Chapter 2:

Anti-Derivatives

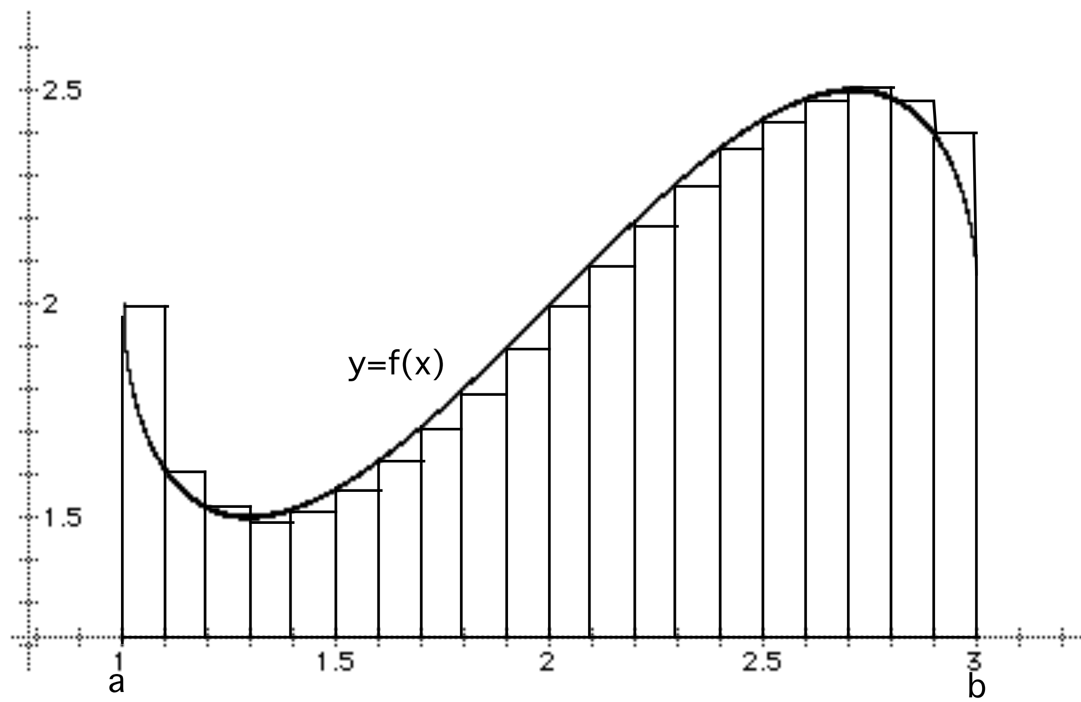
Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations.

- Limits
- Derivatives
- Indefinite Integrals (or Anti-Derivatives)
- Definite Integrals

There are two kinds of Integrals--the Definite Integral and the Indefinite Integral. The Definite Integral was explored first as a way to determine the area bounded by a curve rather than bounded by a polygon.

We know, from Geometry, how to find the exact area of various polygons, but geometry never considered figures where one or more sides is not made of a line segment. Here we want to consider a figure where one side is the curve $y=f(x)$ and the other sides are the x-axis and the lines $x = a$ and $x = b$.



As we can see above, the area can be approximated by rectangles whose height is the y value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. The area of each rectangle would be

$f(x)\cdot\Delta x$ and the total area of n rectangles would be $\sum_{i=1}^{n} f(x_i)\cdot\Delta x$. If we could make an infinite number of rectangles (which would be infinitely thin), we would have the exact area. The rectangles can be drawn several ways--with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But once they become infinitely thin, it will not matter how they were drawn--they will have no width and a height equal to the y-value of the curve.

We can make an infinite number of rectangles mathematically by taking the Limit as n approaches infinity, or

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x.$$

This limit is rewritten as the Definite Integral:

$$\int_a^b f(x)\ dx$$

b is the "upper bound" and a is the "lower bound," and would not mean much if it were not for the following rule. The symbol $\int$ comes from the 17th century S and stands for sum.

For a long time in the mathematical world we did not know that integrals and derivatives were connected. In the mid-1600s Scottish mathematician James Gregory published the first proof of what is now called the Fundamental Theorem of Calculus, changing the math world forever. The Indefinite Integral is often referred to as the Anti-Derivative, because, as an operation, it and the Derivative are inverse operations (just as squares and square roots, or exponential and log functions). In this chapter, we will consider how to reverse the differentiation process. In the next chapter, we will explore the definite integral.

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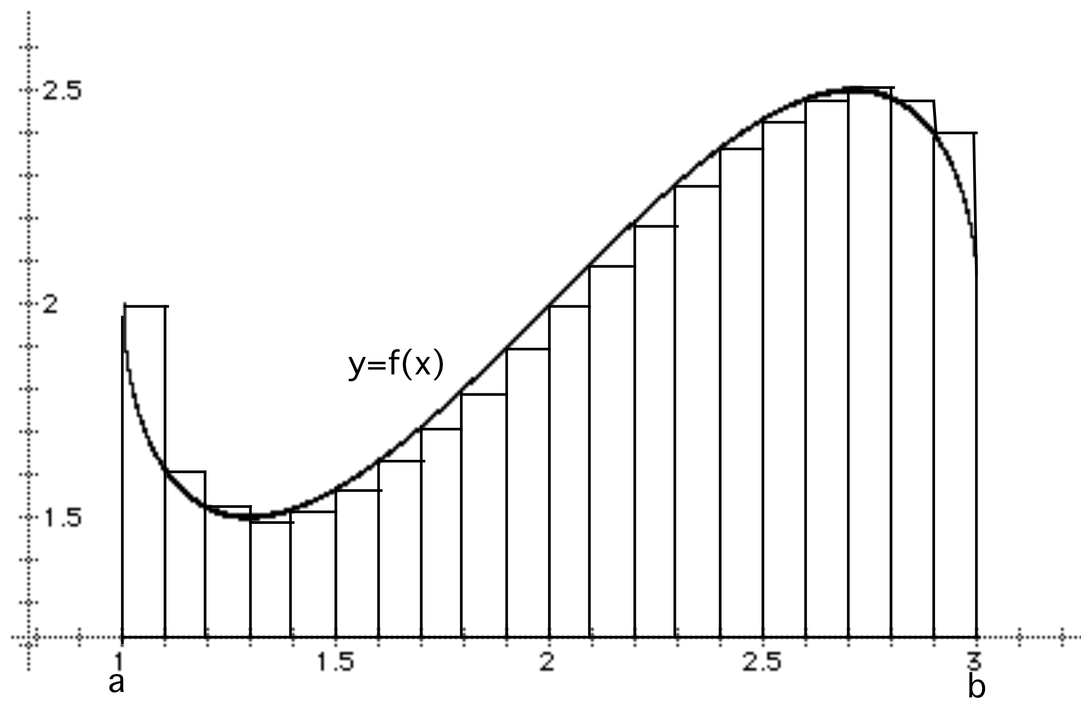

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2.1: Anti-Derivatives--the Power Rule

As we have seen, we can deduce things about a function if its derivative is know. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called Anti-differentiation, or Integration.

Symbol: $\int(f(x))\,dx$ ="the integral of f of x, d-x"

The *dx* is called the differential. For now, we will just treat it as part of the integral symbol. It tells us the independent variable of the function (usually, but not always, x) and, in a sense, is where the increase in the exponent comes from. It does have meaning on its own, but we will explore that later.

Looking at the integral as an anti-derivative, that is, as an operation that reverses the derivative, we should be able to figure out the basic process.

