ hours?
   (b) Is the level of water rising or falling at t = 6? Explain your reasoning.
   (c) How many gallons of water are in the mine at t = 14 hours?
   (d) At what time t, for $0\le t\le 24$, is the volume of water at an absolute maximum? Show the analysis that leads to your answer.

---

2. In 1920, Dr. Quattrin's grandfather Andrea returned to America from Italy after fighting in World War I. He arrived in New York Harbor on the *SS Pannonia* and, despite having established residency in 1913, had to be processed through the Immigration Center at Ellis Island. There were 1123 non-citizen, third-class passengers on the *Pannonia* that had to go through processing. (First- and second-class passengers passed through without processing.) Immigrants entered the processing line at a rate modeled by the function $E(t)=8843\left(\dfrac{t}{5}\right)^4\left(1-\dfrac{t}{10}\right)^5$, where t is measured in hours after the ship began opffloading immigrants. The new arrivals were processed out at a rate of 250 people per hour. The *Pannonia* was the third ship in port, so there were already 2500 people in line when the *Pannonia* passengers got into line.

   (a) How many passengers from the *Pannonia* had gotten in line for processing in the first 6.2 hours?
   (b) Is the rate of change of people entering the processing line increasing or decreasing at $t=6.2$?
   (c) How many people were in line at $t=6.2$?

(d) Assuming no new ships entered the harbor, what was the absolute maximum number of people in the processing line? Justify your answer.

---

3. More than 30% of observed star systems have multiple stars, and 70% of those have more than two stars. When stars are close together, they exchange mass in a process known as accretion. Consider a trinary system where S_1 is larger than S_2, and S_2 is larger than S_3. S_3 will lose mass to S_2, and S_2 will lose mass to S_1 While scientific readings are not available because of the time scale, let us suppose that S_2 loses mass to the larger S_1 at a rate of $L(t)=1+(.01t)^2+.23\sin\left(\frac{\pi}{25}t\right)$ and gains mass from the smaller S_3 at a rate of $G(t)=0.2+0.15\sqrt{t}$ where $0\le t\le 100$ years. $L(t)$ and $G(t)$ are measured in yottatons per year $\left(\mathrm{Y}/_{yr}\right)$. (A yottaton is 10^{26} tons, or 10^{-7} solar masses.)

(a) How much mass does S_2 lose to S_1 on $0\le t\le 100$? State the units.
(b) At $t=50$, is the mass S_2 is gaining from S_3 increasing at an increasing rate? Using the correct units, justify your answer.
(c) At what times on $0\le t\le 100$ is S_2 losing as much mass to S_1 as it is gaining from S_3?
(d) If the mass of S_2 is 1,000,000 yottatons at $t=0$, find the minimum mass of S_2 on the time interval $0\le t\le 100$.

---

4. A diabetic patient takes Metformin twice a day to control her blood sugar. The medication enters the bloodstream at a rate expressed by $M(t)=8-\frac{e^{0.47t}}{t+6}$, where $M(t)$ is measured in centigrams per hour (cg/hr) and *t* is measured in hours for $0\le t\le 12$. The liver cleans the medication out of the bloodstrream at a rate of $L(t)=7-.46t\cos(t)$ cg/hr.

a) How much Metformin enters the bloodstream during this 12-hour time period?

b) After 9 hours, how much Metformin is still in her bloodstream?

c) Find $L'(6)$ and explain the meaning of the answer, using the correct units.

d) Set up, but do not solve, an integral equation that would determine the time when the dose of Metformin has been completely cleaned out of the bloodstream.

---

5. At an intersection in San Francisco, cars turn left at the rate $L(t) = 50\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour for the time interval $0 \le t \le 18$.

(a) To the nearest whole number, find the total number of cars turning left on the time interval given above.

(b) Traffic engineers will consider turn restrictions if $L(t)$ equals or exceeds 125 cars per hour. Find the time interval where $L(t) \ge 125$, and find the average value of $L(t)$ for this time interval. Indicate units of measure.

(c) San Francisco will install a traffic light if there is a two-hour time interval in which the product of the number of cars turning left and the number of cars travelling through the intersection exceeds 160,000. In every two-hour interval, 480 cars travel straight through the intersection. Does this intersection need a traffic light? Explain your reasoning.

---

6. The number of parts per million (ppm), $C(t)$, of chlorine in a pool changes at the rate of $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ ounces per day, where t is measured in days. There are 50 ppm of chlorine in the pool at time $t = 0$. Chlorine should be added to the pool if the level drops below 40 ppm.

(a) Is the amount of chorine increasing or decreasing at $t = 9$? Why or why not?

(b) For what value of t is the amount of chlorine at a minimum? Justify your answer.

(c) When the value of chlorine is at a minimum, does chlorine need to be added? Justify your answer.

---

7. The basement of a house is flooded, and water keeps pouring in at a rate of $w(t)=95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t)=275\sin^2\left(t/3\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

(a) Is the amount of water increasing at $t = 15$? Why or why not?
(b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?
(c) At what time t, for $0 \le t \le 18$, is the amount of water in the basement at an absolute minimum? Show the work that leads to this conclusion.
(d) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

---

8. A tank at a sewage processing plant contains 125 gallons of raw sewage at time $t = 0$. During the time interval $0 \le t \le 12$ hours, sewage is pumped into the tank at the rate $E(t)=2+\dfrac{10}{1+\ln(t+1)}$. During the same time interval, sewage is pumped out at a rate of $L(t)=12\sin\left(\dfrac{t^2}{47}\right)$.

(a) How many gallons of sewage are pumped into the tank during the time interval $0 \le t \le 12$ hours?
(b) Is the level of sewage rising or falling at $t = 6$? Explain your reasoning.
(c) How many gallons of sewage are in the tank at $t = 12$ hours?
(d) At what time t, for $0 \le t \le 12$, is the volume of the sewage at an absolute maximum? Show the analysis that leads to your answer. If the

sewage level ever exceeds 150 gallons, the tank overflows. Is there a time at which the tank overflows? Explain.

---

9. The King Philip Problem

On Ocean Beach at the foot of Noriega is a shipwreck buried in the sand. In 1858, the clipper King Philip ran aground. It was stripped of salvageable material, but the 45% of its hull is still buried on the beach, and it appears every few years as the tide washes the sand in and out.

As the sand washes in or is added by the National Park Service, the height changes at a rate of

$$A(t) = \frac{6t}{1+2t},$$

and, as the tide washes sand out, the height changes at a rate of

$$B(t) = 2 + 7\cos\left(\frac{\pi}{15}t\right)\sin\left(\frac{3\pi}{16}t\right)$$

Both $A(t)$ and $B(t)$ are both measured in inches above the wreck per year for $0 \le t \le 10$. At $t = 0$, the height is 10 inches above the wreck.

(a) Find the total inches of sand above the wreck which are washed out in the first five years. Indicate the correct units.
(b) Is the rate of change of the height increasing or decreasing at $t = 6$ years? Justify your answer.
(c) Write an expression for $H(t)$, the total number of inches above the wreck at time *t*.
(d) What is the absolute minimum number of inches above the wreck over the course of the ten years described?

---

10. Cat Population Problem

The Peninsula Humane Society (PHS) is dedicated to the care and adoption of as many animals who they receive as possible. Since cats breed seasonally, the number of cats and kittens they receive into their facility in a given year varies roughly sinusoidally with time. The data available from 2019 shows the rate $R(\mathrm{t})$, measured in healthy cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$R(\mathrm{t}) = 120 - 88\cos\left[\frac{\pi}{6}(t-2)\right].$$

The rate $A(\mathrm{t})$ at which adoption occur, measured in cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$A(\mathrm{t}) = 125 - 85\cos\left[\frac{\pi}{6}(t-3)\right].$$

On New Years' Day $(\mathrm{t} = 0)$, there were 131 cats in the PHS Nursery waiting to be adopted.

(a) How many cats and kittens were received at PHS in 2019?
(b) Find $A'(10.3)$. Using the correct units, explain the meaning of $A'(10.3)$ in context of the problem.
(c) Find the number of healthy cats and kittens predicted by the models to be in the PHS facility at the end of 2019.
(d) Find the time when the number of healthy cats and kittens in the PHS facility during 2019 was at an absolute maximum. Include the units.

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11. Handout of AP Questions: AB/BC 2002B #2, AB 2005B #2, AB 2006 #2, BC 2011 #2, BC 2015 #1

4.6 Multiple Choice Homework

1. For $t \geq 0$ hours, H is a differentiable function of t that gives the temperature, in degrees Celsius, at an Arctic weather station. Which of the following is the best interpretation of $H'(24)$?

a) The change in temperature during the first day.
b) The change in temperature during the 24th hour.
c) The average rate at which the temperature changed during the 24th hour.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24th day.

---

2. For $t \geq 0$ hours, H is a differentiable function of t that gives the change in temperature, in degrees Celsius per hour, at an Arctic weather station. Which of the following is the best interpretation of $\int_0^t H(x)\,dx$?

a) The change in temperature during the first t hours.
b) The change in temperature during the first day.
c) The average rate at which the temperature changed during the first t hours.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24th day.

---

3. In the classic 2002 Amusement Park problem, equations $E(t)$ and $L(t)$ were given, representing the rate at which people were entering and leaving the park respectively, for time $9 \leq t \leq 23$, the hours during which the park was open, with t=9 corresponding to 9 am. Let us assume that $F(t) = E(t) - L(t)$. Which of the following is the best interpretation of $F(16)$?

a) The number of people in the park at 4 pm.
b) The number of people entering and leaving the park before 4 pm
c) The average number of people in the park between 9 am and 4 pm.
d) The rate at which the number of people in the park is changing at 4 pm.
e) The rate of change of how quickly the number of people in the park is changing at 4 pm.

---

4. The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weight of documents in pounds. Of the following, which is the best interpretation of $C'(500) = 80$?

a) The cost to shred 500 pounds of documents is \$80.

b) The average cost to shred documents is $\frac{80}{500}$ dollar per pound.

c) Increasing the weight of documents by 500 pounds will increase the cost to shred the documents by approximately \$80.

d) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

---

5. An ice field is melting at the rate $M(t) = 4 - \sin^3 t$ Acre-feet per day, where t is measured in days. How many acre-feet of this field will melt from the beginning of Day 1 $(t = 0)$ to the beginning of Day 4 $(t = 3)$?

a) 6.846 b) 10.667 c) 10.951 d) 11.544 e) 11.999

---

6. Let $R(t)$ represent the rate in gal/hr at which water is leaking out of a tank, where is measured in hours. Which of the following expressions represents the average rate of change of gallons of water per hour that leaks out in the first three hours?

a) $\int_0^3 R(t)\,dt$ b) $\frac{1}{3}\int_0^3 R(t)\,dt$ c) $\int_0^3 R'(t)\,dt$

d) $R(3) - R(0)$ e) $\frac{R(3) - R(0)}{3 - 0}$

---

7. The rate of natural gas sales for the year 1993 at a certain gas company is given by $P(t) = t^2 - 400t + 160000$, where $P(t)$ is measured in gallons/day and t is the number of days in 1993 from day 0 to 365. To the nearest gallon, what is the average rate of natural gas sales at this company for the 31 days of January 1993?

a) 4,777,730 b) 4,617,930 c) 154,120

d) 148,965 e) 148,561

---

8. The rate at which ice is melting in a pond is given by $\frac{dV}{dt} = \sqrt{1+2^t}$, where V is the volume of the ice in cubic feet and t is the time in minutes. The amount of ice which has melted in the first five minutes is

a) 14.49 ft^3 b) 14.51 ft^3 c) 14.53 ft^3 d) 14.55 ft^3 e) 14.57 ft^3

---

9. The number of parts per million (ppm), $C(t)$, of chlorine in a pool changes at the rate of $C'(t) = 1 - 3e^{-0.2\sqrt{t}}$ ounces per day, where t is measured in days. There are 10 ppm of chlorine in the pool at time $t = 0$. How many ounces of chlorine are in the pool when $t = 9$?

a) -.646 b) 9.354 c) -9.285

d) 9.285 e) 0.715

---

10. The amount of money in a bank account is increasing at the rate of $R(t) = 10000e^{.06t}$ dollars per year, where t is measured in years. If $t = 0$ corresponds to the year 2005, then what is the approximate total amount of increase from 2005 to 2007.

a) \$21,250 b) \$4,500 c) \$18,350

d) \$32,560 e) \$16,250

---

11. The rate at which water is pumped into a tank is $r(t) = 20e^{0.02t}$, where t is in minutes and $r(t)$ in gallons per minute. Approximately how many gallons of water are pumped into the tank during the first five minutes?

a) 20 b) 22 c) 85 d) 105 e) 150

---

12. Oil is leaking from a tanker at the rate of $R(t) = 2000e^{-0.2t}$ gallons per hour, where t is measured in hours. How much oil leaks out of the tanker from $t = 0$ to $t = 10$?

a) 54 gallons b) 271 gallons c) 865 gallons

d) 8,647 gallons e) 14,778 gallons

4.7 Table Problems

Reasoning from Tabular Data (aka Table Problems) has been one of the most commonly recurring topics on the AP Exam. It has been the first question on the Free Response part of the Exam for the past six years (2014 - 2019).

There are two main kinds of table problems:

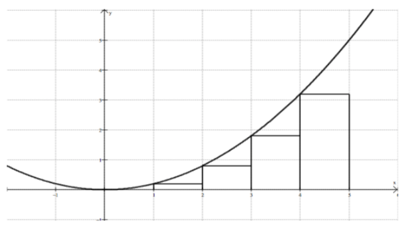
1. Multiple choice questions often have tables of values to plug into the Chain, Product, or Quotient Rule.
   - The trick here is that much of the data are distractors.

2. The most common is a word problem involving an unknown function but with specific data points.
   - These are very much in line with the Accumulation of Rates and Rectilinear Motion problems.

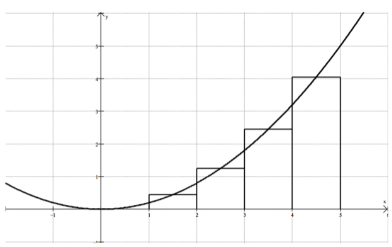
It is this second kind which we will investigate here.

4.6 Approximate Integration – Riemann and Trapezoidal Sums

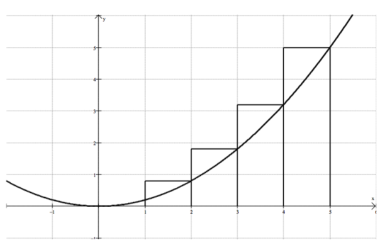
We have been focusing on anti-derivatives of functions where the equation is known. But let us suppose we need to evaluate an integral where either the function is unknown of cannot be anti-differentiated—such as $\int_{-2}^{3} e^{x^2}\,dx$. If we had some exact y-values, we could approximate the area geometrically. This can be done by dividing the area in question into rectangles, and then finding the area of each rectangle. There are three ways to draw the rectangles:



Left-hand Rectangles



Midpoint Rectangles



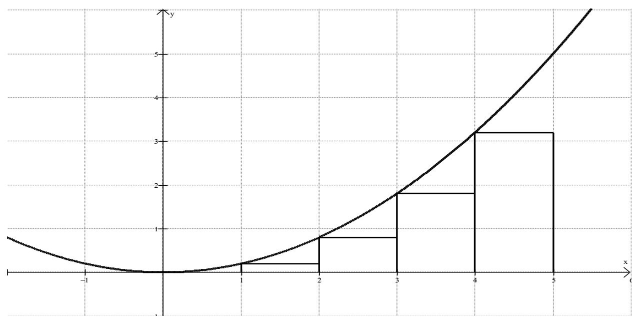
Right-hand Rectangles

Objectives:

Find approximations of integrals using different rectangles.
Use proper notation when dealing with integral approximation.

Ex 1 Use a left end Riemann sum with four equal subintervals to approximate $\int_1^5 f(x)\,dx$ given the table of values below.

x	1	2	3	4	5
$f(x)$	1	4	9	16	25



The heights or each rectangle are the $f(x)$ values on the table, starting from the left, and the widths are the differences between the adjacent x-values.

$$\int_1^5 x^2\,dx \approx 1\cdot(1)+1\cdot(4)+1\cdot(9)+1\cdot(16)=30$$

****Notice the use of $\approx$ and =**

Also notice that the 25 does not get used in the problem as it is not the height of any of the rectangles drawn.

<u>Steps to Approximating an Integral with Rectangles:</u>

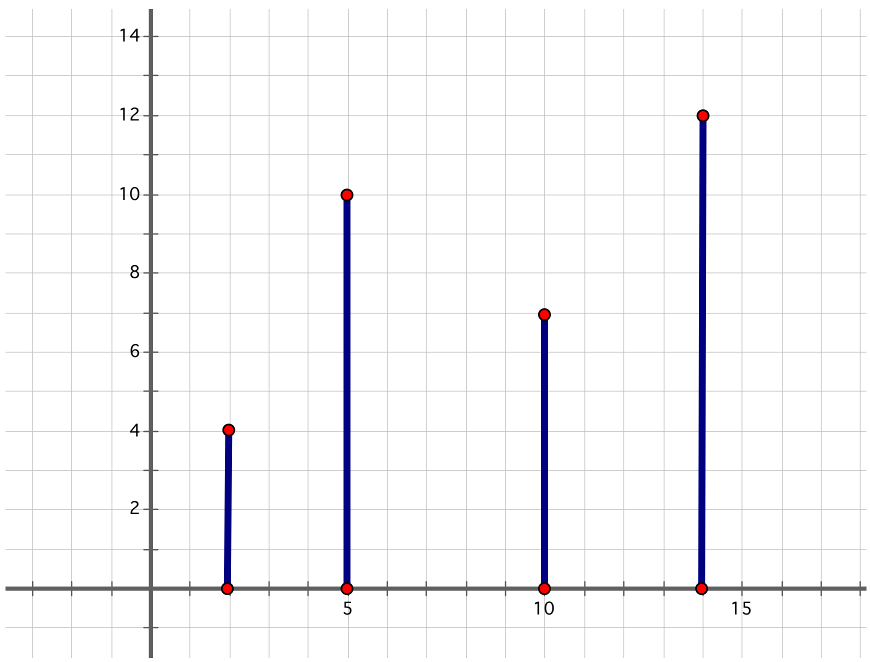
1. Draw the heights represented on the tables.
2. Draw the tops of the rectangles from the left endpoint, right endpoint, or midpoint values.
3. Calculate the areas of each rectangle.
4. Add the areas together for your approximation.
5. State answer using proper notation.

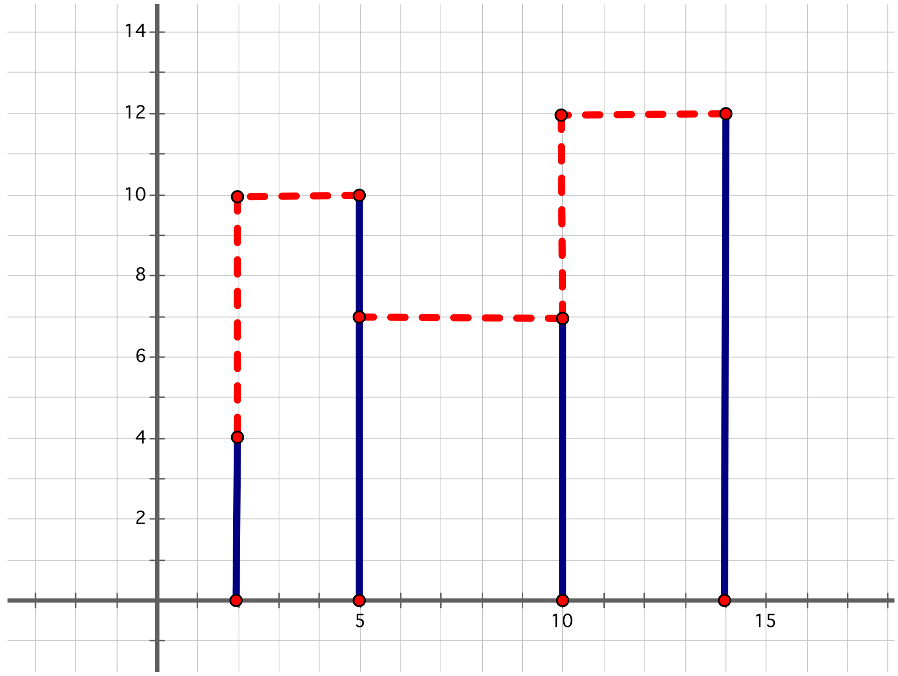
Example 1 had equal width rectangles, but this does not need to be true.

Ex 2   Let f be a differentiable function on the closed interval [2, 14] and which has values as shown on the table below.

x	2	5	10	14
$f(x)$	4	10	7	12

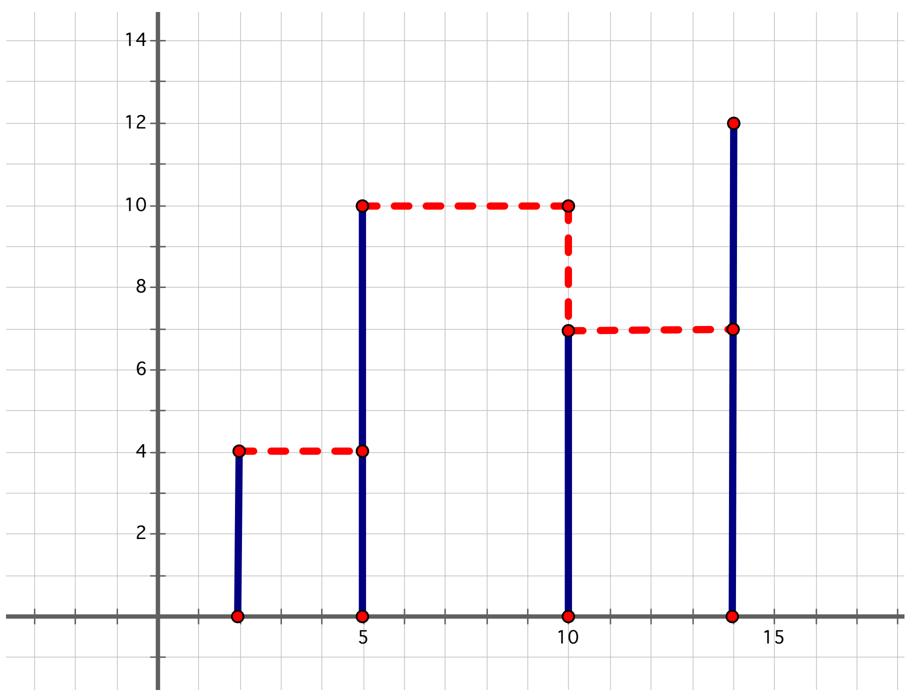
Using the sub-intervals defined by the table values, use the right-hand Riemann sum to approximate $\int_2^{14} f(x)\,dx$.





$$\int_2^{14} f(x)\,dx \approx 3(10)+5(7)+4(12)=113$$

If the question had asked for left hand rectangles, the picture and solution would look like this:



$$\int_2^{14} f(x)dx \approx 3(4)+5(10)+4(7)=90$$

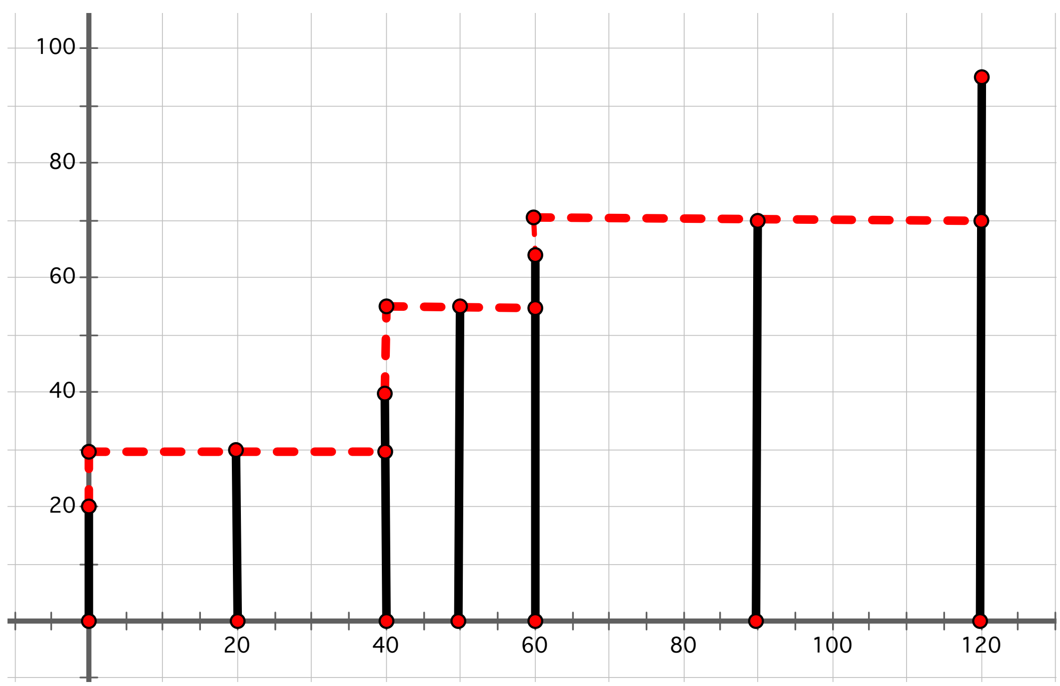
The third kind of Riemann rectangles is where the height comes from the midpoints of the rectangles. Two conditions are needed for Midpoint Rectangles;

1. There need to be an odd number of values.
2. The 2nd, 4th, 6th, etc. x-values must be midpoints between the odd values.

Ex 3 The rate of consumption, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values of $R(t)$ for the time interval $0 \le t \le 90$ is shown below.

t minutes	0	20	40	50	60	90	120
R(t) (Gallons per minute)	20	30	40	55	65	70	95

Use the Riemann sum with Midpoint Rectangles and the subintervals given by the table to approximate the value of $\int_0^{120} R(t)dt$.



$$\int_0^{120} R(t)\,dt \approx 40(30)+20(55)+60(70)=6500$$

Notice that we have half as many rectangles, but each is double width.

Ex 4 Use the midpoint rule and the given data to approximate the value of $\int_0^{2.6} f(x)\,dx$.

x	$f(x)$	x	$f(x)$
0	3.5	1.6	4.7
0.4	2.3	2.1	5.9
0.8	3.2	2.6	4.1
1.2	4.3		

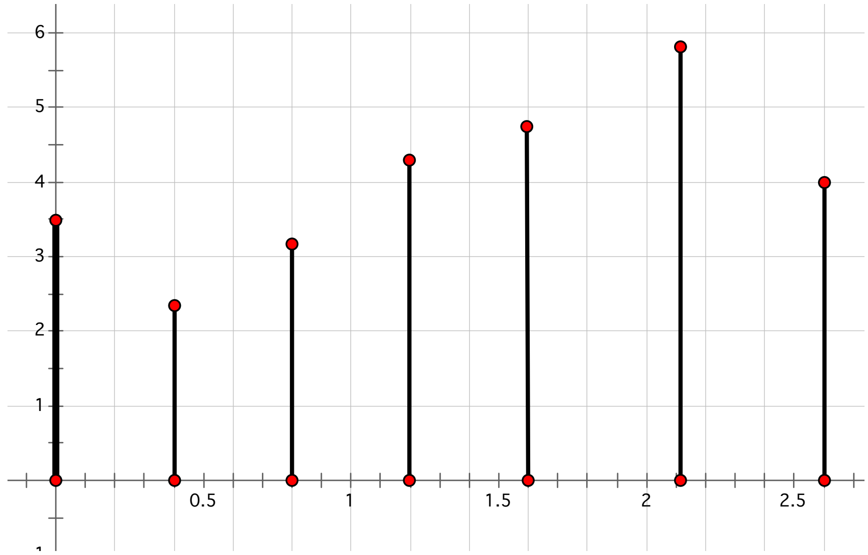
We do not really need the picture to set up the problem:

$$\int_0^{2.6} f(x)\,dx \approx 0.8(2.3)+0.8(4.3)+1.0(5.9)=11.18$$

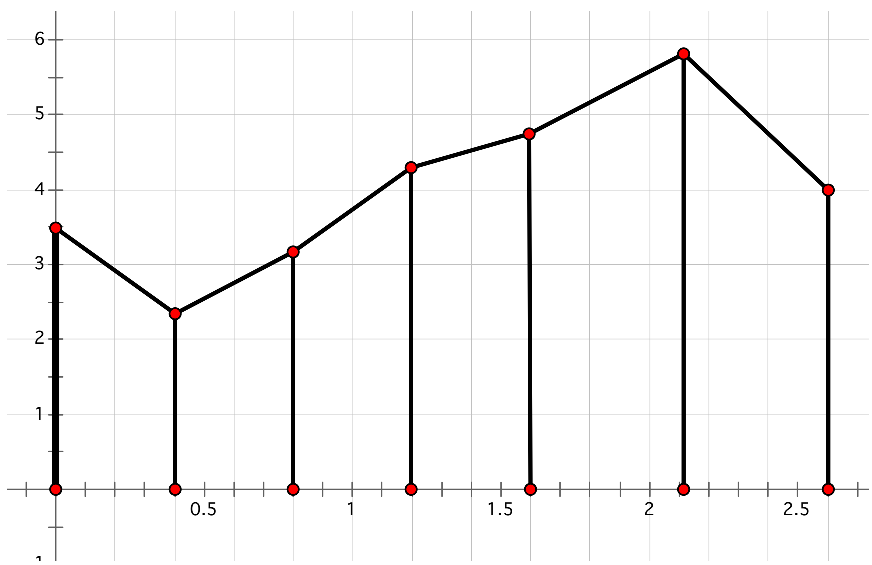
A geometric alternative to rectangles would be to use trapezoids.

Ex 5 Using the table of data from Ex 4, approximate $\int_0^{2.6} f(x)\,dx$ with 6 trapezoids.

We can start by drawing the heights, just as we did with rectangles.



Then we can connect the tops of the heights, making trapezoids:



Remember the area formula for a trapezoid from Geometry:

$$A_{\text{Trap}} = \left(\frac{b_1 + b_2}{2} \right) \cdot h,$$

In this case, b_1 and b_2 are the verticals (i.e., the parallel sides of the trapezoid), and h is the width.

$$\int_0^{2.6} f(x)\,dx \approx (0.4)\left(\frac{3.5+2.3}{2}\right)+(0.4)\left(\frac{2.3+3.2}{2}\right)+(0.4)\left(\frac{3.2+4.3}{2}\right)$$
$$+(0.4)\left(\frac{4.3+4.7}{2}\right)+(0.5)\left(\frac{4.7+5.9}{2}\right)+(0.5)\left(\frac{5.9+4.1}{2}\right)$$

$$=8.15$$

If the table had had equal width subintervals, we could set up the trapezoid area formula and factor out the $\frac{1}{2}$ and the widths. The widths would be $\frac{b-a}{n}$**,** where *n* is the number of trapezoids. Left behind would be the sum of the heights where each height, *except for the first and last*, would appear twice. The result would look like this:

<u>The Trapezoidal Rule*</u>

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]$$

*The Trapezoidal Rule requires ***equal*** sub-intervals

Ex 6 The following table gives values of a continuous function. Approximate the average value of the function using the Trapezoidal Rule.

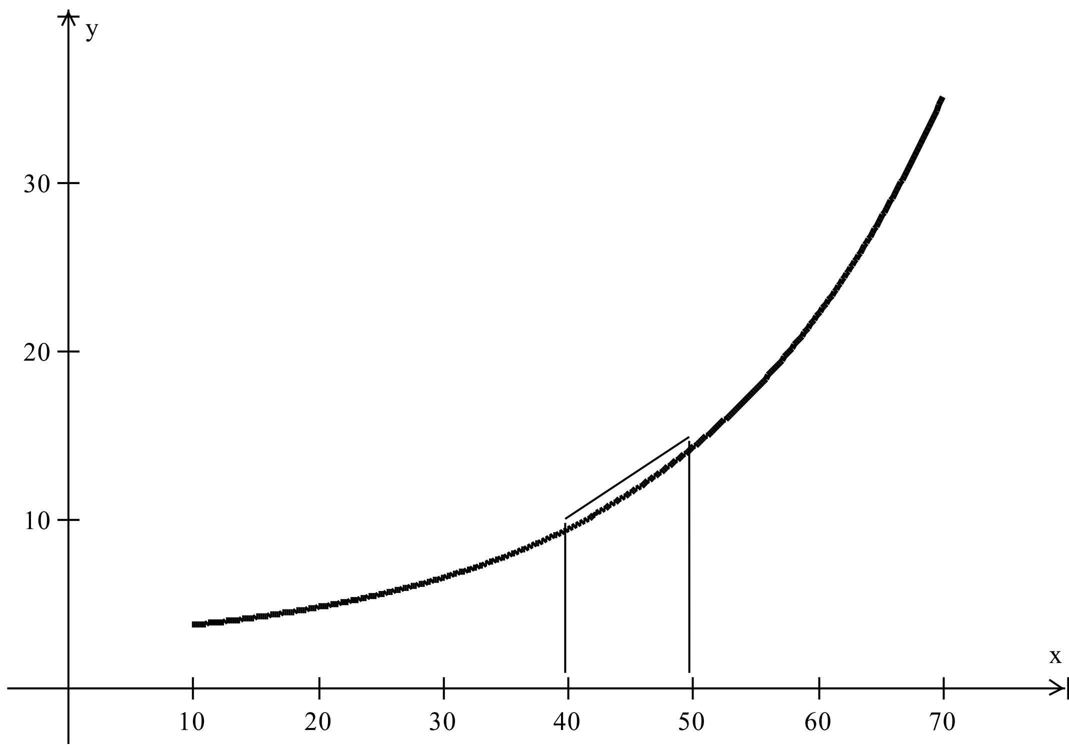
x	10	20	30	40	50	60	70
$f(x)$	3.649	4.718	6.482	9.389	14.182	22.086	35.115

$$f_{avg} = \frac{1}{70-10}\int_{10}^{70} f(x)\,dx$$
$$\approx \frac{1}{70-10}\left[\frac{70-10}{12}\left(\begin{array}{l}3.649+2(4.718)+2(6.482)+\\2(9.389)+2(14.182)+2(22.086)+35.115\end{array}\right)\right]$$
$$=12.707$$

Ex 7   Let us assume that the function that determined the values in the chart above is $f(x) = 2 + e^{.05x}$. Calculate the average value of the function and compare it to your approximations.

$$f_{avg} = \frac{1}{70-10}\int_{10}^{70}\left(2+e^{.05x}\right)dx = 12.489$$

If we want to figure out if our approximations are overestimates or underestimates, we have to look at the graph of the function.