Remember:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

and D_X [constant] is always 0

(or, multiply the power in front and subtract one from the power). If we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by the new power. Also, we do not know, from the derivative, if the original function had a constant that became zero, let alone what the constant was.

The Anti-Power Rule

$$\int\left(x^n\right) dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

The "+ c" is to account for any constant that might have been there before the derivative was taken. NB. This Rule will not work if n = -1, because it would require that we divide by zero. But we know from the Derivative Rules what yields x^{-1} $\left(or\ \frac{1}{x}\right)$ as the derivative--Ln *x*. So we can complete the anti-Power Rule as:

The Anti-Power Rule

$$\int \left(x^n\right)\, dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

$$\int \frac{1}{x}\, dx = Ln \mid x \mid + c$$

Since D_X [f(x)+g(x)] = D_X [f(x)] + D_X [g(x)] and D_X [cx^n] = cD_X [x^n], then

$$\int \left(f(x) + g(x)\right)\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int c\left(f(x)\right)\, dx = c\int f(x)\, dx$$

These allows us to integrate a polynomial by integrating each term separately.

OBJECTIVES
- Find the anti-derivative of a polynomial.
- Integrate functions involving Transcendental operations.
- Use Integration to solve rectilinear motion problems.

Ex 1 $\int \left(3x^2 + 4x + 5\right)\, dx$

$$\int \left(3x^2 + 4x + 5\right)\, dx = 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + c$$

$$= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + c$$

$$= x^3 + 2x^2 + 5x + c$$

Ex 2 $\int\left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)dx$

$$\int\left(x^4+4x^2+5+\frac{1}{x}-\frac{1}{x^5}\right)dx=\frac{x^{4+1}}{4+1}+\frac{4x^{2+1}}{2+1}+\frac{5x^{0+1}}{0+1}+Ln|x|-\frac{x^{-5+1}}{-5+1}+c$$

$$=\frac{1}{5}x^5+\frac{4}{3}x^3+5x+Ln\mid x\mid+\frac{1}{-4x^4}+c$$

Ex 3 $\int\left(x^2+\sqrt[3]{x}-\frac{4}{x}\right)dx$

$$\int\left(x^2+\sqrt[3]{x}-\frac{4}{x}\right)dx=\int\left(x^2+x^{1/3}-\frac{4}{x}\right)dx$$

$$=\frac{x^{2+1}}{2+1}+\frac{x^{1/3+1}}{1/3+1}-4Ln\mid x\mid+c$$

$$=\frac{1}{3}x^3+\frac{3}{4}x^{4/3}-4Ln\mid x\mid+c$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 4 $\int\left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx$

$$\int\left(x^2+\sqrt[3]{x}\right)(2x+1)d\,x=\int\left(2x^3+2x^{4/3}+x^2+x^{1/3}\right)dx$$

$$=\frac{2x^4}{4}+\frac{2x^{7/3}}{7/3}+\frac{x^3}{3}+\frac{x^{4/3}}{4/3}+c$$

$$=\frac{1}{2}x^4+\frac{6}{7}x^{7/3}+\frac{1}{3}x^3+\frac{3}{4}x^{4/3}+c$$

Example 5 is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for c.

Ex 5 $f'(x)=4x^3-6x+3$. Find $f(x)$ if $f(0)=13$.

$$f(x)=\int\left(4x^3-6x+3\right)\,dx$$
$$=x^4-3x^2+3x+c$$
$$f(0)=0^4-3(0)^2+3(0)+c=13$$
$$\therefore\ c=13$$

$$f(x)=x^4-3x^2+3x+13$$

Ex 6 The acceleration of a particle is described by $a(t)=3t^2+8t+1$. Find the distance equation for x(t) if v(0) = 3 and x(0) = 1.

$$v(t)=\int\left(a(t)\right)\,dt=\int\left(3t^2+8t+1\right)\,dt$$
$$=t^3+4t^2+t+c_1$$
$$3=(0)^3+4(0)^2+(0)+c_1$$
$$3=c_1$$
$$v(t)=t^3+4t^2+t+3$$

$$x(t)=\int\left(v(t)\right)\,dt=\int\left(t^3+4t^2+t+3\right)\,dt$$
$$=\frac{1}{4}t^4+\frac{4}{3}t^3+\frac{1}{2}t^2+3t+c_2$$
$$1=\frac{1}{4}(0)^4+\frac{4}{3}(0)^3+\frac{1}{2}(0)^2+3(0)+c_2$$
$$1=c_2$$

$$x(t)=\frac{1}{4}t^4+\frac{4}{3}t^3+\frac{1}{2}t^2+3t+1$$

Ex 7 The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for *x(t)* if $v(1) = 0$ and $x(1) = 3$.

$$
\begin{aligned}
v(t) &= \int (a(t))\ dt = \int (12t^2 - 6t + 4)\ dt \\
&= 4t^3 - 3t^2 + 4t + c_1 \\
0 &= 4(1)^3 - 3(1)^2 + 4(1) + c_1 \\
-5 &= c_1 \\
v(t) &= 4t^3 - 3t^2 + 4t - 5
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \int (v(t))\ dt = \int (4t^3 - 3t^2 + 4t - 5)\ dt \\
&= t^4 - t^3 + 2t^2 - 5t + c_2 \\
3 &= (1)^4 - (1)^3 + 2(1)^2 - 5(1) + c_2 \\
6 &= c_2
\end{aligned}
$$

$$x(t) = t^4 - t^3 + 2t^2 - 5t + 6$$

The proof of the Transcendental Integral Rules can be left to a more formal Calculus course. But since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Derivative Rules

$$\frac{d}{dx}\left[\sin u\right]=\left(\cos u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc u\right]=\left(-\csc u \ \cot u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos u\right]=\left(-\sin u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec u\right]=\left(\sec u \ \tan u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan u\right]=\left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot u\right]=\left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[Ln\ u\right]=\left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right]=a^u\cdot Ln\,a\frac{du}{dx} \qquad \frac{d}{dx}\left[Log_a\ u\right]=\left(\frac{1}{u\cdot Ln\ a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right]=\frac{1}{\sqrt{1-u^2}}\cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right]=\frac{-1}{|\,u\,|\sqrt{u^2-1}}\cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1} u\right]=\frac{-1}{\sqrt{1-u^2}}\cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right]=\frac{1}{|\,u\,|\sqrt{u^2-1}}\cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1} u\right]=\frac{1}{u^2+1}\cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right]=\frac{-1}{u^2+1}\cdot D_u$$

Transcendental Integral Rules

$\int(\cos u)\, du = \sin u + c$

$\int(\sin u)\, du = -\cos u + c$

$\int\left(\sec^2 u\right) du = \tan u + c$

$\int\left(e^u\right) du = e^u + c$

$\int\left(a^u\right) du = \dfrac{a^u}{Ln\ a} + c$

$\int(\csc u \cot u)\, du = -\csc u + c$

$\int(\sec u \tan u)\, du = \sec u + c$

$\int\left(\csc^2 u\right) du = -\cot u + c$

$\int\left(\dfrac{1}{u}\right) du = Ln\ |u| + c$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C \qquad \int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$$

Note that there are only three integrals that yield inverse trig functions where there were six inverse trig derivatives. This is because the other three rules derivative rules are just the negatives of the first three. As we will see later, these three rules are simplified versions of more general rules, but for now we will stick with the three.