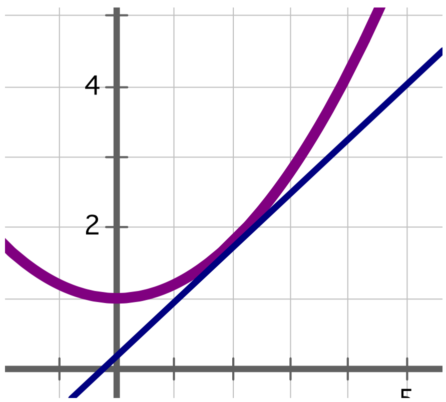
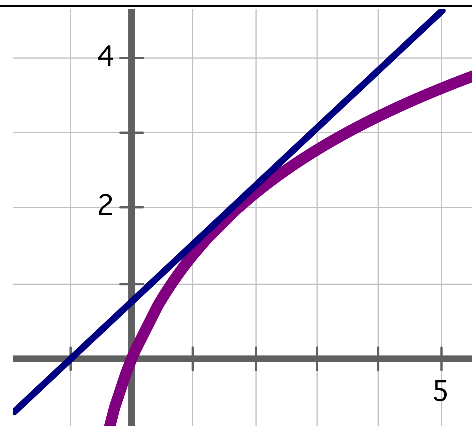


We can see that, rather than the function being increasing or decreasing, over- and under-estimation with trapezoids (as well as with midpoint rectangles) is determined by the concavity of the curve.

Remember from Chapter 1:

<u>Tangent Line Approximations</u>

- Tangent line approximations are an overestimate if the curve is concave down (since your "tangent lines" will be above the curve).
- Tangent line will be an underestimate if the curve is concave up (since your "tangent lines" will be below the curve).

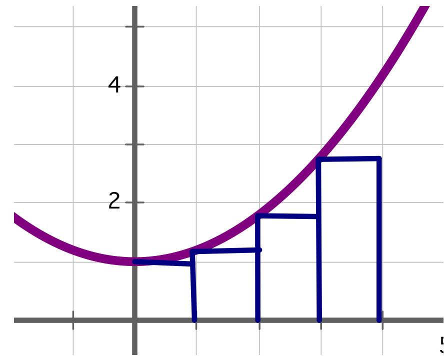
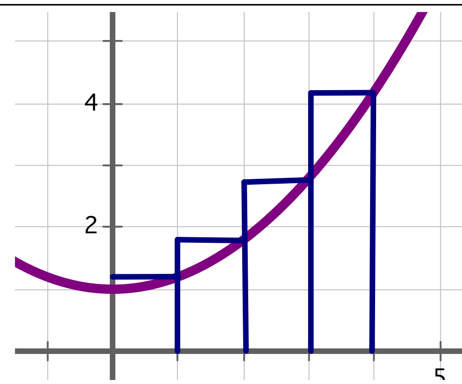
Concave Up	**Concave Down**
 Under-estimate	 Over-estimate

Add to that:

<u>Integral Approximations</u>

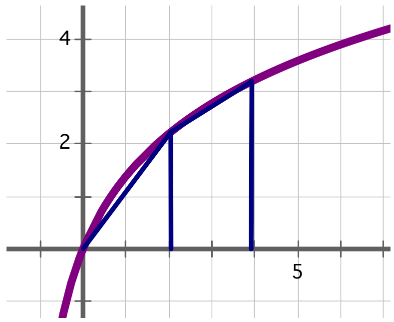
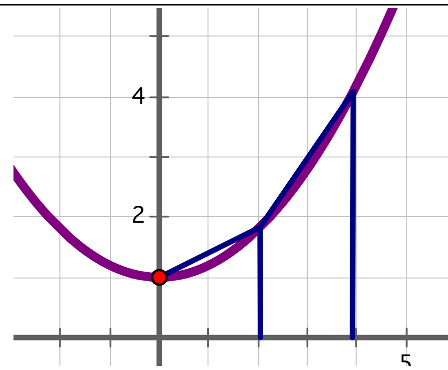
- Left Hand Rectangles are an overestimate if the curve is decreasing and an underestimate if the curve is decreasing.
- Right Hand Rectangles are an underestimate if the curve is decreasing and an overestimate if the curve is decreasing.

These facts might be better understood visually:

	Decreasing	**Increasing**
Left Hand	 Over-estimate	 Under-estimate
Right Hand	 Under-estimate	 Over-estimate

<u>Integral Approximations</u>

- Midpoint Rectangles are an overestimate if the curve is concave down and an underestimate if the curve is concave up.
- Midpoint Rectangles and Trapezoids are an underestimate if the curve is concave down and an overestimate if the curve is concave up.

	Concave Down	**Concave up**
Trapezoidal	 Under-estimate	 Over-estimate

<u>There is not a consistent rule for Midpoint estimates.</u>

4.6 Free Response Homework

1. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

$t(s)$	v (mi/h)	$t(s)$	v (mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

2. The following table gives values of a continuous function. Estimate the average value of the function on $x \in [0, 8]$ using (a) Right-Hand Riemann rectangles, (b) Left-Hand Riemann rectangles, and (c) Midpoint Riemann rectangles.

x	0	1	2	3	4	5	6	7	8
$F(x)$	10	15	17	12	3	-5	8	-2	10

3. Below is a chart showing the rate of a rocket flying according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	10	20	30	40	50	60
$v(t)$ (in km/min)	30	28	32	18	52	48	28

a) Find an approximation for $\int_0^{60} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{30} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

c) Find an approximation for $\int_{30}^{60} v(t)\,dt$ using left rectangles. Make sure you express your answer in correct units.

d) Find an approximation for $\int_0^{40} v(t)\,dt$ using right rectangles. Make sure you express your answer in correct units.

4. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	8	16	24	32	40	48
$V(t)$ (in m^3/min)	26	32	43	24	19	24	26

a) Find an approximation for $\int_0^{48} V(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{16} V(t)\,dt$ using right Riemann rectangles. Make sure you express your answer in correct units.

5. Below is a chart of your speed driving to school in meters/second. Use the information below to find the values in a) and b) below.

t (in seconds)	0	30	90	120	220	300	360
$v(t)$ (in m/sec)	0	21	43	38	30	24	0

a) Find an approximation for $\int_0^{360} v(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{220} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

6. Below is a chart showing the velocity of the Flash as he runs across the country. Use this information to answer each of the following.

t (in seconds)	0	4	8	12	16	20	24
$W(t)$ (in km/second)	10	12	15	19	24	18	7

a) Find an approximation for $\int_0^{24} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units. Describe what this integral means.

b) Find an approximation for $\int_0^{16} v(t)\,dt$ using Trapezoids. Make sure you express your answer in correct units.

7. Below is a chart showing the rate of sewage flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	82	40	38	30	68

a) Find an approximation for $\int_0^{20} V(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{20} V(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

8. Star Formation Rate (SFR) observations of red-shift allow scientists to track the total mass gained in a galaxy by the making of new stars. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

SFR is measured in solar masses per cubic parsec per gigayear (millions of years) and t is measured in gigayears.

a) Use midpoint rectangles to approximate the total star formation. Using the correct units, explain the meaning of your result.

b) Is there a time when $SFR = 0$? Justify your answer.

c) Is there a time when $SFR = 0.0056$? Justify your answer.

9. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the given integral with the specified value of n.

$\int_0^2 \sqrt[4]{1+x^2}\,dx,\ n=8$

10. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the $\int_1^2 \frac{\ln x}{1+x}dx$ with $n=10$.

11. If w'(t) is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

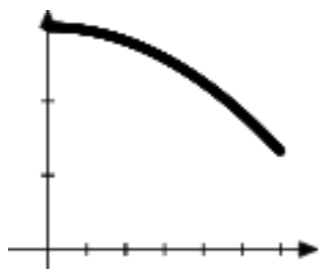
12. If oil leaks from a tank at a rate of r(t) gallons per minute at time t, what does $\int_0^{120} r(t)\,dt$ represent?

13. If x is measured in meters and $f(x)$ is measured in newtons, what are the units for $\int_0^{100} f(x)\,dx$?

14. AP Handout: AB 1998 #3, AB 2001 #2, BC2007 #5

4.6 Multiple Choice Homework

1. The graph of the function f is shown below for $0 \le x \le 3$.



Of the following, which has the smallest value?

a) $\int_1^3 f(x)\,dx$

b) Left Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 6 equal sub intervals.

c) Right Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 6 equal sub intervals.

d) Midpoint Riemann sum approximation of $\int_1^3 f(x)\,dx$ with 6 equal sub intervals.

e) Trapezoidal sum approximation of $\int_1^3 f(x)\,dx$ with 6 equal sub intervals.

Time (sec)	0	10	25	37	46	60
Rate (gal/sec)	500	400	350	280	200	180

2. The table above gives the values for the rate (in gal/sec) at which water flowed into a lake, with readings taken at specific times. A right Riemann sum, with the five subintervals indicated by the data in the table, is used to estimate the total amount of water that flowed into the lake during the time period $0 \le t \le 60$. What is this estimate?

a) 1,910 gal b) 14,100 gal c) 16,930 gal

d) 18,725 gal e) 20,520 gal

---

3. A car is traveling on a straight road such that selected measures of the velocity have values given on the table below.

t	10	20	40	70	80
$v(t)$	90	88	100	90	85

Using four Left Hand Reimann rectangles based on the table, the estimated distance traveled by the car between $t = 10$ and $t = 80$ seconds is

a) 6125 b) 6380 c) 6430 d) 6495 e) 6560

---

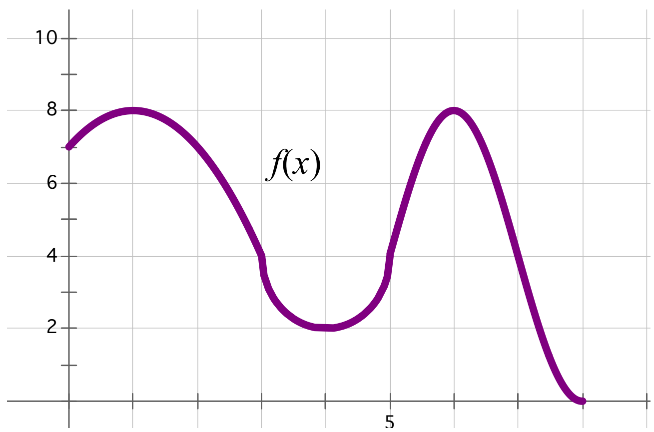
x	3	6	9	12
$f(x)$	12	18	7	5

4. The function f is continuous and differentiable on the closed interval [3,12], what is the right Riemann approximation of $\int_3^{12} f(x)dx$?

a) 69 b) 90 c) 111 d) 126 e) 201

---

5. Consider the function f whose graph is shown below:

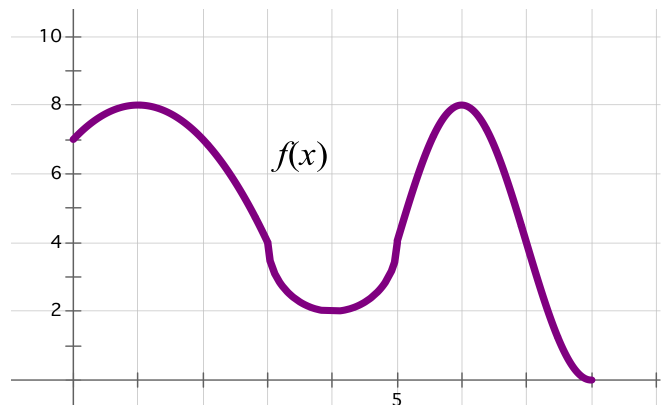


The approximate value of $\int_0^8 f(x)\,dx$, using 4 midpoint rectangles with equal width, is

a) 20 b) 37 c) 40 d) 40.5 e) 44

---

6. Consider the function f whose graph is shown below:

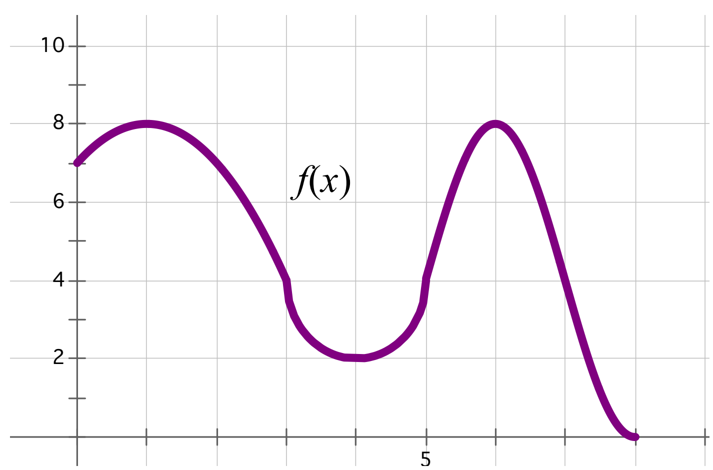


The approximate value of $\int_0^8 f(x)\,dx$, using eight right-hand rectangles with equal width, is

a) 18.5 b) 37 c) 40 d) 40.5 e) 44

---

7. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight left-hand rectangles with equal width, is

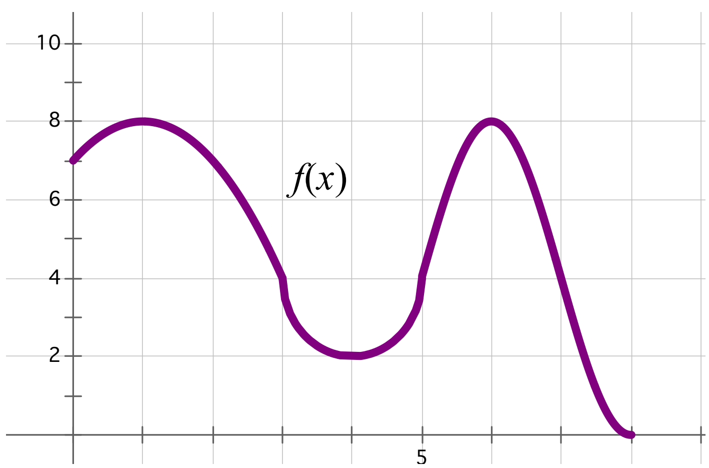
a) 23 b) 37 c) 40 d) 40.5 e) 44

x	2	5	10	14
$f(x)$	12	28	34	30

8. Let f be a differentiable function on the closed interval [2, 14] and which has values as shown on the table above. Using the sub-intervals defined by the table values and using right hand Riemann sums, $\int_2^{14} f(x)\,dx =$

a) 296 b) 312 c) 343 d) 374 e) 390

9. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight trapezoids with equal width, is

a) 37 b) 40 c) 40.5 d) 44 e) 48

10. The following table lists the known values of a function $f(x)$.

x	1	2	3	4	5
$f(x)$	0	1.1	1.4	1.2	1.5

If the Trapezoidal Rule is used to approximate $\int_1^5 f(x)\,dx$ the result is

a) 4.1 b) 4.3 c) 4.5 d) 4.7 e) 4.9

---

t	0	1	2	3	4
$H(t)$	0	1.3	1.5	2.1	2.6

11. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of each day for four days. The data, where $H(t)$ is measured in inches per day and t is measured in days, are listed above. Using the trapezoidal rule, which of the following represents an estimate of of the average rate of growth of the plant over the four-day period?

a) $\frac{1}{4}(0+1.3+1.5+2.1+2.6)$

b) $\frac{1}{4}\left[\frac{1}{2}(0+1.3+1.5+2.1+2.6)\right]$

c) $\frac{1}{4}\left[\frac{1}{2}(0+2(1.3)+2(1.5)+2(2.1)+(2.6))\right]$

d) $\frac{1}{4}\left[\frac{1}{2}(0+2(1.3)+2(1.5)+2(2.1)+2(2.6))\right]$

e) $\frac{1}{4}\left[\frac{1}{4}(0+2(1.3)+2(1.5)+2(2.1)+2.6)\right]$

Remember:

Instantaneous rate of change: $\frac{dx}{dt}$

Average rate of change: $\frac{f(b)-f(a)}{b-a}$

Average value: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Total change: $\int_a^b R(t)\,dt$ or $\int_a^t \left[\text{incoming rate} - \text{outgoing rate}\right] dx$

Total rate of change: $\text{incoming rate} - \text{outgoing rate}$

Total Amount: $Total(t) = \text{initial value} + \int_a^t \left[\text{incoming rate} - \text{outgoing rate}\right] dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(\text{Rate of Change})$ is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(\text{Rate of Change})$ is positive

Objective

Analyze the relationship between rates of change and integrals in light of data given in a tabular format.

Let us consider a problem similar to the ones in the Accumulation of Rates Section, but with a table of data for part of the problem.

Ex 1 At 6am at the *Popular Potatoes* potato chip factory, there are already 5 tons of potatoes in the factory. More potatoes are delivered from 6am ($t=6$) until noon ($t=12$) at a rate modeled by

$$P(t) = 9 - \frac{9\sin(x-2)}{x-2} \text{ tons of potatoes per hour.}$$

Workers arrive at 6am and begin to process the potatoes to turn them into potato chips. Their supervisor measures their rate of output every hour and records her findings in the chart below.

t = time after midnight in hours	6	9	13	14	16
$C(t)$= Rate of potatoes processed in tons/hour	7.9	6.5	3.9	3.1	1.3

The supervisor determines that the workers' rate of processing is a decreasing function throughout the day.

(a) How many tons of potatoes arrive at the *Popular Potatoes* factory between 6am and noon?
(b) Use a Left Riemann sum with subintervals indicated by the table to approximate $\int_6^{16} C(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.
(c) Is your approximation in part (b) an under- or over-approximation? Explain.
(d) The workers end their shift at 4pm. At that time, are there still potatoes in the factory left to process? Explain your reasoning.

(a) How many tons of potatoes arrive at the *Popular Potatoes* factory between 6am and noon?

$$\int_6^{12} P(t)\,dt = 54.899 \text{ } tons\,of \text{ } potatos$$

(b) Use a Left Riemann sum with subintervals indicated by the table to approximate $\int_6^{16} C(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

$$\int_6^{16} C(t)\,dt \approx 3(7.9)+4(6.5)+1(3.9)+2(3.1)=59.8$$

Approximately 59.8 tons of potatoes were processed into chips between 6 a.m. and 4 p.m.

(c) Is your approximation in part (b) an under- or over-approximation? Explain.

The approximation is an over-estimate because the data on the table show a decreasing function and left-hand Riemann rectangles over-estimate a decreasing function.

(d) The workers end their shift at 4pm. At that time, are there still potatoes in the factory left to process? Explain your reasoning.

Yes, there were still potatoes left at the end of the day because, while 59.8 tons of potatoes were processed, there were 59.899 tons on site—5 tons at the beginning of the day and 54.899 tons which were delivered.

Ex 2 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{20} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Find the value of $\int_0^{20} V'(t)\,dt$ and explain the meaning of this value in the context of the problem.

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

$$V'(7) = \frac{V(10)-V(6)}{10-6} = \frac{48-83}{4} = 8.75\,{}^{gal}\!/\!_{\min^2}$$

(b) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{20} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

$$\int_0^{20} V(t)\,dt \approx 4\left(\frac{83+68}{2}\right)+2\left(\frac{68+83}{2}\right)+4\left(\frac{83+48}{2}\right)+3\left(\frac{48+38}{2}\right)$$
$$+2\left(\frac{38+30}{2}\right)+5\left(\frac{30+38}{2}\right)$$
$$=1082$$

Approximately 1082 gallons of sewage flowed through the pipeline between $t=0$ and $t=20$ minutes.

(c) Find the value of $\int_0^{20} V'(t)\,dt$ and explain the meaning of this value in the context of the problem.

$$\int_0^{20} V'(t)\,dt = V(20) - V(0) = 38 - 83 = -45\, {}^{gal}\!/_{\min}$$

The total change of the rate of flow between $t = 0$ and $t = 20$ minutes is -45 gallons per minute.

The Mean Value and Rolle's Theorems

The Mean Value Theorem is an interesting piece of the history of Calculus that was used to prove a lot of what we take for granted. The Mean Value Theorem was used to prove that a derivative being positive or negative told you that the function was increasing or decreasing, respectively. Of course, this led directly to the first derivative test and the intervals of concavity.

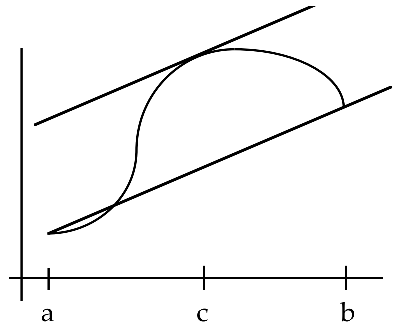
Mean Value Theorem

If f is a function that satisfies these two hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)

Then there is a number c in the interval (a,b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Again, translating from math to English, this just says that, if you have a smooth, continuous curve, the slope of the line connecting the endpoints has to equal the slope of a tangent somewhere in that interval. Alternatively, it says that the secant line through the endpoints has the same slope as a tangent line.



MEAN VALUE THEOREM

Ex 3 Show that the function $f(x) = x^3 - 4x^2 + 1,\ [-1, 3]$ satisfies all the conditions of the Mean Value Theorem and find c.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c) = 3c^2 - 8c \qquad \frac{f(3) - f(-1)}{3 - (-1)} = \frac{-8 - (-4)}{4} = -1$$

$$3c^2 - 8c = -1$$
$$3c^2 - 8c + 1 = 0$$
$$c = \frac{8 \pm \sqrt{8^2 - 4(3)(1)}}{2(3)}$$

$$c = 2.535 \ or \ 0.131$$

Rolle's Theorem is a specific a case of the Mean Value theorem, though Joseph-Louis Lagrange used it to prove the Mean Value Theorem. Therefore, Rolle's Theorem was used to prove all of the rules we have used to interpret derivatives for the last couple of years. It was a very useful theorem, but it is now something of a historical curiosity.

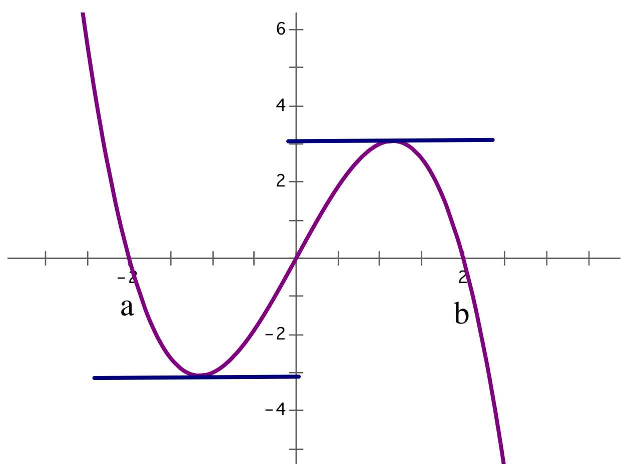
Rolle's Theorem

If f is a function that satisfies these three hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)
3. $f(a)=f(b)$

Then there is a number c in the interval (a,b) such that $f'(c)=0$.

Written in this typically mathematical way, it is a bit confusing, but it basically says that if you have a continuous, smooth curve with the initial point and the ending point at the same height, there is some point in the curve that has a derivative of zero. If you look at this from a graphical perspective, it should be pretty obvious.



Ex 4 Show that the function $f(x)=x^2-4x+1,\ [0,4]$ satisfies all the conditions of the Mean Value Theorem and find c.

Polynomials are continuous throughout their domain, so the first condition is satisfied.

Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c)=\frac{f(b)-f(a)}{b-a}\rightarrow f'(c)=\frac{f(4)-f(0)}{4-1}=0$$

$$f'(c)=2c-4=0$$

$$c=2$$

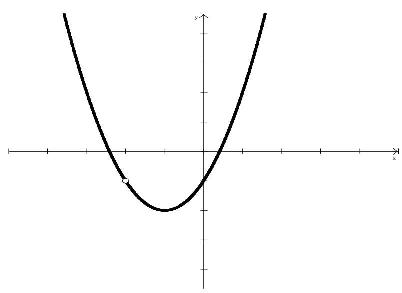
We need to consider when the MVT might not apply to a problem—that is, when is a function not continuous or not differentiable. We saw the first in PreCalculus, but did not always name it "discontinuity." Continuity basically means a function's graph has no breaks in it. The formal definition involved limits and we will explore that in the Limit Chapter.

Since all the families of functions investigated in PreCalculus were continuous in their domain, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

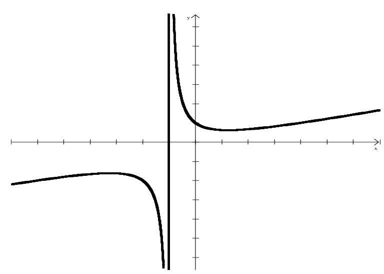
Removable Discontinuity
($\lim\limits_{x\to a} f(x)$ does exist)

Essential Discontinuity
($\lim\limits_{x\to a} f(x)$ does not exist)

f(a) does not exist

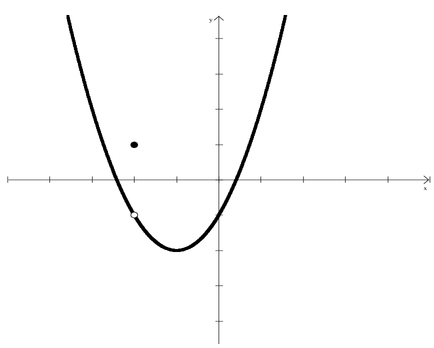


Point of Exclusion (POE)

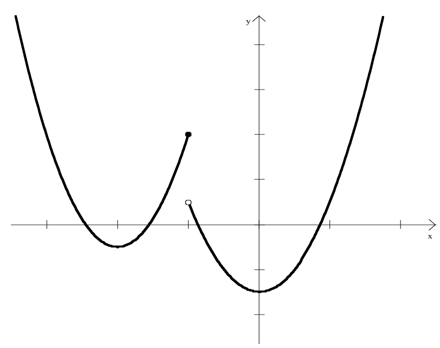


Vertical asymptotes

f(a) exists



Point of Displacement (POD)



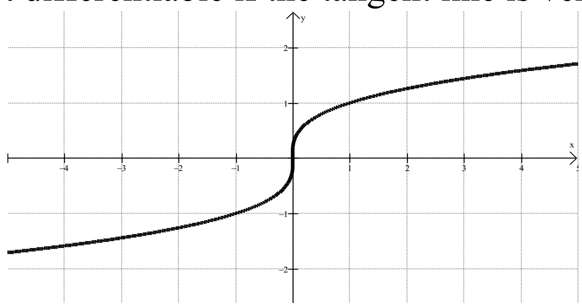
Jump Discontinuity

"Not differentiable" simply means the derivative does not exist. This can happen one of three ways.

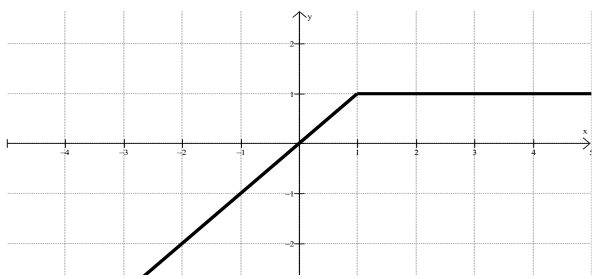
1. A function is not differentiable if it is not continuous.

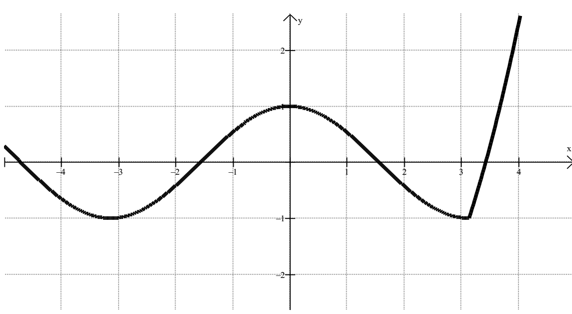
For pictures, see above.

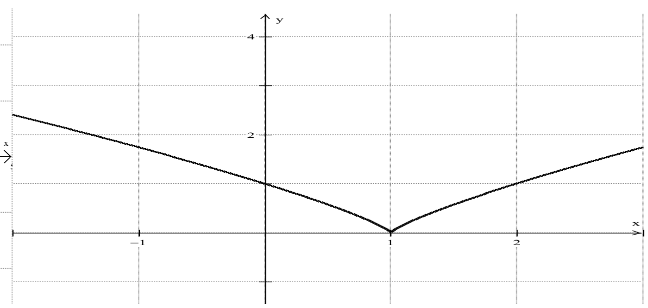
2. A function is not differentiable if the tangent line is vertical.



3. A function is not differentiable if it is not "smooth."







NB. All the families of functions studied in PreCaculus are continuous and differentiable in their domains.

The MVT often arises in AP Table Problems like this one:

Ex 5   Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(d)   Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$

(d)   Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$.

Because $V(t)$ is a continuous and differentiable function, the Mean Value Theorem applies. Since $V(0)=83=V(6)$, there must be a c-value between $t=0$ and $t=6$ where $V'(t)=0$. Similarly, since $V(13)=38=V(26)$, there must be a c-value between $t=13$ and $t=20$ where $V'(t)=0$. There might be more c-values where $V'(t)=0$, but the MVT guarantees at least two.

<u>Other Theorems that can be confused with</u>
<u>The Mean Value Theorem</u>

The Average Value Theorem: The average value of a function *f* on a closed interval [*a*, *b*] is defined as $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Average Rate of Change: The average rate of change of a function *f* on a closed interval [*a*, *b*] is defined as $= \frac{f(b)-f(a)}{b-a}$.

The Intermediate Value Theorem: If *f* is continuous on the closed interval [*a*, *b*] then $f(x)$ attains every height between $f(a)$ and $f(b)$.

One consequence of the Intermediate Value Theorem is that if $f(a)$ and $f(b)$ are opposite signs, there is a zero in the closed interval.

Ex 6 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function. Explain why there are at least two times when $V(t)=74$.

Because $V(t)$ is continuous and $68<74<83$, according to the IMT, there must be at least one time between $t=0$ and $t=4$ and at least one time between $t=4$ and $t=6$ when $V(t)=74$.

4.7 Free Response Homework

1. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	8	16	24	32	40	48
$V(t)$ (in m^3/min)	26	32	43	24	19	24	26

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Find $\int_8^{40} V'(t)\,dt$

(c) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{48} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{48}\int_0^{48} V(t)\,dt$ in the context of the problem.

---

t days	0	1	2	3	4
$H(t)$ in mm per day	0	1.3	1.5	2.1	2.6

2. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of each day for four days. The data, where $H(t)$ is measured in millimeters per day and t is measured in days, are listed above.

(a) Estimate $H'(3)$. Show the work that leads to your answer. Indicate the units.

(b) Explain how one would know that the plant's growth is not increasing at a decreasing rate

(c) Use right-hand rectangles with subintervals indicated by the table to approximate $\int_0^4 H(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{4}\int_0^4 H(t)\,dt$ in the context of the problem.

---

3. The rate of consumption of fuel, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values of $R(t)$ for the time interval $0 \le t \le 90$ is shown below.

t minutes	0	20	40	50	60	90
$R(t)$ (gallons per minute)	20	30	40	55	65	70

(a) Estimate $R'(30)$. Show the work that leads to your answer. Indicate the units.

(b) Use right hand Riemann rectangles to approximate $\int_0^{90} R(t)dt$ and indicate units of measure. Explain the meaning of $\int_0^{90} R(t)dt$ in terms of the fuel consumption.

(c) Use left hand rectangles to find $\frac{1}{70}\int_{20}^{90} R(t)dt$. Using the correct units, explain the meaning of $\frac{1}{70}\int_{20}^{90} R(t)dt$ in terms of the fuel consumption.

---

4. A diabetic patient tests his blood glucose level every morning. After being put on insulin, the data below show the glucose levels in milligrams per deciliter (mg/dL) over one week.

t days	1	2	3	4	5	6	7
$G(t)$ (mg/dL)	233	198	185	168	147	130	147

Let $G(t)$ represent the glucose level where t is measured in days.

(a) Estimate $G'(3.7)$. Using the correct units, explain the meaning of the result.

(b) Use midpoint Riemann rectangles to approximate $\int_1^7 G(t)dt$. Using the correct units, explain the meaning of $\frac{1}{7}\int_1^7 G(t)dt$ in terms of the patient's glucose levels.

(c) Ignoring the last data point, $M(t) = 237.6e^{-0.082t}$ is a model of $G(t)$. Find $M'(3.7)$. Is $M(t)$ decreasing at an increasing rate? Show the work that leads to your conclusion.

---

5. Diabetic patients take a test called an A1c every three months which measures the three-month average percentage of glycated hemoglobin (that is, hemoglobin covered in glucose).

Month	0	3	6	9	12	15	18	21
$A(t)$ (as a %)	10.2	10.0	10.5	9.1	8.0	8.9	8.3	8.6

Let $A(t)$ represent the A1c score, measured as a percentage.

(a) Find $\int_0^{21} A'(t)dt$. Show the work that leads to your answer. Explain the meaning of $\int_0^{21} A'(t)dt$ in terms of A1c scores.

(b) Use right-hand Riemann rectangles to approximate $\int_0^{21} A(t)dt$. Using the correct units, explain the meaning of $\frac{1}{21}\int_0^{21} A(t)dt$ in terms of the patient's A1c score.

(c) $B(t) = -.305x + 10.571 + .1\sin \pi x$ is a model of $A(t)$. Find $\frac{1}{21}\int_0^{21} B(t)dt$.

---

t months	0	1	2	3	4	5	6
$P(t)$	160.3	192.8	345.7	746.1	944.2	873.0	1128.6

t months	7	8	9	10	11	12
$P(t)$	928.3	851.3	751.3	535.5	216.4	150.7

6. A family leases solar panels on their house. At the end of the year, they receive a report, including the tables above, which shows the monthly production $P(t)$ of electricity, in kilowatts per month (kW/month), from the panels.

(a) Use right-hand Riemann rectangles to approximate $\int_0^{12} P(t)dt$. Indicate the units.

(b) $k(t)=660-489\cos\frac{\pi}{6}t$ is a model of $P(t)$. Find $\int_0^{12} k(t)dt$.

(c) Using the model $k(t)=660-489\cos\frac{\pi}{6}t$, show that the production is decreasing at $t=9$. Is the production decreasing at an increasig rate?