Ex 8 $\int(\sin x + 3\cos x)\, dx$

$$\int(\sin x + 3\cos x)\, dx = \int(\sin x)\, dx + 3\int(\cos x)\, dx$$
$$= -\cos x + 3\sin x + c$$

Ex 9 $\int\left(e^x + 4 + 3\csc^2 x\right) dx$

$$\int\left(e^x + 4 + 3\csc^2 x\right) dx = \int\left(e^x\right) dx + 4\int dx + 3\int\left(\csc^2 x\right) dx$$

$$= e^x + 4x - 3\cot x + c$$

Trig Inverse Integral Rules

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{du}{u^2+a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

Ex 10 If $\int \frac{dx}{u^2+4}$

$$\int \frac{dx}{u^2+4} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

Ex 11 If $\frac{dy}{dx} = \sec x(\sec x + \tan x)$, find $y(x)$ if $y(0)=0$.

$$y = \int (\sec x(\sec x + \tan x))dx = \int (\sec^2 x)\, dx + \int (\sec x \tan x)\, dx$$

$$= \tan x + \sec x + c$$

$$0 = \tan 0 + \sec 0 + c$$
$$0 = 0 + 1 + c$$
$$c = -1$$

$$y = \tan x + \sec x - 1$$

2.1 Free Response Homework

Perform the Anti-differentiation.

1. $\int\left(6x^2-2x+3\right)\,dx$

2. $\int\left(x^3+3x^2-2x+4\right)\,dx$

3. $\int\frac{2}{\sqrt[3]{x}}\,dx$

4. $\int\left(8x^4-4x^3+9x^2+2x+1\right)\,dx$

5. $\int x^3\left(4x^2+5\right)\,dx$

6. $\int(4x-1)(3x+8)\,dx$

7. $\int\left(\sqrt{x}-\frac{6}{\sqrt{x}}\right)dx$

8. $\int\left(\frac{x^2+\sqrt{x}+3}{x}\right)dx$

9. $\int(x+1)^3\,dx$

10. $\int(4x-3)^2\,dx$

11. $\int\left(\sqrt{x}+3\sqrt[2]{x^3}-\frac{6}{\sqrt{x}}\right)dx$

12. $\int\left(\frac{4x^3+\sqrt{x}+3}{x^2}\right)dx$

13. $\int\left(x^2+5x+6\right)dx$

14. $\int\left(\frac{x^2-4x+7}{x}\right)dx$

15. $\int\frac{x^5-7x^3+2x-9}{2x}\,dx$

16. $\int\frac{x^3+3x^2+3x+1}{x+1}\,dx$

17. $\int\left(y^2+5\right)^2\,dy$

18. $\int\left(4t^2+1\right)\left(3t^3+7\right)dt$

Solve the initial value problems.

19. $f'(x)=3x^2-6x+3$. Find $f(x)$, if $f(0)=2$.

20. $f'(x)=x^3+x^2-x+3$. Find $f(x)$, if $f(1)=0$.

21. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find $f(x)$, if $f(4) = 1$.

22. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for $x(t)$, if $v(1) = 1$ and $x(1) = 3$.

23. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

2.1 Multiple Choice Homework

1. $\int \frac{1}{x^2}\, dx =$

a) $\ln x^2 + C$ b) $-\ln x^2 + C$ c) $x^{-1} + C$

d) $-x^{-1} + C$ e) $-2x^{-3} + C$

---

2. $\int x(10 + 8x^4)\, dx =$

a) $5x^2 + \frac{4}{3}x^6 + C$ b) $5x^2 + \frac{8}{5}x^5 + C$ c) $10x + \frac{4}{3}x^6 + C$

d) $5x^2 + 8x^6 + C$ e) $5x^2 + \frac{8}{7}x^6 + C$

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3. $\int x\sqrt{3x}\ dx =$

a) $\frac{2\sqrt{3}}{5}x^{5/2} + C$ b) $\frac{5\sqrt{3}}{2}x^{5/2} + C$ c) $\frac{\sqrt{3}}{2}x^{1/2} + C$

d) $2\sqrt{3x} + C$ e) $\frac{5\sqrt{3}}{2}x^{3/2} + C$

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4. $\int (x-1)\sqrt{x}\ dx =$

a) $\frac{3}{2}\sqrt{x} - \frac{1}{\sqrt{x}} + c$ b) $\frac{2}{3}x^{3/2} + \frac{1}{2}x^{1/2} + c$ c) $\frac{1}{2}x^2 - x + c$

d) $\frac{2}{5}x^{5/2} - \frac{2}{3}x^{3/2} + c$ e) $\frac{1}{2}x^2 + 2x^{3/2} + c$

---

5. A particle is moving upward along the y-axis until it reaches the origin and then it moves downward such that $v(t) = 8 - 2t$ for $t \geq 0$. The position of the particle at time t is given by

a) $y(t) = -t^2 + 8t - 16$ b) $y(t) = -t^2 + 8t + 16$

c) $y(t) = 2t^2 - 8t - 16$ d) $y(t) = 8t - t^2$

e) $y(t) = 8t - 2t^2$

---

6. If a particle's acceleration is given by $a(t) = 12t + 4$ and $v(1) = 5$ and $y(0) = 2$, then $y(2) =$

a) 20 b) 10 c) 4 d) 16 e) 12

2.2: Integration by Substitution--the Chain Rule

The other three derivative rules--The Product, Quotient and Chain Rules--are a little more complicated to reverse than the Power Rule. This is because they yield a more complicated function as a derivative, one which usually has several algebraic simplifications. The Integral of a Rational Function is particularly difficult to unravel because, as we saw, a Rational derivative can be obtained by differentiating a composite function with a Log or a radical, or by differentiating another rational function. Reversing the Product Rule is as complicated, though for other reasons. We will leave both these subjects for a traditional Calculus Class. The Chain Rule is another matter.

Composite functions are among the most pervasive situations in math. Though not as simple at reverse as the Power Rule, the overwhelming importance of this rule makes it imperative that we address it here.

Remember:

The Chain Rule: $$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\cdot g'(x)$$

The derivative of a composite turns into a product of a composite and a non-composite. So if we have a product to integrate, it might be that the product came from the Chain Rule. The integration is not done by a formula so much as a process that might or might not work. We make an educated guess and hope it works out. You will learn other processes in Calculus for when it does not work.

Integration by Substitution (The Unchain Rule)

0) Notice that you are trying to integrate a product (or quoptient).
1) Identify the inside function of the composite and call it u.
2) Find du from u.
3) If necessary, multiply a constant inside the integral to create du, and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
4) Substitute u and du into the equation.
5) Perform the integration by Anti-Power (or Transcendental Rules, in next section.)
6) Resubstitute the *x*-equivalent for u.

This is one of those mathematical processes that makes little sense when first seen. But after seeing several examples, the meaning suddenly becomes clear. **Be Patient.**

OBJECTIVE
Use the Unchain Rule to integrate composite, product expressions.