---

t months	0	1	2	3	4	5	6
$C_e(t)$	390.7	660	667.1	538.4	420.5	412.1	347.8
$C_g(t)$	87.6	84.6	109	116	79.8	53.9	42.9

t months	7	8	9	10	11	12
$C_e(t)$	287.5	303.1	322.4	342.5	390.3	384.2
$C_g(t)$	24.9	25.6	18	20.3	48.9	91.8

7. Dr. Quattrin analyzes his PG&E bill to track his consumption of both electricity ($C_e(t)$) and gas ($C_g(t)$) over the course of a year. The tables above are the result. $C_e(t)$ is measures in kilowatts (kW) and $C_g(t)$ is measured in therms (thm).

(a) Approximate $C_e'(3.4)$ and $C_g'(3.4)$. Use correct units, explain the meaning of these estimations in terms of increasing and/or decreasing consumption of each commodity at $t=3.4$.

(b) Use right hand Riemann rectangles to approximate $\int_0^{12} C_e(t)dt$. Indicate the units.

(c) Use midpoint Riemann rectangles to approximate $\int_0^{12} C_g(t)dt$. Indicate the units.

(d) Using the correct units, explain the meaning of $\frac{1}{12}\int_0^{12} C_g(t)dt$.

---

8. Dr. Quattrin decides to lease solar panels from Sunrun Solar. After a year, he reanalyzes his PG&E bill to track both his consumption of electricity ($C_e(t)$) and his production of electricity ($P_e(t)$) over the course of a year. The tables below show the consumption of electricity, measured in kilowatts (kWs).

t months	0	1	2	3	4	5	6
$C_e(t)$	326.5	660.0	667.1	538.4	420.5	412.1	347.8

t months	7	8	9	10	11	12
$C_e(t)$	287.5	303.1	322.4	342.5	390.3	384.2

$P_e(t) = 407 - 374.2\cos\frac{\pi}{6}t$ models the production in kW per month that PG&E buys back.

(a) How much power does PG&E buy back from the Quattrins over the course of the year? Indicate the units.
(b) Using the trapezoidal sum, approximate the amount of power the Quattrins consume over the course of the year. Based on the estimates to (a) and (b), does Dr. Quattrin owe PG&E for electricity at the end of the year or does PG&E owe Dr. Quattrin a refund?
(c) Electricity costs \$0.28 per kW. Write an expression for amount due on the PG&E bill at time t months.

9. Dr. Quattrin's paternal grandmother's family originated in the Alpine town of Sauris, Italy, where the temperature in January changes at a rate of $W(t)$ degrees Celsius per hour. $W(t)$ is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight ($t = 0$), the temperature in Sauris is $-8°C$.

t	0	1	3	6	8

(in hours after midnight)					
$W(t)$ (in degrees Celsius per hour)	-2.6	-3.1	-1.2	1.9	2.5

a) At approximately what rate is the rate of change of the temperature changing at 2am ($t=2$)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Set up, but do not solve, an integral equation which would determine the temperature in Sauris at 1pm.

---

10. On May 15, the weather in the town of Apcalc changes at a rate of $W(t)$ degrees Fahrenheit per hour. $W(t)$ is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight, t=0, the weather in Apcalc is 40 degrees Fahrenheit.

t (in hours since midnight)	0	1	3	6	8
$W(t)$ (in degrees Fahrenheit per hour)	-2.4	-2.1	-1.2	1.8	4.5

a) At approximately what rate is the rate of change of the temperature changing at 2am (t=2)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Is there a time when the rate of change of the temperature equals 7? Justify your answer.

d) Is there a time in $0 \le t \le 8$ when $W(t) = 0$? Justify your answer.

---

11. Star Formation Rate (*SFR*) observations of red-shift allow scientists to track the gains and approximate future gains. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

Assume the *SFR* data represents a continuous and differentiable function. *SFR* is measured in solar masses per giga-years (millions of years) per cubic parsecs and time t is measured in giga-years.

a) Use midpoint rectangles to approximate the total star formation. Using the correct units, explain the meaning of your result.

b) Is there a time when *SFR* = 0? Justify your answer.

c) Is there a time when *SFR* = 0.0056? Justify your answer.

---

12. The Callaghan Ranch Harvest Problem

In 1867, Dr. Q's great-great grandfather Michael Callaghan bought 160 acres of public land in Solano County to raise wheat and sheep. By 1890, the ranch had grown to 1210 acres. Prior to the advent of the combine harvester, reaping and threshing were done by different machines. The table below shows the rate at which wheat on the ranch was gathered by a McCormick Reaper on a given 12-hour workday.

t	0	2	5	9	12
$R(t)$	3.3	2.5	1.6	2.7	1.6

Time t is measured in hours after 6am $(t=0)$ and $R(t)$ is measured in acres per hour. The wheat was then delivered to the barn and put through the threshing machine. At the start of the day, there are 2.4 acres of wheat in the barn.

(a) Is the rate at which the wheat is being harvested at 9:30 $(t=3.5)$ increasng or decreasing? Explain the result in context of the problem, using the appropriate units.

(b) Use a right-hand sum to approximation to find the total amount of wheat harvested on this particular day.

(c) The thresher processed the wheat into grain and hay at a rate of $P(t)=\dfrac{2.1x}{\sqrt{x^2+1}}$ acres per hour. How much wheat has been thresher by noon $(t=6)$ when the farm hands break for lunch?

(d) Write an equation for $0 \le t \le 12$ which would determine the amount of wheat in the barn at any time t. Using your estimate of the amount of wheat reaped in b) above, find the amount of unthreshed wheat is in the barn at the end of the day.

(e) Assume $R(t)$ can be modeled by the equation $Q(t)=3.3-\dfrac{3}{20}t-.9\sin\left(\dfrac{\pi}{7}t\right)$. What is the minimum amount of unthreshed wheat in the barn?

13. The Yuma Desalting Problem Ib

The desalting plant at Yuma, AZ, removes alkaline (salt) products from the Colorado River the make the water better for irrigation downstream in Mexico. Data from a Pilot Run of the plant shows that water enters the plant at a rate $W(t)$ as shown on the table below:

t in Month	0	1	2	3	4	5	6	7	8	9	10
$W(t)$ in foot-acres per month	0	2375	3189	3411	3207	2169	2269	2151	2167	3022	2293

The rate $P(t)$ of outflow of processed water, in foot-acre per month is modeled by

$$P(t) = -0.55t^4 + 15t^3 - 158t^2 + 722t + 1032$$

For $0 \le t \le 10$. Based on supplies available, not all the water gets processed before returning to the Colorado River.

a) Using a Midpoint Reimann Sum, approximate the volume of water that enters the plant during these ten months.

b) Set up an equation for $U(t)$ which would define the amount of unprocessed water that exits the plant. Using your answer in part a), approximate $U(10)$. Indicate units.

c) Approximate $W'(6)$. Using the correct units, explain the meaning of your answer.

d) Assuming $W(t)$ can be modeled by $E(t) = 2800 + 750\sin\left(\frac{2\pi}{11}t\right)$, find the time at which there is an absolute maximum amount of unprocessed water flowing through the plant for $0 \le t \le 10$. Justify your answer.

---

14. The 49er Sack Leader Problem II

NB's Games completed	0	19	21	41	57
$B(g)$ in sacks per game	0	0.68	0.62	0.98	1.16

The table above shows Nick Bosa's sack rate, in sacks per game, over his first four seasons.

(a) Using a Right-hand Reimann Sum, determine the approximate number of sacks Nick Bosa had during these four years. Round to the neares whole number.

(b) Using the data on the table, estimate $B'(32)$. Based on this estimate and the data on the table, was Bosa's sack total increasing at an increasing or decreasing rate? Using the correct units, explain your answer.

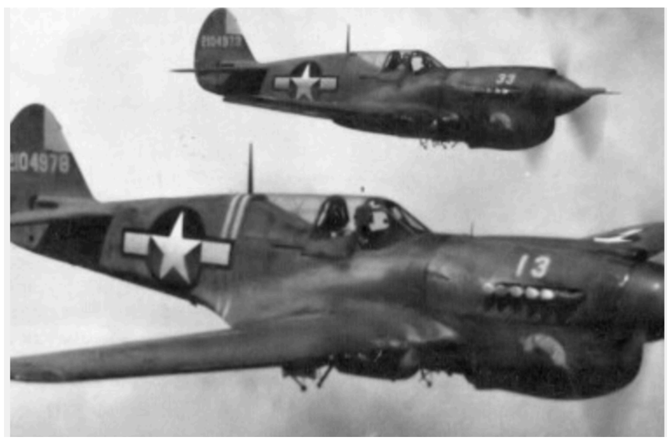
(c) Disregarding his injury-shortened 2020 season, $N(g) = 0.178\sqrt{g}$ models Bosa's sack rate per game during the first four years of his career. Find $\frac{1}{57}\int_0^{57} N(g)\,dg$. Using the correct units, explain the the result in context of the problem.

(d) Assume that $A(g) = -.001g^2 + 0.073g - 0.004$ models the rate of Aldon Smith's sacks over his first 50 games from 2011 to 2014. During what game is the difference between Bosa's and Smith's sack totals at a maximum? Show your calculations.

---

16. WWII Aircraft Problem I

t in Months	6	18	30	42	54	66
$A(t)$ in thousand of Axis-built planes per month	1.06	1.3	1.4	2.1	3.5	5.7
$U(t)$ in thousand of US-built planes per month	0.18	0.5	1.6	4.0	7.2	8.1



The table above shows the rates, in thousand of military planes per month, at which aircraft were built during WWII, where t is measured in months after the beginning of 1939. $A(t)$ represents the rate at which Germany and Japan were building aircraft and $U(t)$ is the rate at which the United States was building aircraft. During the first six months of the War, the Axis powers had produced 4.9 thousand planes and the US had produced 1.2 thousand planes.

(a) Using a Right-Hand Reimann Sum, approximate the number of aircraft build by the Axis powers during these 66 months.

(b) Approximate $U'(20)$. Using the correct units, explain the meaning of this result.

(c) Assume $X(t) = 0.757e^{0.028t}$ models the data for $A(t)$ and $S(t) = 0.164e^{0.067t}$ models the data for $U(t)$. Set up and equation that would model the difference between the Axis and the US aircraft production.

(d) Find the maximum difference between the Axis and the US aircraft production.

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16. AP Handout: AP Calc AB/BC 2004B #3, AP Calc AB 2002 #6

4.7 Multiple Choice Homework

1. Let f be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true of at least one value of x between a and b?

I. $f(x) = 0$ II. $f'(x) = 0$ III. $f''(x) = 0$

a) I only b) II only c) III only

d) II and III only e) I, II, and III

---

2. Which of the following functions **fail** to meet the conditions of the Mean Value Theorem?

I. $3x^{2/3} - 1$ on $[-1, 2]$ II. $|3x - 2|$ on $[1, 2]$

III. $4x^3 - 2x + 3$ on $[-1, 2]$

a) I only b) II only c) III only d) I and II only e) II and III only

---

3. $y = -2x^2 + x - 2$ is defined on $x \in [1, 3]$. Find the c-value determined by the Mean Value Theorem.

a) $x = \frac{1}{2}$ b) $x = \frac{5}{4}$ c) $x = \frac{3}{2}$ d) $x = 2$

e) $x = \frac{9}{4}$

---

4. Let $g(x)$ be a continuous function on the interval $x \in [0, 1]$ and let $g(1) = 0$ and $g(0) = 1$. Which of the following statements is not necessarily true?

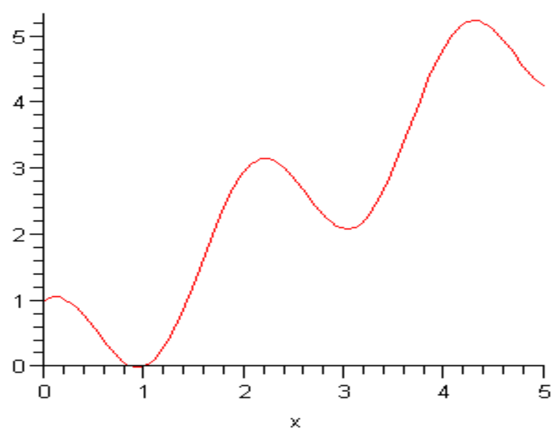
a) The exist a number c on $[0, 1]$ such that $g(c) \geq g(x)$ for all $x \in [0, 1]$

b) For all a and b in $[0, 1]$, if $a = b$, then $g(a) = g(b)$.

c) The exist a number c on $[0, 1]$ such that $g(c) = \frac{1}{2}$

d) The exist a number c on $[0, 1]$ such that $g(c) = \frac{3}{2}$

e) For all c in $(0, 1)$, $\lim_{x \to c} g(x) = g(c)$



5. The graph of a differentiable function f is shown above on the closed interval [0, 5]. How many values of x in the open interval (0, 5) satisfy the conclusion of the Mean Value Theorem for f on [0, 5] ?

a) Two b) Three c) Four d) Five

---

6. To which of the following phrases does the equation $Y = \frac{f(b) - f(a)}{b - a}$ pertain?

a) The Mean Value Theorem

b) The Average Value Theorem

c) The Average Rate of Change

d) The Intermediate Value Theorem

e) Rolle's Theorem

---

t hours	0	2	4	6	8	10	12	14
$E(t)$	10	4	10	5	7	3	7	5

7. The table above shows particular values of a differentiable function $E(t)$. At how many times on the interval $t \in [0, 12]$ does the Mean Value Theorem guarantee that $E'(t) = 0$

a) None b) One c) Two d) Three

4.8: AP-Style Rectilinear Motion Problems

The major emphasis in this section is AP style FRQs that present the given information in either algebraic, tabular, or graphical form.

Key Ideas:

- Know the relations among position, velocity, and acceleration.
- Velocity has direction (positive or negative) but speed does not.

Common Sub-Topics:

- Given velocity, find acceleration at a given time.
- Given velocity, find position at a given time.
- Finding time when particle switches direction
- Total distance versus displacement
- Speeding up or slowing down

OBJECTIVES
Use the derivative and antiderivative to make conclusions about motion. Analyze motion information presented in algebraic, graphical, or tabular format.

Typical non-calculator algebraic problems were addressed in the last section. Here we will concentrate on the **calculator-required** problems.

Calculator-Required Algebraic Format

Ex 1 An object moves along the y-axis with coordinate position $y(t)$ and velocity $v(t) = -1.5 + \sqrt{t} - \dfrac{\cos\left(e^{0.9t}\right)}{t+1}$ for $t \geq 0$ seconds. At time $t = 0$, the object's position is $y = -1.3$ feet.

a) At what time(s) on $t \in [0, 2.5]$, if any, does the particle switch directions? Indicate units.

b) Approximate the particle's acceleration at $t = 2$ seconds. Indicate units

c) What is the total distance traveled by the particle on $t \in [0, 2.5]$.
d) What is the position of the particle on $t = 5.9$?

---

a) At what time(s) on $t \in [0, 2.5]$, if any, does the particle switch directions? Indicate units.

Graph the function and find the zeros by calculator: $t = 1.131,\ 1.538,\ \textit{and}\ 2.285$ seconds.

b) Approximate the particle's acceleration at $t = 2$ seconds. Indicate units

Use Math 8: $a(2) = v'(2) = 0.042\ {ft}/{sec^2}$

c) What is the total distance traveled by the particle on $t \in [0, 2.5]$.

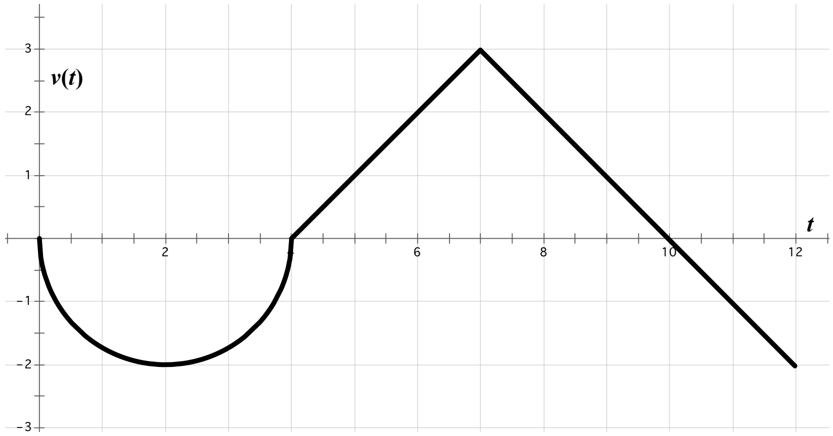
$$Dist = \int_0^{2.5} |v(t)|\,dt = 1.116$$

d) What is the position of the particle on $t = 5.9$?

$$Position = -1.3 + \int_0^{5.9} v(t)\,dt = -0.456$$

Graphical Format

Ex 2 A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t \in [0,\ 12]$ is shown below.



(a) At what times, if any, does the particle switch directions?

(b) At what time on $t \in [0,\ 12]$ is the *speed* the greatest?

(c) What is the total distance traveled by the particle on $t \in [0,\ 12]$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

(a) At what times, if any, does the particle switch directions?

$v(t) = 0$ and switches signs at $t = 4$ and 10.

(b) At what time on $t \in [0,\ 12]$ is the *speed* the greatest?