Ex 1 $\int\left(3x^2\left(x^3+5\right)^{10}\right)dx$

$\left(x^3+5\right)^{10}$ is the composite function. $u=x^3+5$, $du=3x^2\,dx$

$$\int\left(3x^2\left(x^3+5\right)^{10}\right)dx=\int\left(u^{10}\right)du$$

$$=\frac{u^{11}}{11}+c$$

$$=\frac{1}{11}\left(x^3+5\right)^{11}+c$$

Ex 2 $\int\left(x\left(x^2+5\right)^3\right)dx$

$\left(x^2+5\right)^3$ is the composite function. So $u=x^2+5$, $du=2x\,dx$

$$\int\left(x\left(x^2+5\right)^3\right)dx=\frac{1}{2}\int\left(x^2+5\right)^3(2x\,dx)$$

$$=\frac{1}{2}\int\left(u^3\right)du$$

$$=\frac{1}{2}\cdot\frac{u^4}{4}+c$$

$$=\frac{1}{8}\left(x^2+5\right)^4+c$$

Ex 3 $\int \left(\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5} \right) dx$

$\sqrt[4]{x^4+2x^2-5}$ is the composite function. So

$$u = x^4+2x^2-5$$
$$du = \left(4x^3+4x\right) dx = 4\left(x^3+x\right) dx$$

$$\int \left(\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5} \right) dx = \frac{1}{4}\int \left(\sqrt[4]{x^4+2x^2-5} \right) 4\left(x^3+x\right) dx$$
$$= \frac{1}{4}\int \left(\sqrt[4]{u}\right) du$$
$$= \frac{1}{4}\int \left(u^{1/4}\right) du$$
$$= \frac{1}{4}\frac{u^{5/4}}{5/4}+c$$

$$= \frac{1}{5}\left(x^4+2x^2-5\right)^{5/4}+c$$

Ex 4 $\int \left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3} \right) dx$

$$u = x^3+2x^2-5x+2$$
$$du = \left(3x^2+4x-5\right) dx$$

$$\int \left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3} \right) dx = \int \left(x^3+2x^2-5x+2\right)^{-3}\left(\left(3x^2+4x-5\right) dx\right)$$
$$= \int \left(u^{-3}\right) du$$
$$= \frac{u^{-2}}{-2}+c$$

$$= \frac{-1}{2\left(x^3+2x^2-5x+2\right)^2}+c$$

Of course, the Unchain Rule will apply to the transcendental functions quite well.

Ex 5 $\int(\sin 5x)\,dx$

$$u = 5x$$
$$du = 5dx$$

$$\begin{aligned}\int(\sin 5x)\,dx &= \frac{1}{5}\int(\sin 5x)\ 5\ dx \\ &= \frac{1}{5}\int(\sin u)\,du \\ &= \frac{1}{5}(-\cos u) + c \\ &= -\frac{1}{5}\cos\ 5x\ + c\end{aligned}$$

Ex 6 $\int\left(\sin^6 x\ \cos x\right)dx$

$$u = \sin x$$
$$du = \cos\ x\ dx$$

$$\begin{aligned}\int\left(\sin^6 x\ \cos x\right)dx &= \int\left(u^6\right)du \\ &= \frac{1}{7}u^7 + c \\ &= \frac{1}{7}\sin^7\ x\ + c\end{aligned}$$

Ex 7 $\int\left(x^5 \sin x^6\right)dx$

$$u = x^6$$
$$du = 6x^5 dx$$

$$\begin{aligned}\int\left(x^5 \sin x^6\right)dx &= \frac{1}{6}\int\left(\sin x^6\ \right)\left(6x^5 dx\right) \\ &= \frac{1}{6}\int(\sin u)\,du \\ &= -\frac{1}{6}\cos u + c \\ &= -\frac{1}{6}\cos x^6 + c\end{aligned}$$

Ex 8 $\int\left(\cot^3 x \ \csc^2 \ x\right) dx$

$$u = \cot x$$
$$du = -\csc^2 \ x \ dx$$

$$\int\left(\cot^3 x \ \csc^2 \ x\right) dx = -\int\left(\cot^3 x \ \right)\left(-\csc^2 \ x \ dx\right)$$
$$= -\int\left(u^3\right)du$$
$$= -\frac{1}{4}u^4 + c$$

$$= -\frac{1}{4}\cot^4 x + c$$

Ex 9 $\int\left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right) dx$

$$u = \sqrt{x} = x^{1/2}$$
$$du = \frac{1}{2}x^{-1/2} \ dx = \frac{1}{2x^{1/2}}dx$$

$$\int\left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right) dx = 2\int\left(\cos\sqrt{x}\ \right)\left(\frac{1}{2\sqrt{x}}dx\right)$$
$$= 2\int\left(\cos u\right) du$$
$$= 2\sin u + c$$

$$= 2\sin\sqrt{x} + c$$

Ex 10 $\int \left(xe^{x^2+1} \right) dx$

$$u = x^2 + 1$$
$$du = 2x\ dx$$

$$\int \left(xe^{x^2+1} \right) dx = \frac{1}{2} \int \left(e^{x^2+1} \right) (2x\ dx)$$
$$= \frac{1}{2} \int \left(e^u \right) du$$
$$= \frac{1}{2} e^u + c$$
$$= \frac{1}{2} e^{x^2+1} + c$$

Ex 11 $\int \frac{x}{\sqrt{1-x^4}}\ dx$

$$u = x^2$$
$$du = 2x\ dx$$

$$\int \frac{x}{\sqrt{1-x^4}}\ dx = \frac{1}{2} \int \frac{1}{\sqrt{1-\left(x^2\right)^2}} (2x\ dx)$$
$$= \frac{1}{2} \int \frac{1}{\sqrt{1-u^2}}\ du$$
$$= \frac{1}{2} \sin^{-1} u + c$$
$$= \frac{1}{2} \sin^{-1} x^2 + c$$

Ex 12 $\int\left(xe^{x^2}+4x^2-3\sin 5x\right)\,dx$

$$\int\left(xe^{x^2}+4x^2-3\sin 5x\right)\,dx=\int\left(xe^{x^2}\right)\,dx+\int\left(4x^2\right)\,dx+\int(-3\sin 5x)\,dx$$

$$=\frac{1}{2}\int e^{x^2}\left(2xdx\right)\;+4\int x^2\;dx-\frac{3}{5}\int\sin 5x\;\left(5dx\right)$$

$u_1=x^2$ $\qquad$ $u_2=5x$

$du_1=2x\;dx$ $\qquad$ $du_2=5\;dx$

$$=\frac{1}{2}\int e^{u_1}\,du_1\;+4\int x^2\;dx-\frac{3}{5}\int\sin u_2\,du_2$$

$$=\frac{1}{2}e^{u_1}\;+4\left(\frac{x^3}{3}\right)\;-\frac{3}{5}\left(-\cos u_2\right)+c$$

$$=\frac{1}{2}e^{x^2}\;+\frac{4}{3}x^3\;+\frac{3}{5}\cos 5x+c$$

2.2 Homework Set A

Perform the Anti-differentiation.

1. $\int (5x+3)^3\, dx$

2. $\int \left(x^3 (x^4+5)^{24} \right) dx$

3. $\int (1+x^3)^2\, dx$

4. $\int (2-x)^{2/3}\, dx$

5. $\int \left(x\sqrt{2x^2+3} \right) dx$

6. $\int \dfrac{dx}{(5x+2)^3}$

7. $\int \dfrac{x^3}{\sqrt{1+x^4}}\, dx$

8. $\int \dfrac{x+1}{\sqrt[3]{x^2+2x+3}}\, dx$

9. $\int \left(x^5 - \sin(3x) + xe^{x^2} \right) dx$

10. $\int \left(x^2 \sec^2(x^3) + \dfrac{\ln^3 x}{x} \right) dx$

11. $\int \left(x^4 \cos x^5 \right) dx$

12. $\int (\sin(7x+1)) dx$

13. $\int \left(\sec^2(3x-1) \right) dx$

14. $\int \left(\dfrac{\sin \sqrt{x}}{\sqrt{x}} \right) dx$

15. $\int \left(\tan^4 x \sec^2 x \right) dx$

16. $\int \dfrac{\ln x}{x}\, dx$

17. $\int \left(e^{6x} \right) dx$

18. $\int \dfrac{\cos 2x}{\sin^3 2x}\, dx$

19. $\int \dfrac{x \ln (x^2+1)}{x^2+1}\, dx$

20. $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

21. $\int \left(\sqrt{\cot x}\, \csc^2 x \right) dx$

22. $\int \dfrac{1}{x^2} \left(\sin \dfrac{1}{x} \right) \left(\cos \dfrac{1}{x} \right) dx$