The range of $v(t)$ is $v \in [-2,\ 3]$. Speed is $|v(t)|$ so the greatest speed is 3

(c) What is the total distance traveled by the particle on $t \in [0,\ 12]$

Total distance traveled = $\int_0^{12} |v(t)| dt$.

$$\begin{aligned}\int_0^{12} |v(t)| dt &= -\int_0^4 v(t)dt + \int_4^{10} v(t)dt - \int_{10}^{12} v(t)dt \\ &= -(-2\pi) + 9 - (-2) \\ &= 11 + 2\pi\end{aligned}$$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

$$x(8) = 6 + \int_0^8 v(t)dt = 6 + (-2\pi) + \frac{9}{2} = \frac{21}{2} - 2\pi$$

Tabular Format

t	0	.3	.7	1.3	1.7	2.2	2.8	3.3	4
$v(t)$	0	14.1	9.5	17.1	13.3	15.6	12.7	13.7	12.0

Ex 3: Pat takes her bike on a 4-hour ride. She records her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 4$ hours, as shown in the table above. For $0 \le t \le 4$, $v(t) > 0$.

(a) Use the data in the table to approximate Pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\frac{3\sin(2\pi t) + 8t}{t^2 + 2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time t = 1.7? Give a reason for your answer.

(a) Use the data in the table to approximate pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

$$a(1.5) \approx \frac{v(1.7) - v(1.3)}{1.7 - 1.3} = \frac{13.3 - 17.1}{1.7 - 1.3} = -\frac{3.8}{.4} = -9.5 \ {}^{mi}\!/_{hr^2}$$

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\ dt$ using a left-hand Riemann sum using the values from the table.

$\int_0^4 v(t)\ dt$ would be the approximate number of miles Pat traveled during her four-hour ride.

$$\begin{aligned}\int_0^4 v(t)\ dt &\approx .3(0)+.4(14.1)+.5(9.5)+.4(17.1)+.5(13.3)+.6(15.6)+.5(12.7)+.7(13.7)\\ &= 49.18\ miles\end{aligned}$$

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t)=9\sqrt{\dfrac{3\sin(2\pi t)+8t}{t^2+2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

$$AveVelocity=\frac{1}{4-0}\int_0^4 9\sqrt{\frac{3\sin(2\pi t)+8t}{t^2+2}}dt=13.350\ mph$$

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time $t=1.7$? Give a reason for your answer.

$a(1.7)=f'(1.7)=-3.288\ {}^{mi}\!/\!_{hr^2}$

Since it was stated that "For $0 \le t \le 4$, $v(t)>0$," Pat's speed is decreasing because the velocity and acceleration have opposite signs at $t=1.7$.

4.8 Free Response Homework

1. A particle is moving along the x-axis so that its velocity is given by $v(t) = -1.5 + \dfrac{4.1x}{\sqrt{x^2+1}} - 3.4\sin(0.3x)$ on $0 \le t \le 10$. The position of the particle at $t=0$ is $x=-2.4$.

a) At what time(s) on $0 \le t \le 10$, if any, does the particle switch directions?
b) What is the acceleration at $t=3.4$?
c) What is the total distance traveled by the particle on $t \in [3, 9]$.
d) What is the position of the particle on $t=5.9$?

---

2. A particle is moving along the x-axis so that its velocity is given by $v(t) = 2 + 3.6\sqrt{t}\sin t - \dfrac{e^{0.3t}}{t+4}$ on $0 \le t \le 10$. The position of the particle at $t=1$ is $x=-4.3$.

a) At what time(s) on $t \in [0, 10]$, if any, does the particle switch directions?
b) Find the acceleration equation at $t=5.4$.
c) What is the total distance traveled by the particle on $t \in [2, 9]$.
d) What is the position of the particle on $t=6.9$?

---

3. An object moves along the y-axis with coordinate position $y(t)$ and velocity $v(t) = -1.5 + \sqrt{t} - \dfrac{\cos(e^{0.9t})}{t+1}$ for $t \ge 0$ seconds. At time $t=0$, the object's position is $y = -1.3$ feet.

a) At what time(s) on $t \in [0, 2.5]$, if any, does the particle switch directions? Indicate units.
b) Approximate the particle's acceleration at $t=2$ seconds. Indicate units
c) What is the total distance traveled by the particle on $t \in [0, 2.5]$.
d) What is the position of the particle on $t=5.9$?

---

4. A particle is moving along the x-axis so that its velocity is given by $v(t)=\ln(t+3)-e^{\frac{t}{2}-1}\cos t$ on $t\in[0,\ 8]$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 8]$, if any, does the particle switch directions?

b) Where is the particle when it is furthest to the left?

c) What is the total distance traveled by the particle on $t\in[3,\ 6]$.

d) Find the acceleration equation. At what time interval within $t\in[0,\ 8]$, if any, is the acceleration negative?

---

5. A particle is moving along the x-axis so that its velocity is given by $E(t)=5-416\left(\frac{t}{5}\right)^4\left(1-\frac{t}{10}\right)^5$ on $0\le t\le 10$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 10]$, if any, does the particle switch directions?

b) Find the acceleration at $t=7.3$.

c) What is the total displacement of the particle on $t\in[1,\ 6]$.

d) What is the total distance traveled by the particle on $t\in[2,\ 9]$.

---

6. A particle is moving along the x-axis so that its velocity is given by $v(t)=2.6\sqrt{t}\cos t-\dfrac{e^{0.5t}}{t+6}$ on $0\le t\le 9$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 9]$, if any, does the particle switch directions?

b) Find the acceleration equation at $t=3.4$.

c) What is the total distance traveled by the particle on $t\in[3,\ 8]$.

d) What is the position of the particle on $t=6.9$?

---

7. A particle is moving along the x-axis so that its velocity is given by $v(t)=\ln(t+3)-e^{\frac{t}{2}-1}\cos t$ on $t\in[0,\ 8]$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 8]$, if any, does the particle switch directions?

b) Where is the particle when it is furthest to the left?

c) What is the total distance traveled by the particle on $t\in[3,\ 6]$.

d) Find the acceleration equation. At what time interval within $t\in[0,\ 8]$, if any, is the acceleration negative?

---

8. A particle is moving along the x-axis so that its velocity is given by $v(t)=3-\frac{3}{20}t-.9\sin\left(\frac{\pi}{7}t\right)-\frac{2.1x}{\sqrt{x^2+1}}$ on $t\in[0,\ 12]$. The position of the particle at $t=0$ is $x=2.4$.

a) At what time(s) on $t\in[0,\ 12]$, if any, does the particle switch directions?

b) Find the acceleration at $t=7.3$.

c) What is the total distance traveled by the particle on $t\in[0,\ 10]$.

d) What is the position of the particle on $t=10.4$.

---

t	0	1	4	6	9	10	13	15	18
$v(t)$	55	70	68	55	40	38	46	50	70

9. A car is traveling on a straight road. Values of the continuous and differentiable function $v(t)$ are given on the table above. $v(t)$ is measured in feet per minute and time t is measured in minutes.

a) Approximate the acceleration at $t=7$. Indicate the units.

b) Using left-hand rectangles, approximate $\int_0^{18} v(t)\,dt$. Using the correct units, explain the meaning of the approximation.

c) How many times on the interval $0 \le t \le 18$ is $a(t)=0$? Explain your reasoning.

d) Assume the data are modeled by $P(t)=.05t^3-1.07t^2+3.89t+62$. Use the model to find the average velocity of the car on the interval $0 \le t \le 18$.

---

t in minutes	0	8	15	23	33	45	53
$v(t)$ in mph	0	8.5	7.5	10.1	9.3	7.1	4.1
$v(t)$ in mi/min	0	0.142	0.125	0.168	0.155	0.118	0.068

10. Mr. Evans has run the 7.4-mile Bay-to-Breakers many times (always with cloths on). His best time was 53 minutes. The table above shows estimates of his velocity at different times along the course from The Embarcadero to Ocean Beach. Assume the data represents a continuous and differentiable function.

a) Approximate Mr. Evans' acceleration at $t=30$.

b) Given you result in a), was his speed increasing or decreasing at $t=10$? Explain, using the correct units.

c) Find a trapezoidal approximation for $\int_0^{53} v(t)\,dt$. Why isn't the answer 7.4 miles?

d) Mr. Lannan did the same run. (He likes to take his time and gawk at the costumes.) His velocity is modeled by $L(t)=0.023\left(5+4\sin\frac{\pi}{16}t\right)$. According to this model, does Mr. Lannan finish in under 65 minutes? Explain your reasoning.

---

11. Mr. Alverado takes the Cross Country team out for a morning run and tracks his pace. The data table below shows his pace $p(t)$ in minutes per mile and his velocity $v(t)$ miles per minute at 15-minute intervals.

t (in minutes)	0	15	30	45	60
$p(t)$ (in min/mile)	8:07	7:34	8:16	8:07	7:14
$v(t)$ (in mi/min)	0.123	0.132	0.121	0.123	0.138

Both $p(t)$ and $v(t)$ are continuous and differentiable functions.

a) Find an approximation for $\int_0^{60} v(t)\,dt$ using midpoint rectangles. Explain the meaning of the result, using the correct units.

b) Using correct units, explain the meaning of $\frac{1}{60}\int_0^{60} p(t)\,dt$.

c) Approximate the acceleration at $t = 37$ minutes.

d) Is there a time during which the pace reaches a maximum? Explain your reasoning.

---

t in hours	0	12	24	36	48
$v(t)$ in km/hr	21	26.3	31.4	36.8	41.5

12. A Gravitational Slingshot Effect is sometimes used by space probes like Voyager 2 in order to increase its velocity without expending fuel. By flying close to the planet Saturn in a parabolic arc, the velocities on the table above were achieved by a probe. (In the original *Star Trek* episode "Tomorrow is Yesterday," the Enterprise used this effect around a black hole to time-travel to 1967.)

a) Approximate the probe's acceleration at $t=30$.

b) Use a trapezoidal approximation for $\int_0^{48} v(t)\,dt$. Using the correct units, explain the meaning of this result.

c) Using your answer in b), approximate the average velocity of the probe between $t=0$ and $t=48$? Indicate the correct units.

d) The data on the table can be approximated by the equation $v(t)=0.000027x^2+0.4396x+21$. Based on this equation, find the total distance traveled by the probe between $t=0$ and $t=48$ hours. Indicate the units.

---

13. Below is a chart of your speed driving to school in meters/second. Use the information below to find the values in a) and b) below.

t (in seconds)	0	30	90	120	220	300	360
$v(t)$ (in m/sec)	0	21	43	38	30	24	0

a) Approximate your acceleration at $t=100$.

b) Given you result in a), are you speeding up or slowing down at $t=100$? Explain, using the correct units.

c) Find an approximation for $\int_0^{360} v(t)\,dt$ using right Riemann rectangles. Using the correct units, explain the meaning of your result.

---

t	0	0.08	0.2	0.33	0.55	0.75	0.86	1
$H(t)$	60	5	63	25	75	28	70	6

14. Traffic flow on HWY 280 from San Francisco to San Jose during rush hour is notoriously variable because of bottlenecks caused by intersecting with HWYs

380, 92, and 85. The table above shows the average speed of traffic, in miles per hour, at time *t*, where *t* is measured in hours after entering the highway.

a) Approximate $H'(0.6)$. Using the correct units, explain the meaning of $H'(0.6)$.

b) Set up a left-hand Reimann Sum to approximate $\int_0^1 H(t)dt$. Using the correct units, explain the meaning of $\int_0^1 H(t)dt$.

c) Assume that $S(t)=40-35\cos\left(\frac{\pi}{9}(t-5.5)\right)$ would accurqtely model the data on the table. Set up, but do not solve, an integral equation that would determine the time at which the car has been driving 50 miles.

d) Set up, but do not solve, an integral equation that would determine the average $S(t)$ between $t=0.33$ and $t=0.55$.

---

15. On the first offensive play of the 2020 49ers vs Jets game, running back Raheem Mostert set a record as the ballcarrier to hit the highest velocity in a game. According to Pro Football Focus (PPF), he achieved a velocity of 23.09 mph on his 80-yard touchdown run. (He actually went 87 yards from his starting position in the backfield.) As a football fan and stats advocate, Dr. Quattrin used replay and a stopwatch to build the table of approximate velocities (below) during the run.

Time t in seconds	0	1.1	2	3.74	4.82	6.18	7.51	9.19	11.95
$v(t)$ in yards/sec	0	4.38	6.24	9.75	11.28	10.52	8.35	7.02	4.62

a) Approximate Mostert's acceleration at $t = 3$seconds. Indicate the units.
b) Use a Right-hand Riemann Sum to approximate the total number of yards run. [EC. Why is this more than 87 yards?]
c) One model of the data on the table is $M(t) = 6.2te^{-0.21t}$. According to this model, find Mostert's approximate acceleration at $t = 3$ seconds.
d) According to the model $M(t) = 6.2te^{-0.21t}$, find Mostert's maximum velocity. Indicate the units.

---

16. The Pony Express achieved mythic status in the Old West despite only running for 19 months. The series of riders took mail from St. Joseph, MO, to Sacramento, CA—1966 miles in 10 days. One rider named Pony Bob Haslam impressed with both the fastest ride and the longest ride on record. The fastest with a 120-mile-ride from Friday's Station to Bucklands Station, Nevada. He made the trip in 8 hours 10 minutes (while wounded), carrying Lincoln's Inaugural Address. Estimates of his distances and average velocity at different times along the route are given in the table below.

Time t in hours	0	2.5	4.8	6.3	8.2
$v(t)$ in mi/hr	0	15.6	13.5	18.7	11.6

a) Approximate Bob's acceleration at $t = 3$ hours. Indicate units.

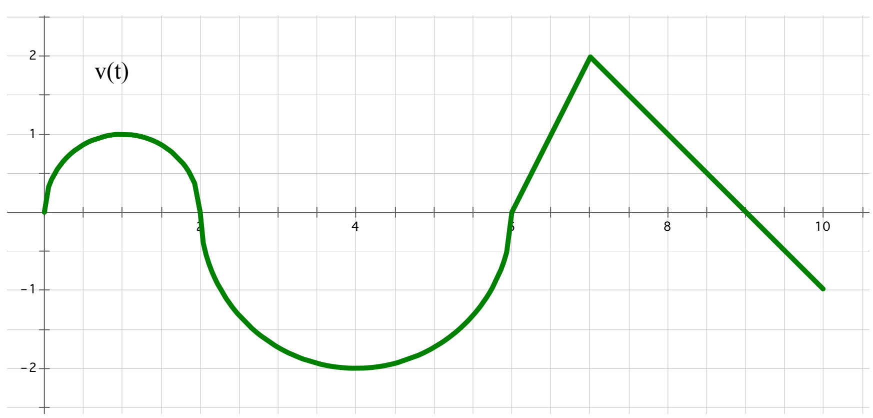
b) Use a Right-hand Riemann Sum to approximate $\int_0^{8.2} v(t)\, dt$.

c) Using your answer from part b), estimate Bob's average velocity during this ride.

d) Assume that $H(t) = 6.2te^{-0.21t}$ is an accurate model of Bob's velocity. Does the model indicate that his velocity is increasing at an increasing rate at $t = 3$ hours? Using the correct units, explain your reasoning.

---

17. A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t \in [0, 10]$ is shown below.



(a) At what times, if any, does the particle switch directions?

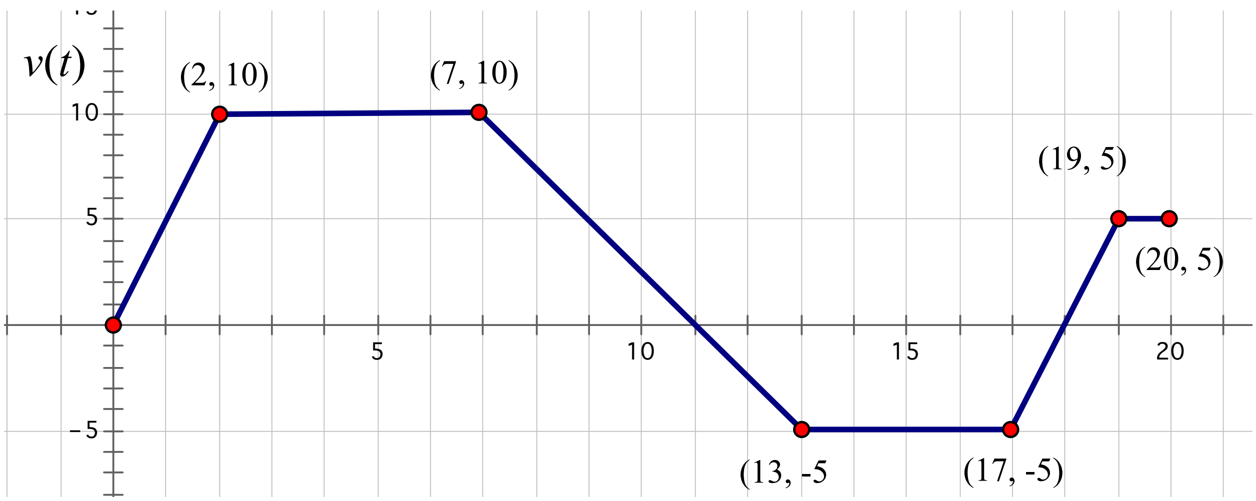
(b) At what time on $t \in [0, 10]$ is the *speed* the greatest?

(c) What is the total distance traveled by the particle on $t \in [0, 10]$.

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

---

18. A couple take their new dog Skadi to run around at Fort Funston. She immediately runs away and back toward them several times. For $0 \le t \le 20$, Skadi's velocity is modeled by the piecewise-linear function defined by the graph below

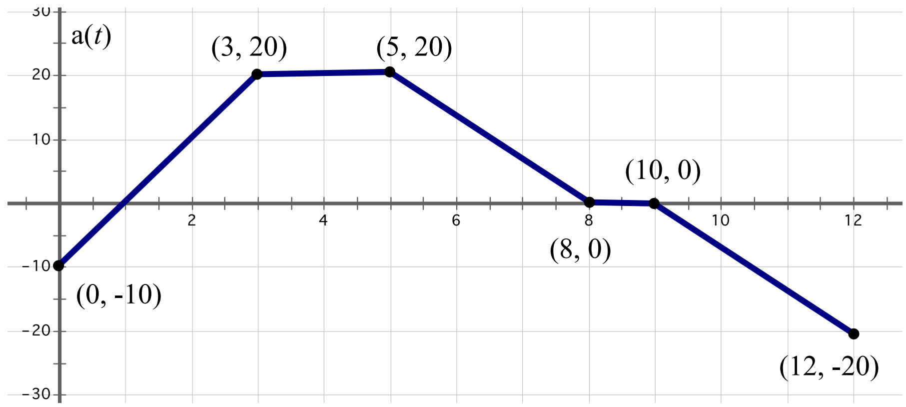


where $v(t)$ is measured in feet per second and t is measured in seconds.

a) At what times in the interval $0 \le t \le 20$, if any, does Skadi change direction? Give a reason for your answer.
b) At what time in the interval $0 \le t \le 20$, what is the farthest Skadi gets from the couple?
c) Find the total distance Skadi travels during the time interval $0 \le t \le 20$.
d) Write expressions for Skadi's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from the couple that are valid for the time interval $7 \le t \le 13$.

---

19. A car is driving west on the highway so that its acceleration $a(t)$ is given by the continuous function whose graph at time $t \in [0, 12]$ is shown below.



The graph is comprised of five line segments. $a(t)$ is measured in miles per hour^2 and time t is measured in hours.

a) At what time on $t \in [0, 12]$ is the acceleration the greatest?

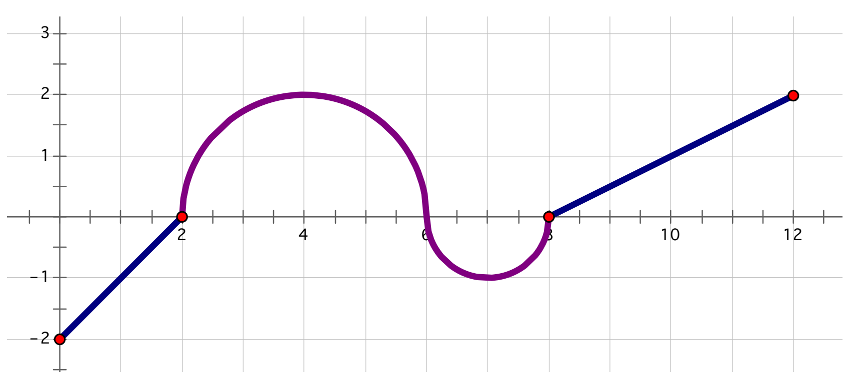
b) If $v(0) = 25$, find $a'(6.3)$. Indicate units.

c) If $v(0) = 25$, find $v(3)$.

d) If $v(0) = 25$, what is the maximum velocity on $t \in [0, 12]$?

---

20. A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t \in [0, 12]$ is shown below.



The graph is comprised of two line segments and two semicircles.

a) At what times, if any, does the particle switch directions?

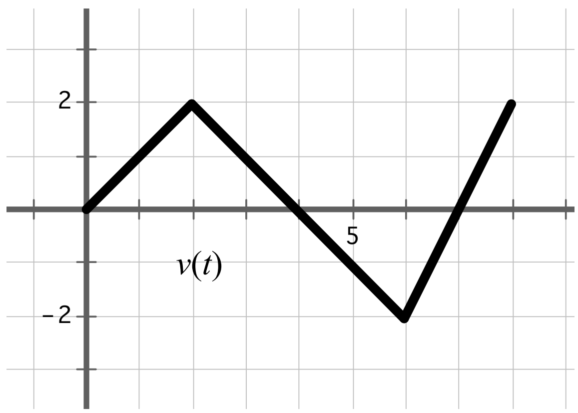
b) At what time on $t \in [0, 12]$ is the *speed* the greatest?

c) What is the total distance traveled by the particle on $t \in [0, 12]$.

d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

---

21. A car's velocity $v(t)$, in miles per minute, is modeled by the continuous function whose piecewise-linear graph at time $t \in [0, 8]$ is shown below.



a) Find the acceleration at $t = 3.5$. Indicate the units.

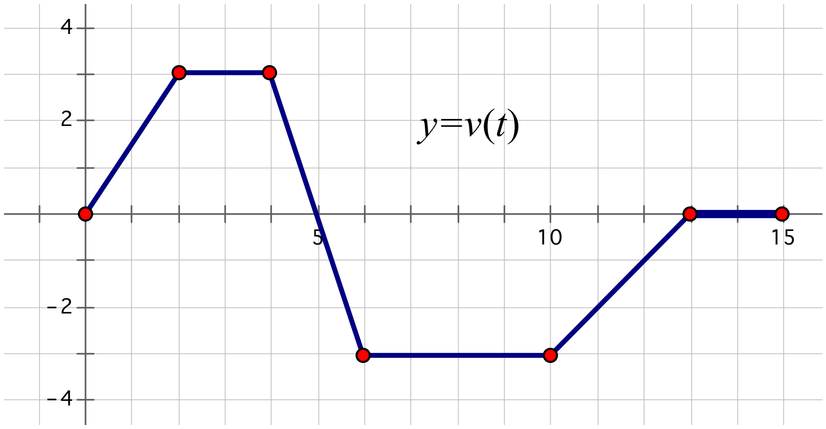
b) At what time(s) on $t \in [0, 8]$ does the car switch directions? Explain.

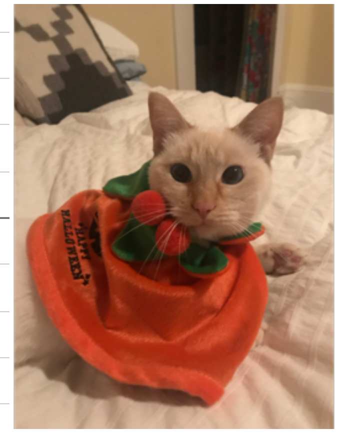
c) Find the total distance traveled by the particle on $t \in [0, 8]$.

d) Find the car's average rate of change of the velocity on $t \in [3, 8]$. How many times on $t \in [3, 8]$ is the instantaneous velocity equal to the average velocity? [extra credit is you find the time(s).]

---

22. A cat named Dolly chases her favorite mouse toy back and forth across the floor. Her velocity, in m/sec, as a function of time in seconds is shown in the graph below.



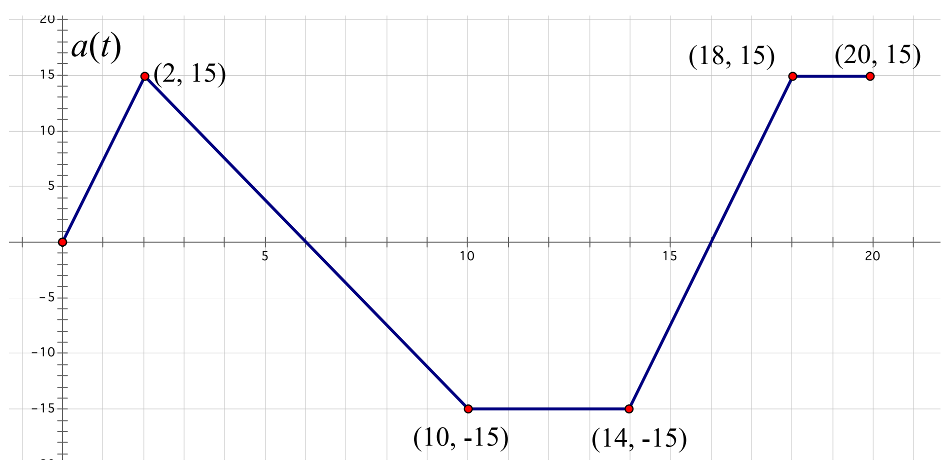


a) What is Dolly's acceleration between $t = 4$ seconds and $t = 6$ seconds?

b) At what time (after she starts moving) is Dolly's displacement equal to zero?

c) Dolly catches her toy after 13 seconds. What is the total distance that she travels while chasing her toy?

d) When is Dolly's acceleration zero?



23. A pickup truck is traveling along a straight country road with a velocity of $50\ ^{\text{ft}}/_{\text{sec}}$ at time $t = 0$ second. For $0 \le t \le 20$ seconds, the truck's acceleration $a(t)$, in $^{\text{ft}}/_{\text{sec}^2}$, is given by the piecewise linear function defined by the graph above.

a) Is the velocity increasing at $t = 2$ seconds? Why or why not?

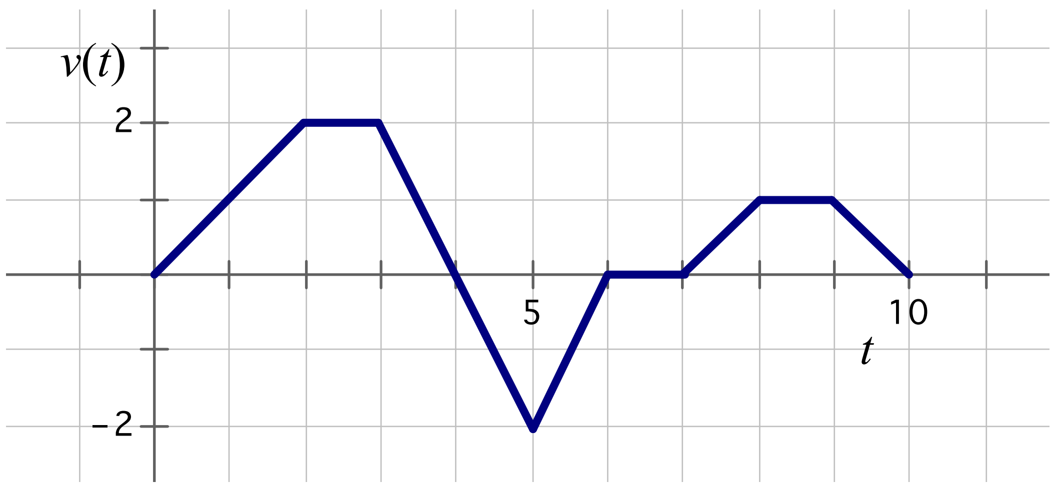
b) On what time in the interval $0 \le t \le 20$, other than at $t = 0$, is the velocity of the truck $50\ {}^{ft}/_{sec}$? Why?

c) Find the truck's absolute maximum velocity on $0 \le t \le 20$. Justify your answer.

d) On the time in the interval $0 \le t \le 20$, if any, is the truck's. velocity equal to zero? Justify your answer.

---

24. A car is driving along a straight road such that its velocity, in miles per minute, is shown in the graph below.



a) Find the times at which the car is stopped.
b) How many miles from its starting point is the car at $t = 6$?
c) How many miles has the car traveled between $t = 0$ and $t = 10$?
d) Find the acceleration at $t = 3.5$. Indicate the units.

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25. AP Packet: AB13 #2, AB14 #4, AB15 #3, AB16 #2, AB 19 #2

5.1 Multiple Choice Homework

1. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Find the value of t at which the acceleration is zero.

a) $-\frac{2}{3}$ b) $-\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{3}$ e) $-\frac{1}{2}$

---

2. A particle moves on the x-axis with velocity given by $v(t) = 3t^4 - 11t^2 + 9t - 2$ for $-3 \le t \le 3$. How many times does the particle change direction as t increases from –3 to 3?

a) Zero b) One c) Two

d) Three e) Four

---

3. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

4. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the acceleration of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

5. Find the acceleration at time $t = 9$ seconds if the position (in cm.) of a particle moving along a line is $s(t) = 6t^3 - 7t^2 - 9t + 2$.

a) 310 cm/sec^2 b) 310 cm/sec c) 1323 cm/sec^2

d) 1323 cm/sec e) −1323 cm/sec

---

6. The acceleration of a particle is given by $a(t) = 4e^{2t}$. When $t = 0$, the position of the particle is $x = 2$ and $v = -2$. Determine the position of the particle at $t = \frac{1}{2}$.

a) $e - 3$ b) $e - 2$ c) $e - 1$ d) e e) $e + 1$

---

7. A particle moves along a straight line with its position at any time $t \geq 0$ given by $s(t) = \int_0^t (x^3 - 2x^2 + x)\,dx$, where s is measured in meters and t is in seconds. The maximum velocity attained by the particle on $0 \leq t \leq 3$ is

a) $\frac{1}{3}$ m/s b) $\frac{4}{27}$ m/s c) $\frac{27}{4}$ m/s d) 12 m/s

---

8. A particle moves along the x-axis with acceleration at any time t given as $a(t) = 3t^2 + 4t + 6$. If the particle's initial velocity is 10 and its initial position is 2, what is the position function?

a) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 12$

b) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 10t + 2$

c) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 2$

d) $x(t) = 3t^4 + t^3 + t^2 + 10t + 2$

---

9. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Which of the following statements is/are true?

I. The particle is moving right at $t = \frac{2}{3}$.

II. The particle is paused at $t = \frac{1}{3}$

III. The particle is speeding up at $t = 1$.

a) I only b) II only c) III only

d) I and II only e) I and III only

---

10. A particle moves along the y-axis so that at any time $t \geq 0$, it velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \dfrac{\pi}{2}$ is $y = 3$, the particle's position at time $t = 0$ is

a) -4 b) 2 c) 3 d) 4 e) 6

Definite Integral Chapter Practice Test

1. Find $\int_1^4 \frac{6}{\sqrt{x}}\,dx$

a) 12 b) 8 c) 47 d) 24 e) 48

---

2. $\int_0^5 \frac{dx}{\sqrt{3x+1}}$

a) $\frac{1}{2}$ b) $\frac{2}{3}$ c) 1 d) 2 e) 6

---

3. $\int_1^2 \frac{1}{\sqrt{1-\frac{1}{4}t^2}}\,dt$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{6}$ d) π e) $\frac{2\pi}{3}$

---

4. For $t \geq 0$ hours, H is a differentiable function of t that gives the change in temperature, in degrees Celsius per hour, at an Arctic weather station. Which of the following is the best interpretation of $\frac{1}{t}\int_0^t H(x)\,dx$?

a) The change in temperature during the first t hours.
b) The change in temperature during the first day.
c) The average change the temperature during the first t hours.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24^{th} day.

---

5. Which of the following is equal to $\int_0^{\pi} \cos x \, dx$?

a) $\int_0^{\pi} \sin x \, dx$ b) $\int_{-\pi/2}^{\pi/2} \cos x \, dx$ c) $\int_{-\pi/2}^{\pi/2} \sin x \, dx$

d) $\int_{\pi}^{2\pi} \sin x \, dx$ e) $\int_{\pi/2}^{3\pi/2} \cos x \, dx$

---

6. The following table lists the known values of a function $f(x)$.

x	1	2	3	4	5
$f(x)$	0	1.1	1.4	1.2	1.5

If the Right-Hand Riemann Sum is used to approximate $\int_1^5 f(x) \, dx$, the result is

a) 3.7 b) 4.5 c) 4.6 d) 5.2

e) none of these

---

7. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1,\ 2]$

a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

---

1. $\displaystyle\int_0^{\pi/9}\left(\tan 3x+\frac{x}{4+x^2}\right)dx$. Show the anti-derivatives.

---

2. Find the average value of $y=\dfrac{4}{x}\ln^3 x$ on $x\in[1,e]$. Show the anti-derivative.

---

3. Find the area between $y=xe^{3x^2}$ and the x-axis on $x\in[-2,1]$. Show the anti-derivative.

4. Find the area on $x \in [0, 2]$ under $f(x) = \dfrac{1}{x^2+9} + \sin 4x$. Show the anti-derivative.

5. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + .8t\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

---

a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

---

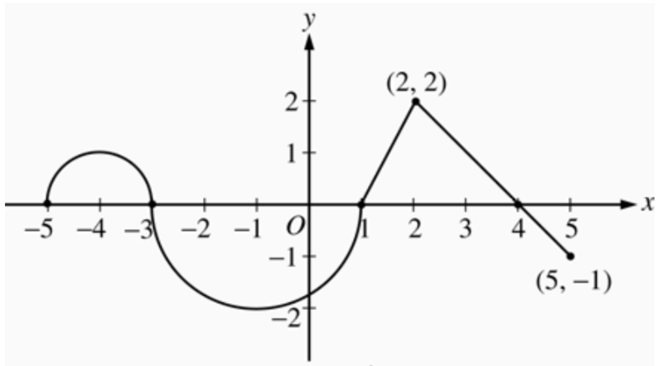
b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

.

d) How many pounds of bananas are on the display table at time $t = 8$?

6. The graph of $f'(x)$, defined on $x \in [-5,\ 5]$, is shown below. $f'(x)$ consists of two semi-circles and two line segments.



Graph of f'

Also, let $f(2) = 3$.

---

a) For $-5 < x < 5$, find all values x at which f has a relative maximum. Justify your answer.

---

b) Write the equation of the line tangent to $f(x)$ at $x = 2$.