23. $\int \frac{x}{1+x^4}\,dx$

24. $\int \frac{\cos x}{\sqrt{1-\sin^2 x}}\,dx$

2.2 Multiple Choice Homework

1. $\int \frac{x}{x^2-4}\,dx =$

a) $\frac{-1}{4\left(x^2-4\right)^2}+C$ b) $\frac{1}{2\left(x^2-4\right)}+C$ c) $\frac{1}{2}\ln\left|x^2-4\right|+C$

d) $2\ln\left|x^2-4\right|+C$ e) $\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$

---

2. $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx =$

a) $\ln\sqrt{x}+C$ b) $x+C$ c) e^x+C

d) $\frac{1}{2}e^{2\sqrt{x}}+C$ e) $e^{\sqrt{x}}+C$

---

3. When using the substitution $u=\sqrt{1+x}$, an anti-derivative of $\int 60x\sqrt{1+x}\,dx$ is

a) $20u^3-60u+C$ b) $15u^4-30u^2+C$ c) $30u^4-60u^2+C$

d) $24u^5-40u^3+C$ e) $12u^6-20u^4+C$

---

4. $\int \frac{3x^2}{\sqrt{x^3+3}}dx$

a) $2\sqrt{x^3+3}+c$ b) $\frac{3}{2}\sqrt{x^3+3}+c$ c) $\sqrt{x^3+3}+c$

d) $\ln\sqrt{x^3+3}+c$ e) $\ln\left(x^3+3\right)+c$

---

5. $\int x\left(x^2-1\right)^4 dx =$

a) $\frac{1}{10}x^2\left(x^2-1\right)^5+C$ b) $\frac{1}{10}\left(x^2-1\right)^5+C$ c) $\frac{1}{5}\left(x^3-x\right)^5+C$

d) $\frac{1}{5}\left(x^2-1\right)^5+C$ e) $\frac{1}{5}\left(x^2-x\right)^5+C$

---

6. $\int 4x^2\sqrt{3+x^3}\,dx$

a) $\frac{16\left(3+x^3\right)^{3/2}}{9}+c$ b) $\frac{8\left(3+x^3\right)^{3/2}}{9}+c$ c) $\frac{8\left(3+x^3\right)^{3/2}}{3}+c$

d) $\frac{4}{3\left(3+x^3\right)^{1/2}}+c$ e) $\frac{8}{3\left(3+x^3\right)^{1/2}}+c$

---

7. $\int\left(x^3+2+\frac{1}{x^2+1}\right)\,dx=$

a) $\frac{x^4}{4}+2x+\tan^{-1}x+C$

b) $x^4+2+\tan^{-1}x+C$

c) $\frac{x^4}{4}+2x+\frac{3}{x^3+3}+C$

d) $\frac{x^4}{4}+2x+\tan^{-1}2x^2+C$

e) $4+2x+\tan^{-1}x+C$

---

8. $\int\cos(3-2x)\,dx=$

a) $\sin(3-2x)+C$

b) $-\sin(3-2x)+C$

c) $\frac{1}{2}\sin(3-2x)+C$

d) $-\frac{1}{2}\sin(3-2x)+C$

e) $-\frac{1}{5}\sin(3-2x)+C$

---

9. $\int\frac{x-2}{x-1}\,dx=$

a) $-\ln|x-1|+C$

b) $x+\ln|x-1|+C$

c) $x-\ln|x-1|+C$

d) $x+\sqrt{x-1}+C$

e) $x-\sqrt{x-1}+C$

---

10. $\int \frac{e^{x^2} - 2x}{e^{x^2}} \, dx =$

a) $x - e^{x^2} + C$ b) $x - e^{-x^2} + C$ c) $x + e^{-x^2} + C$

d) $-e^{x^2} + C$ e) $-e^{-x^2} + C$

---

11. $\int 6 \sin x \cos^2 x \, dx =$

a) $2\sin^3 x + C$ b) $-2\sin^3 x + C$ c) $2\cos^3 x + C$

d) $-2\cos^3 x + C$ e) $3\sin^2 x \cos^2 x + C$

---

12. $\int \frac{4x}{1+x^2} \, dx =$

a) $4 \arctan x + C$ b) $\frac{4}{x} \arctan x + C$ c) $\frac{1}{2}\ln\left(1+x^2\right) + C$

d) $2\ln\left(1+x^2\right) + C$ e) $2x^2 + 4\ln|x| + C$

---

13. $\int \frac{x}{4+x^2} dx$

a) $\tan^{-1}\frac{x}{2} + c$ b) $\ln\left(4+x^2\right) + c$ c) $\tan^{-1} x + c$

d) $\frac{1}{2}\ln\left(4+x^2\right) + c$ e) $\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

14. $\int\left(2^x - 4e^{2\ln x}\right)dx$ $\int\left(2^x - 4e^{2\ln x}\right)dx$

a) $2^x \ln 2 - \frac{4}{3}e^{2\ln x} + c$ b) $x2^{x-1} - \frac{4}{3}x^3 + c$ c) $\frac{2^x}{\ln 2} - \frac{4}{3}e^{2\ln x} + c$

d) $x2^{x-1} - \frac{4}{3}e^{2\ln x} + c$ e) $\frac{2^x}{\ln 2} - \frac{4}{3}x^3 + c$

---

15. The anti-derivative of $2\tan x$

a) $2\ln|\sec x| + c$ b) $2\sec^2 x + c$ c) $\ln|\sec^2 x| + c$

d) $2\ln|\cos x| + c$ e) $\ln|2\sec x| + c$

---

16. Which of the following statements are true?

I. $\int\left(x^5 \sin x^6\right)dx = -\frac{1}{6}\cos x^6 + c$ II. $\int \tan x\, dx = \sec^2 x + c$

III. $\int\left(\left(x^3 + x\right)\sqrt[4]{x^4 + 2x^2 - 5}\right)dx = \frac{1}{5}\left(x^4 + 2x^2 - 5\right)^{5/4} + c$

a) I only b) II only c) III only d) I and II only e) II and III only

ab) I and III only ac) I, II, and III ad) None of these

---

17. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

a) $2e^{\sqrt{x}}+c$ b) $\frac{1}{2}e^{\sqrt{x}}+c$ c) $e^{\sqrt{x}}+c$

d) $2\sqrt{x}\ e^{\sqrt{x}}+c$ e) $\frac{e^{\sqrt{x}}}{2\sqrt{x}}+c$

---

18. If $x'(t)=2t\cos t^2$, find $x(t)$ when $x\left(\sqrt{\frac{\pi}{2}}\right)=3$

a) $x(t)=-4t^2\sin t^2$

b) $x(t)=-4t^2\sin t^2+2\cos t^2$

c) $x(t)=\sin t^2+3$

d) $x(t)=-\sin t^2+4$

e) $x(t)=\sin t^2+2$

---

19. A particle moves along the y-axis so that at any time $t\geq 0$, it velocity is given $v(t)=\sin(2t)$. If the position of the particle at time $t=\frac{\pi}{2}$ is $y=3$, the particle's position at time $t=0$ is

a) -4 b) 2 c) 3 d) 4 e) 6

2.2 Homework Set B

1. $\int (2x+5)(x^2+5x+6)^6\,dx$

2. $\int 3t^2(t^3+1)^5\,dt$

3. $\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}}\,dm$

4. $\int \frac{3x^2}{(1+x^3)^5}\,dx$

5. $\int (4s+1)^5\,ds$

6. $\int \frac{5t}{t^2+1}\,dt$

7. $\int \frac{3m^2}{m^3+8}\,dm$

8. $\int (181x+1)^5\,dx$

9. $\int \frac{v^2}{5-v^3}\,dv$

10. $\int \left(x^5-\sin(3x)+xe^{x^2}\right)dx$

11. $\int \frac{\cos x}{1+\sin x}\,dx$

12. $\int x^2\sec^2(x^3)+2xe^{x^2}\,dx$

13. $\int \sec^2(2x)\,dx$

14. $\int \frac{\csc^2(e^{-x})}{e^x}\,dx$

15. $\int \frac{\sec(\ln x)\tan(\ln x)}{3x}\,dx$

16. $\int \left(x^5+\frac{7}{x^2}-e^{2x}+\sec^2 x\right)dx$

17. $\int e^x\csc e^x\cot e^x\,dx$

18. $\int (e^x-2)(e^x-1)\,dx$

19. $\int (\sec^2 y\tan^5 y)\,dy$. Verify that your integration is correct by taking the derivative of your answer.

20. $\int \left(\cos\theta e^{\sin\theta} + \frac{\theta}{\theta^2+1} \right) d\theta$. Verify that your integration is correct by taking the derivative of your answer.

21. $\int t \sec^2\left(4t^2\right) \sqrt{\tan\left(4t^2\right)}\, dt$. Verify that your integration is correct by taking the derivative of your answer.

22. $\int x^2 \sin x^3 \, dx$

23. $\int t e^{5t^2+1} \, dt$

24. $\int \left(e^y + 1\right)^2 dy$

25. $\int x \sec^2 x^2 \sqrt{\tan x^2} \, dx$

26. $\int \sin(3t)\cos^5(3t)\, dt$

27. $\int x \cos x^2 e^{\sin x^2} \, dx$

28. $\int \tan\theta \ln(\sec\theta)\, d\theta$

29. $\int \left(e^{4y} + 2y^2 - 7\cos 3y\right) dy$

30. $\int \frac{\sin(x+4)}{\cos^7(x+4)} dx$

31. $\int \left(\frac{2x}{x^2+5} - \sec^2(3x) + xe^{x^2} - \pi \right) dx$

32. $\int e^{2t} \sec^2 e^{2t} dt$

33. $\int \frac{18 \ln m}{m} dm$

34. $\int \frac{2y\cos\left(y^2\right)}{\sin^4\left(y^2\right)} dy$

2.3 Separable Differential Equations

Vocabulary:

Differential Equation (differential equation) – an equation that contains an unknown function and one or more of its derivatives.

General Solution – The solution obtained from solving a differential equation. It still has the +C in it.

Initial Condition – Constraint placed on a differential equation; sometimes called an initial value.

Particular Solution – Solution obtained from solving a differential equation when an initial condition allows you to solve for C.

Separable Differential Equation – A differential equation in which all terms with y's can be moved to the left side of an equals sign (=), and in which all terms with x's can be moved to the right side of an equals sign (=), by multiplication and division only.

OBJECTIVES
Given a separable differential equation, find the general solution. Given a separable differential equation and an initial condition, find a particular solution.

Ex 1 Find the general solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

$\frac{dy}{dx} = -\frac{x}{y}$	Start here.
$ydy = -xdx$	Separate all the *y* terms to the left side of the equation and all of the *x* terms to the right side of the equation.
$\int ydy = \int -xdx$	Integrate both sides.
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$	You only need *C* on one side of the equation and we put it on the side containing the *x*.
$y^2 = -x^2 + C$	Multiply both sides by 2. Note: 2*C* is still a constant, so we'll continue to note it just by *C*.
$x^2 + y^2 = C$	This equation should seem familiar. It's the family of circles centered at the origin with radius $\sqrt{C}$.
$y = \pm\sqrt{C - x^2}$	Isolate *y*.

Usually, if it's possible, we will solve our equation for *y* – so our solution can be written as $y = \pm\sqrt{C - x^2}$.

Also to note, we could check our solution by taking the derivative of our solution.
$x^2 + y^2 = C \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$.

Steps to Solving a Differential Equation:

1. Separate the variables. Note: Leave constants on the right side of the equation.
2. Integrate both sides of the equation. Note: only write the +*C* on the right side of the equation. Here's why – you will have a constant on either of your equation when you integrate both sides of your differential Equation, but you would wind up subtracting one from the other eventually, and two constants subtracted from one another is still a constant, so we only write +*C* on one side of the equation.
3. Solve for *y*, if possible. If you integrate and get a natural log in your result, solve for *y*. If there is no natural log, solve for *C*. Note: $e^{\ln|y|} = y$ because *e* raised to any power is automatically positive, so the absolute values are not necessary.
4. Plug in the initial condition, if you are given one, and solve for *C*.

Note: Solve for C immediately if the left integral does not result in a Ln. Simplify before solving if there is a Ln.

Ex 2 Find the general solution to the differential equation $\frac{dm}{dt} = mt$.

$\frac{dm}{dt} = mt$ Start here.

$\frac{dm}{m} = tdt$ Separate all the *m* terms to the left side of the equation and all of the *t* terms to the right side of the equation.

$\int \frac{dm}{m} = \int tdt$ Integrate both sides.

$\ln|m| = \frac{1}{2}t^2 + C$ You only need *C* on one side of the equation and we put it on the side containing the *x*.

$e^{\ln|m|} = e^{\frac{1}{2}t^2 + C}$ $\quad$ e both sides of the equation to solve for y.

$m = e^{\frac{1}{2}t^2} e^C$

$m = Ke^{\frac{1}{2}t^2}$ $\quad$ e^C is still a constant, so we will just note it as K.

Ex 3 Find the particular solution to $y' = 2xy - 3y$, given $y(3) = 2$.

$$\frac{dy}{dx} = (2x-3)y$$

$$\frac{dy}{y} = (2x-3)dx$$

$$\ln|y| = x^2 - 3x + C$$

$$y = e^{x^2-3x+C} = e^{x^2-3x}e^C = Ke^{x^2-3x}$$

$$y(3) = 2 \rightarrow 2 = Ke^0$$

$$K = 2$$

$$y = 2e^{x^2-3x}$$

Ex 4 Solve the differential equation $\frac{dy}{dx} = \frac{9x^2+2x+5}{2y+e^y}$

$$\frac{dy}{dx} = \frac{9x^2+2x+5}{2y+e^y}$$

$$(2y+e^y)dy = (9x^2+2x+5)dx$$

$$y^2 + e^y = 3x^3 + x^2 + 5x + C$$

$$y^2 + e^y - 3x^3 - x^2 - 5x = C$$

This is the general solution.

Ex 5 Find the particular solution to $\frac{dr}{dt}=\frac{3t^2-\sin t}{4r}$ given that $r(0)=3$.

$$\begin{aligned}
\frac{dr}{dt}&=\frac{3t^2-\sin t}{4r}\\
4rdr&=\left(3t^2-\sin t\right)dt\\
2r^2&=t^3+\cos t+C\\
2\cdot 3^2&=0^3+\cos 0+C\Rightarrow C=17\\
2r^2&=t^3+\cos t+17\\
r&=\pm\sqrt{\frac{t^3}{2}+\frac{1}{2}\cos t+\frac{17}{2}}\\
r&=\sqrt{\frac{t^3}{2}+\frac{1}{2}\cos t+\frac{17}{2}}
\end{aligned}$$

Why do we only need the positive *r*?