---

c) On what interval(s), if any, is the graph of $f(x)$ both increasing and concave up? Justify your answer.

d) Find the absolute maximum value of $f(x)$ on $x \in [-5, 5]$. Justify your answer.

4.0 Multiple Choice Answers:

1.	A	2.	D	3.	C	4.	A	5.	B	6.	B

4.1 Free Response Answers

1. $\frac{364}{3}$ 2. 107.5 3. 0 4. 280.605 5. $\frac{7}{8}$

6. 0 7. 1 8. $3\ln 3$ 9. 2 10. $\frac{1}{\sqrt{2}}-1$

11. $84\frac{2}{3}$ 12. 2 13. $\frac{1}{\sqrt{2}}$ 14. 3 15. $7\ln 2-1$

16. $\frac{48844}{33}$ 17. -5 18. -5 19. -11 20. 12

21. 1 22. 7 23. 4 24. 27

25. -22 26. $\frac{53}{2}$ 27. 15 28. $\frac{37}{10}$

29. $\frac{16}{3}$ 30. $\frac{2\sqrt{3}}{3}$ 31. $\frac{4}{\pi}$

32. $\frac{1}{2}\ln 3$ 33. $30\frac{4}{9}$ 34. $46\frac{4}{5}$ 35. 225.753°F

35. 8.201 hrs 37. 63.045°F 38. $y^2\sin y$

39. $\sqrt{1+2x}$ 40. $-\cos(x^2)$ 41. $-\frac{1}{x^2}\arctan{}^{1}/_{x}$

42. $\frac{\cos\sqrt{x}}{2x}$ 43. $\sqrt{257}$ 44. $2x\ln\left(x^4+1\right)$ 45. $e^{5\sqrt{x}}\left(\frac{1}{2}\sqrt{x}\right)$

46. $4x^3\ln x$ 47. $m\cos^2 m$ 48. $\dfrac{y}{\ln^4 y}$ 49.

$-\left(e^{3x}+e^x+1\right)$

4.1 Multiple Choice Answers:

1.	E	2.	B	3.	E	4.	C	5.	B	6.	D
7.	D	8.	B	9.	C	10.	B	11.	B	12.	E

4.2 Free Response Answers:

1.	$\dfrac{182}{9}$	2.	4	3.	$e-e^{1/2}$	4.	1
5.	2	6.	$\dfrac{1}{2}\ln 3$	7.	$\ln 3$	8.	$\ln\left(\dfrac{\ln 4}{\ln 2}\right)$
9.	$\tan^{-1}2-\dfrac{\pi}{4}$	10.	$\dfrac{9}{128}$	11.	-2	12.	$\dfrac{107}{10}$
13.	0	14.	$-\dfrac{1}{8}\left(e^{-25}-e^{-1}\right)$	15.	$-\dfrac{3}{25}$	16.	$\dfrac{2}{17}$

4.2 Multiple Choice Answers:

1.	B	2.	E	3.	C	4.	E	5.	A	6.	A
7.	C										

4.3 Free Response Answers:

1.	11.833	2.	4.917	3.	35.5	4.	3
5.	$\ln 2$	6.	2.704	7.	1.627	8.	9.519

9. 2.707 10. 9.804 11. (a) $-\frac{3}{2}m$ and (b) $\frac{41}{6}m$

12. (a) $-\frac{10}{3}m$ and (b) $\frac{98}{3}m$

13. $\int_0^a \left(e^{-x^2}-x\right)dx - \int_a^2 \left(e^{-x^2}-x\right)dx = 3.080$, where $a = 0.65291864$

14. $\int_0^b \left(e^{-x^2}-2x\right)dx - \int_b^2 \left(e^{-x^2}-2x\right)dx = 5.305$, where $b = 0.41936482$

15. 2.235

16. 13.030

4.3 Multiple Choice Answers:

1. B 2. C 3. B 4. C 5. B 6. A

7. C

4.4 Free Response Answers:

1a. $f(0) = \int_3^0 g(t)dt = -3.5$; $f'(0) = g(0) = 1$; $f''(0) = g'(0) = 1$

1b. .225 1c. $x \in (-2, \ -1)$ 1d. $x = -1$

2a. $g(2) = \int_{-2}^2 f(t)dt = -8$, $g'(2) = f(2) = -\frac{2}{3}$, and $g''(2) = f'(2) = \frac{2}{3}$.

2b. $\frac{5}{12}$ 2c. $x \in (-2, \ -1) \cup (6, \ 8)$ 2d. $x = 3$

3a. $k(4) = -2\pi$, $k'(4) = 0$, and $k''(4)$ *dne*.

3b. $\dfrac{11.5-2\pi}{15}$ 3c. $x \in (0,\ 2) \cup (12,\ 15)$ 3d. $x = 4$

4a. $g(3) = 3;\quad g'(3) = 2;\quad g''(3) = -\dfrac{1}{2}$ 4b. $\dfrac{7}{3}$

4c. Two. 4d. $x = 2$ and 5

5a. $h(2) = 2\pi;\quad h'(2) = 2;\quad h''(2) = 0$ 5b. π

5c. $x \in (-3,\ -2)$ 5d. 2π

6a. $g(-5) = -\dfrac{1}{2};\quad g'(-5) = 3$ 6b. $g(2) = -1.7$

6c. $x = 5$ 6d. $x \in (-2,\ 0)$

7a. $g(-5) = 8;\quad g'(-5) = -2$ 7b. $g(2) = 9$

7c. $x = -1$ 7d. $x \in (1,\ 2)$

8a. $h(2) = -\dfrac{\pi}{2};\quad h'(2) = 0;\quad h''(2) = dne$ 8b. $\dfrac{\pi}{4}$

8c. $x \in (1,\ 2)$ 8d. $2 - \dfrac{\pi}{2}$

9a. $g(4) = \pi - 8;\quad g'(2) = 0;\quad g''(4) = dne$ 9b. $\dfrac{2}{7} - \dfrac{13\pi}{28}$

9c. $x \in (2,\ 5)$ 9d. $x = -7$

10a. $g(0) = 2\pi - 8;\quad g'(0) = -2;\quad g''(0) = dne$ 10b. $y - (2\pi - 8) = -2(x - 0)$

10c. $x \in (-4,\ -2)$ 10d. $x = 7$

11a. $g(4) = 14;\quad g'(4) = -2;\quad g''(4) = 0$ 11b. 1

11c. 1 11d. 10

12. See AP Central

4.4 Multiple Choice Answers:

1. B 2. A 3. B 4. A 5. C 6. D

4.5 Free Response Answers:

1a) $11{,}977.905\ gal$ 1b) Falling 1c) $3748.733\ \text{gal}$ 1d) $\text{t} = 24$

2a) 971 or 972 2b) decreasing 2c) 1921 or 1922

2d) 2500.

36a) 133.333 yottatons 3b) yes

3c) $t = 28.000,\ \ 50.376,\ \ 77.948,\ \ 91.465$ 3d) 999,987.137 yottatons.

4a) $76.844\ \ cg$ 4b) $4.143\ \ cg$ 4c) decreasing

4d) $\int_0^t \left(8 - \frac{e^{0.47t}}{x+6}\right) - (7 - .46x\cos(x))dx = 0$

5a. $1382\ \ cars$ 5b) 2880.737

5c) Yes

66a) The amount of chorine is decreasing. b) $30.173\ \ \text{days}$

6c) yes, chlorine needs to be added.

7a) no, the water is NOT increasing. 7b) 1310 7c) $t = 6.495$

7d) $\int_0^k R(t)\, dt = 1310$

8a) 70.571 8b) Falling 8c) 122.026 gal 8d) No.

7a) $13.304\ inches$__________7b) increasing

7c) $H(t) = 15 + \int_0^t [A(x) - B(x)]\, dx$

7d) 3.963 inches below the top of the ship.

8a) $1440\ \ cats$

8b) At $t = 10.3\ \ months$, the rate at which healthy cats are being adopted is decreasing by 28.008 cats per month per month.

8c) $C(t) = 131 + \int_0^{12} R(t) - A(t)\, dt = 71\ \ cats$

8d) $t =\ 8.410$ months.

4.5 Multiple Choice Answers:

1.	E	2.	A	3.	D	4.	D	5.	B	6.	E
7.	C	8.	C	9.	E	10.	A	11.	D	12.	D

4.6 Free Response Answers:

1. 4940 miles

2a. 2b. 7.25 2c. 30

3a. 1880 km 3b. 840 km 3c. 1880 km 3d) 1300

4a) $1280 m^2/\text{min}$ b) $600 m^2/\text{min}$

5a) 10,900 m b) 6850 m

6a) 392 km b) 252 km

7a) 1130 gal b)1134 gal

8a. 0.032 b) Yes, MVT c) No, IVT

9a) 2.414 b) 2.411

10a) 0.147 b) 0.147

11. $\int_5^{10} w'(t)\, dt$ represents the change of the child's weight in pounds from ages 5 to 10 years

12. $\int_0^{120} r(t)\, dt$ represents how much oil, in gallons, has leaked from the tank in the first 120 minutes.

13. Newtons motors, or joules

14. See AP Central

4.6 Multiple Choice Answers:

1. C 2. C 3. E 4. B 5. C 6. B

7. E 8. D 9. C 10. C 11. C

<u>4.7 Free Response Answers:</u>

1a) $\frac{3}{4}\ m^3/\min^2$ 1b) -8 1c) $1344\ m^3$

1d) $\frac{1}{48}\int_0^{48} V(t)\,dt$ represents the average rate, in m^3/min, of the flow of water through the pipeline between $t = 0$ and $t = 48$ minutes.

2a) $0.55 cm/day^2$. 2b) increasing at an increasing rate

2c) 9.9 mm.

2d) average number of millimeters per day that the plan grew.

3a) $\frac{1}{2} gallons/\min^2$

3b) 4700 gallons; total consumption of gallons of fuel between $t = 0$ and $t = 90$ minutes.

3c) $50\ gal/\min$; The average rate of consumption of fuel, in gallons per minute, between $t = 0$ and $t = 90$ minutes is 50.

4a) -17. At $t = 3.7$, the patient's blood glucose is decreasing by 17 mg/dL per day.

b) 992. The patient's average morning blood glucose level in mg/dL.

c) decreasing at an increasing rate

5a) -1.7 b) $= 189.3$; $\frac{1}{21}\int_0^{21} A(t)\,dt \approx 9.01$ is the patient's average A1c score per month from $t = 0$ to $t = 21$months.

c) 7.372.

6a) 766.3 kW b) 7920 kW

c) $k'(9) = -256.040$. The production is decreasing at $t = 9$.

7a) decreasing 7b) 5075.3 kW 7c) 692.6 therms.
7d) the average gas consumption in therms per month during these 12 months.

8a) 4884 kW 8b) 5099.55 kW; Dr Quattrin owes PG&E

8c) $E(t) = 0.28\int_0^t C_e(x)\,dx - 0.28\int_0^t P_e(x)\,dx$

---

9 a) $0.45\ {}^\circ C/hr^2$

9b) $2.1^\circ C$. The temperature in Sauris on this night has risen approximately $2.1^\circ C$ between midnight and 8:00am.

9c) $T(1pm) = T(13) = -8 + \int_0^{13} W(t)dt.$

10a) $0.45\ {}^\circ\text{F/hr}^2$

b) The total temperature change between midnight and 8am is 9.9°F.

c) Yes.

d) Yes

11a) 0.032

b) Yes.

c) While there might be a time when *SFR* = 0.0056, it is not guaranteed.

12a) $-0.3\ {}^{acres}\!/_{hr^2}$. b) $25.4\ acres$ c) $10.674\ acres$

d) $4.613\ acres$ e) 0

13a) 26,456 foot-acres

b) 6202.667 foot-acres.

c) $-9\ {}^{\text{ft}-\text{acres}}\!/_{\text{mo}^2}$. The rate of water entering the plant is decreasing at 9 foot-acres per month per month when $t = 6$ months .

d) $t = 10$ months.

14a) 52 sacks. b) approximately .0175 sacks per game per game.

c) .896 sacks per game d) 7.597 more sacks

15a) 172.9 b) increasing at an approximate rate of 0.092 thousand aircraft per month per month at $t = 20$ months.

c) $D(t) = 3.7 + \int_{6}^{66} [X(x) - S(x)]dx$ d) 56.831 thousand aircraft.

4.7 Multiple Choice Answers:

1. B 2. A 3. D 4. D 5. D

6. C 7. C

4.8 Free Response Answers

1a) $t = 0.600, 2.480, \textit{and } 7.623$ 1b) $a(3.4) = -0.442$

1c) 3.728 1d) -5.023

2a) $t = 3.389, 6.189, \textit{and } 9.490$ 2b) $a(5.4) = 4.607$

2c) 38.869 2d) -1.511

3a) $t = 1.131, 1.538, \textit{and } 2.285$ 3b) $a(2) = 0.042$

3c) 1.116 3d) -0.456

4a) $t = 0.516 \textit{ and } 7.718$ 4b) -0.217

4c) 8.455 4d) $t \in [3.256 \textit{ and } 6.304]$

5a) $t = 2.292 \textit{ and } 6.752$ 5b) $\mathrm{a}(7.3) = 3.536$

5c) -18.957 5d) 31.831

6a) $t = 1.482, 4.898, \textit{and } 7.411$ 6b) $\mathrm{a}(3.4) = 0.314$

6c) 15.899 6d) $x = -4.477$

7a) $t = 5.160 \textit{ and } 7.718$ 7b) -1.817 7c) 8.455

7d) $a(t) = \frac{1}{t+3} + e^{\frac{t}{2}+1}\left(\sin t - \frac{1}{2}\cos t\right);\ x \in [\,3.664,\ 6.738]$

8a) $t = 2.214,\ 7.526,\ and\ 11.480$ 8b) 0.245

8c) 4.704 8d) 3.552

9a) $-5\ {ft}/{\min^2}$ 9b) The car traveled 962 feet in these 18 minutes.

9c) Twice 9d) $54.35\ {ft}/{\min}$

10a) $-0.08 \text{mi}/\text{hr}^2$ 10b) Decreasing

10c) 6.672 miles. 10d) Yes

11a) The team ran approximately 7.65 miles in these 60 minutes.

11b) The result would equal the time, on average, it would take to complete one mile.

11c) $\frac{2}{15000}{mi}/{\min^2}$ 11d) Yes

12a) $0.45\ {mi}/{\min^2}$ 12b) 1509km

12c) $31.438\ {km}/{hr}$ 12d) $1515.415\ km$

13a) $-\frac{1}{6}\ {m}/{\sec^2}$ 13b) Slowing down 13c) 13,190 meters

14a) $-23.5\ mph\ per\ hour$ 14b) 46.97 14c) 50

14d) $\frac{1}{0.55-0.33}\int_{0.33}^{0.55}\left[40 - 35\cos\left(\frac{\pi}{9}(t-5.5)\right)\right]dt$

15a) $2.006\, {}^{yds}\!/\!_{sec^2}$ 15b) 89.288 yds 15c) 1.222 15d) 10.861

16a) $-0.913\, {}^{mi}\!/\!_{hr^2}$ 16b) 120.14 miles 16c) $14.651\, {}^{mi}\!/\!_{hr}$

16d) no

17a) $t = 4$ *and* 10 17b) 3 17c) $\frac{5\pi}{2} + \frac{7}{2}$ 17d) $\frac{17}{2} - 2\pi$

18a) $t = 11$ *and* 18 18b) $t = 11$ 18c) 160 ft

18d) $a(t) = -\frac{5}{2}$, $v(t) = -\frac{5}{2}t + 11$, and $x(t) = -\frac{5}{2}t^2 + 11t + \frac{199}{4}$

19a) $t \in [3, 5]$ 19b) $-\frac{20}{3}\, {}^{mi}\!/\!_{hr^3}$

19c) $40\, {}^{mi}\!/\!_{hr}$ 19d) $110\, {}^{mi}\!/\!_{hr}$.

20a) $t = 2, 6$ *and* 8. 20b) $t = 0, 4$ *and* 12.

20c) $6 + \frac{5\pi}{2}$ 20d) $6 + \frac{3\pi}{2}$

21a) -1 21b) $t = 4$ *and* 7 21c) 8

21d) $\frac{1}{5}$; twice. 21EC. $t = 3.8$ *and* 6.1

22a) $-3\, {}^{m}\!/\!_{s^2}$ 22b) $t = 9$

22c) 28.5m 22d) $t \in [2, 4] \cup [6, 10] \cup [13, 15]$

23a) Yes, because $a(2) = v'(2) = 15 > 0$. 23b) $t = 15.487$

23c) $95\, {}^{ft}\!/\!_{sec}$ 23d) yes

24a) $t = 0, 4, [6,7], 10$ 24b) 3 miles 24c) 9 miles

24d) $-2\ {mi}/{hr^2}$

25. See AP Central

Integral Practice Test Answers

1. A 2. D 3. E 4. C 5. C 6. D

7. B

1. 0.246 2. $\frac{1}{e-1} = 0.582$ 3. 27128.812

4. $A = 1.336$ where $\int f(x)\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} - \frac{1}{4}\cos 4x + c$

5a. $20.051\ lbs$ 5b. $f'(7) = -8.120\ lbs/hr/hr$

5c. Decreasing 5d. $23.347\ lbs$

6a. $x = -3\ \ and\ \ 4$ 6b. $y - 3 = 2(x-2)$

6c. $x \in (-5,\ -4) \cup (1,\ 2)$ 6d. $\frac{5}{2} + 2\pi$