What would have happened if you had solved for *r* <u>before</u> plugging in the initial condition?

$$\begin{aligned}
r^2&=\frac{t^3}{2}+\frac{1}{2}\cos t+C\\
r&=\pm\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+C}\\
3&=\pm\sqrt{\frac{0^3}{3}+\frac{1}{2}\cos 0+C}\Rightarrow C=\frac{17}{2}\\
r&=\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+\frac{17}{2}}
\end{aligned}$$

Again, you can check your solution by taking the derivative of your solution.

Ex 6 Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = \frac{y+1}{x^2+9}$ and suppose the point $(0, -3)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Determine if the point $(0, -3)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = \frac{y+1}{x^2+9}$ at $(0, -3)$.

---

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{y+1}{x^2+9}\right] = \frac{\left(x^2+9\right)\frac{dy}{dx} - (y+1)(2x)}{\left(x^2+9\right)^2}$$

$$= \frac{\left(x^2+9\right)\left[\frac{y+1}{x^2+9}\right] - (y+1)(2x)}{\left(x^2+9\right)^2} = \frac{(y+1) - (y+1)(2x)}{\left(x^2+9\right)^2} = \frac{(y+1)(1-2x)}{\left(x^2+9\right)^2}$$

---

b) Determine if the point $(0, -3)$ is at a maximum, a minimum, or neither.

At the point $(0, -3)$, $\frac{dy}{dx} = \frac{(-3)+1}{0^2+9} = -\frac{2}{9} \neq 0$, therefore, neither.

---

c) Find the particular solution to $\frac{dy}{dx} = \frac{y+1}{x^2+9}$ at $(0, -3)$.

$$\frac{dy}{dx} = \frac{y+1}{x^2+9}$$

$$\frac{1}{y+1}dy = \frac{1}{x^2+9}dx$$

$$\int \frac{1}{y+1}dy = \int \frac{1}{x^2+9}dx$$

$$\ln|y+1| = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

$$|y+1| = e^{\frac{1}{3}\tan^{-1}\frac{x}{3} + c}$$

$$y+1 = ke^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

$$(0, -3) \rightarrow -3+1 = ke^{\frac{1}{3}\tan^{-1}\frac{0}{3}} \rightarrow -2 = k$$

$$y+1 = -2e^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

$$y = -1 - 2e^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

2.3 Free Response Homework

Find the general solution each of the following differential equations.

1. $\frac{dy}{dx} = \frac{y}{x}$

2. $\frac{dy}{dx} = xy^2$

3. $\left(x^2+1\right)\frac{dy}{dx} = xy$

4. $\frac{dy}{dx} = \left(3y^2\right)\frac{1}{1+x^2}$

5. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3-3}}{y^2}$

6. $\frac{dy}{dt} = \frac{\sec^2 t}{ye^{5y^2}}$

7. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

8. $\frac{dy}{dx} = \frac{x^2+1}{\sec y \tan y}$

9. $\frac{dy}{dx} = 4xy^3$

10. $\frac{dy}{dx} = y^2 \cos x$

11. $\frac{dv}{dt} = 2 + 2v + t + tv$

12. $\frac{dy}{dt} = \frac{t}{y\sqrt{y^2+1}}$

13. $\frac{d\theta}{dr} = \frac{1+\sqrt{r}}{\sqrt{\theta}}$

Find the solution of the differential equation that satisfies the given initial condition.

14. $\frac{dy}{dx} = xy^2;\ y(0) = 5$

15. $\frac{dy}{dx} = \frac{2x}{y};\ y(0) = 1$

16. $\frac{dy}{dx} = \frac{2x^3}{3y^2};\ y\left(\sqrt{2}\right) = 0$

17. $\frac{dy}{dx} = \left(x^2+1\right)\left(2-y\right);\ y(1) = 3$

18. $\frac{dy}{dx} = \left(y^2+1\right);\ y(1) = 0$

19. $\frac{dy}{dx} = \frac{y^2+1}{xy};\ y(0) = -1$

20. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$; $u(0) = -5$

21. $\frac{dy}{dx} = yx - y\sin x$; $y(0) = 5e$

22. Solve the initial-value problem $\frac{dy}{dx} = \frac{\sin x}{\sin y}$; $y(0) = \frac{\pi}{2}$, and graph the solution.

23. Solve the equation $e^{-y}\frac{dy}{dx} + \cos x = 0$ and graph several members of the family
of solutions. How does the solution curve change as the constant C varies?

24. Find an equation of the curve that satisfies $\frac{dy}{dx} = 4x^3 y$ and whose y-intercept is 7.

25. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = y^2(6 - 2x)$, and suppose the point $\left(3,\ -\frac{1}{3}\right)$ is on the graph of $y = f(x)$.

a) Use implicit differation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $\left(3,\ -\frac{1}{3}\right)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = y^2(6 - 2x)$ at $\left(3,\ -\frac{1}{3}\right)$.

26. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = xy + y$, and suppose the point $(-1,\ 2)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(-1,\ 2)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = xy + y$ at $(-1,\ 2)$.

27. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = \frac{3x^2}{y+2}$ and suppose the point $(0,\ 1)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(0,\ 1)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = \frac{3x^2}{y+2}$ at $(0,\ 1)$.

28. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = (x-1)(y+2)$ and suppose the point $(1, 0)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(1, 3)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = (x-1)(y+2)$ at $(1, 3)$.

2.3 Multiple Choice Homework

1. If $\frac{dy}{dx} = \sin x \cos^3 x$ and if $y = 1$ when $x = \pi$, what is the value of y when $x = 0$?

a) 0 b) 1 c) 1.5 d) 2 e) 2.5

---

2. If $\frac{dy}{dx} = \cos x \sin^2 x$ and if $y = 0$ when $x = \pi$, what is the value of y when $x =$ 0?

a) –1 b) $-\frac{1}{3}$ c) 0 d) $\frac{1}{3}$ e 1

---

3. Identify is the first mistake (if any) in this process:

$$\frac{dy}{dx} = xy + x$$

Step 1: $\frac{1}{y+1}dy = xdx$

Step 2: $\ln|y+1| = x^2 + c$

Step 3: $|y+1| = e^{x^2} + c$

Step 4: $y = e^{x^2} + c$

a) Step 1 b) Step 2 c) Step 3

d) Step 4 e) There is no mistake.

---

4. Identify is the mistake (if any) in this process:

$$\frac{dy}{dx} = 6x^2y^2$$

Step 1: $\frac{1}{y^2}dy = 6x^2dx$

Step 2: $\ln|y^2| = 2x^3 + c$

Step 3: $y^2 = e^{2x^3+c}$

Step 4: $y = \pm\sqrt{ke^{2x^3}}$

a) Step 1 b) Step 2 c) Step 3

d) Step 4 e) There is no mistake.

5. The solution to the differential equation $\frac{dy}{dx} = 8xy$ with initial condition $y(0) = 5$ is

a) $\ln(4x^2 + 5)$ b) $e^{4x^2} + 5$ c) $e^{4x^2} + 4$

d) $5\ln(4x^2)$ e) $5e^{4x^2}$

2.4: Integration by Substitution--Back Substitution

Sometimes when applying the Chain Rule, the other factor is not the du, or there are extra *x*'s that must be replaced with some form of *u*. The method of choosing u to equal the inside of the composite function remains the same, but there is more substitution necessary. This is best understood in an example:

Ex 1: $\int \left(x^3 \left(x^2+4\right)^{3/2} \right) dx$

$$u = x^2 + 4$$
$$x^2 = u - 4$$
$$du = 2x\ dx$$

$$\begin{aligned}\int \left(x^3 \left(x^2+4\right)^{3/2} \right) dx &= \frac{1}{2}\int \left(x^2 \left(x^2+4\right)^{3/2} \right)(2x\ dx) \\ &= \frac{1}{2}\int \left((u-4)u^{3/2} \right)\ du \\ &= \frac{1}{2}\int \left(u^{5/2} - 4u^{3/2} \right) du \\ &= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - \frac{4u^{5/2}}{5/2} \right) + c \\ &= \frac{1}{7}\left(x^2+4\right)^{7/2} - \frac{4}{5}\left(x^2+4\right)^{5/2} + c\end{aligned}$$

Ex 2: $\int (x+1)\sqrt{x-1}dx$

$$u = x - 1$$
$$x = u + 1$$
$$du = \ dx$$

$$\int (x+1)\sqrt{x-1}\ dx = \int ((u+1)+1)\sqrt{u}\ du$$
$$= \int (u+2)u^{1/2}du$$
$$= \int \left(u^{3/2} + 2u^{1/2}\right)du$$
$$= \frac{u^{5/2}}{5/2} + \frac{2u^{3/2}}{3/2} + c$$
$$= \frac{2}{5}(x-1)^{5/2} + \frac{4}{3}(x-1)^{3/2} + c$$

<u>Integration by Back Substitution (The Unchain Rule)</u>

0) Notice that you are trying to integrate a product.
1) Identify the inside function of the composite and call it u.
2) Find du from u.
3) If necessary, multiply a constant inside the integral to create du, and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
   - 3a) Identify any "extra *x*'s."
   - 3b) Isolate *x* in the scratch work
4) Substitute u and du as well as the extra *x*'s into the equation.
5) Perform the integration by Anti-Power (or Transcendental Rules, in next section.)
6) Resubstitute the *x*-equivalent for u.

Ex 3: $\int (x+2)(x-3)^4 dx$

$$u = x-3$$
$$x = u+3$$
$$du = dx$$

$$\begin{aligned}\int (x+2)(x-3)^4 dx &= \int \big((u+3)+2\big)u^4 \; du \\ &= \int (u+5)u^4 du \\ &= \int \left(u^5 + 5u^4\right) du \\ &= \frac{u^6}{6} + \frac{5u^5}{5} + c\end{aligned}$$

$$= \frac{1}{6}(x-3)^6 + (x-3)^5 + c$$

Ex 4: $\int \frac{x^2+4}{x+2} dx$

There are two ways to approach this problem. One could use polynomial long division to simplify before integrating:

$$\int \frac{x^2+4}{x+2} dx = \int x - 2 + \frac{8}{x+2} dx$$

Then

$$u = x+2$$
$$du = dx$$

$$\begin{aligned}\int \left(x - 2 + \frac{8}{x+2} \right) dx &= \int (x-2) dx + 8 \int \frac{1}{u} du \\ &= \frac{x^2}{2} - 2x + 8 \ln|u| + c \\ &= \frac{x^2}{2} - 2x + 8 \ln|x+2| + c\end{aligned}$$

An alternative would be to make the denominator *u* and use back-substitution:

Ex 4 (again): $\int \frac{x^2+4}{x+2}dx$

$$u = x+2$$
$$x = u-2$$
$$du = dx$$

$$\int \frac{x^2+4}{x+2}dx = \int \frac{(u-2)^2+4}{u}dx$$
$$= \int \frac{u^2-4u+4+4}{u}du$$
$$= \int \frac{u^2-4u+8}{u}du$$
$$= \int \left(u-4+\frac{8}{u}\right)du$$
$$= \frac{u^2}{2}-4u+8\ln|u|+c$$
$$= \frac{(x+2)^2}{2}-4(x+2)+8\ln|x+2|+c$$

The answer looks different from what the answer was the first time, but, with FOILing and adding like terms, it can be shown that these are the same answer.

As a rule of thumb, doing algebra simplification before Calculus will generally make the problem shorter and more simple. In this case, Polynomial Long Division made the problem easier than Back-Substitution.

2.4 Free Response Homework

1. $\int x\sqrt{4-x}\,dx$

2. $\int \left(x^5\right)\sqrt{x^3+4}\,dx$

3. $\int \frac{x+5}{2x+3}dx$

4. $\int x^3\left(x^2+1\right)^{12}dx$

5. $\int \frac{\left(3+\ln x\right)^2\left(2-\ln x\right)}{x}\,dx$

6. $\int \sqrt{4-\sqrt{x}}\,dx$

7. $\int \left(x^5\left(x^2+4\right)^2\right)dx$

8. $\int \sqrt{x+3}\,\left(x+1\right)^2\,dx$

9. $\int (t-1)(2t+4)^5\,dt$

10. $\int (z-3)(3z-1)^3\,dz$

11. $\int \frac{y^5}{\sqrt{y^3+5}}dy$

12. $\int \frac{w^5}{w^2+4}dw$

13. $\int \frac{x^5}{\left(x^2-1\right)^{5/2}}\,dx$

14. $\int \frac{x^7}{\left(x^4+4\right)^{3/2}}\,dx$

15. $\int (x+2)\sqrt[3]{x-1}\,dx$

16. $\int \sqrt{4-x}\,\left(2x+5\right)dx$

17. $\int \left(e^{2x}\sqrt{e^x+1}\right)dx$

2.4 Multiple Choice Homework

1. $\int \left(x^3\right)\sqrt{1+x^2}\,dx$

a) $\frac{x^4}{2}\cdot\frac{\left(1+x^2\right)^{3/2}}{3}+c$

b) $\frac{1}{2}\left(1+x^2\right)^{1/2}+\frac{1}{3}\left(1+x^2\right)^{3/2}+c$

c) $-\frac{1}{3}\left(1+x^2\right)^{3/2}+\frac{1}{5}\left(1+x^2\right)^{5/2}+c$

d) $\frac{1}{3}\left(1+x^2\right)^{3/2}-\frac{1}{5}\left(1+x^2\right)^{5/2}+c$

e) $\frac{1}{3}\left(1+x^2\right)^{3/2}+c$

---

2. $\int \left(x^5\right)\sqrt{1+x^2}\,dx$

a) $\frac{x^6\left(1+x^2\right)^{3/2}}{18}+c$

b) $\frac{1}{3}\left(1+x^2\right)^{3/2}+\frac{2}{7}\left(1+x^2\right)^{7/2}+c$

c) $\frac{1}{7}\left(1+x^2\right)^{7/2}-\frac{2}{5}\left(1+x^2\right)^{5/2}+\frac{1}{3}\left(1+x^2\right)^{3/2}+c$

d) $\frac{2}{7}\left(1+x^2\right)^{7/2}-\frac{4}{5}\left(1+x^2\right)^{5/2}+\frac{2}{3}\left(1+x^2\right)^{3/2}+c$

e) $\frac{1}{7}\left(1+x^2\right)^{7/2}+\frac{2}{5}\left(1+x^2\right)^{5/2}+\frac{1}{3}\left(1+x^2\right)^{3/2}+c$

---

3. $\int \left(\frac{x^3}{9-x^2} \right) dx =$

a) $\frac{x^3}{3} \cdot \frac{\left(9-x^2\right)^{3/2}}{9} + c$

b) $\frac{9}{2} \ln\left|9-x^2\right| + \frac{1}{2}\left(9-x^2\right) + c$

c) $\frac{9}{2}\left(9-x^2\right)^2 - \frac{1}{2}\left(9-x^2\right) + c$

d) $\frac{9}{2}\left(9-x^2\right) + \frac{1}{2}\left(9-x^2\right)^2 + c$

e) $\frac{x^4}{36} - \frac{x^2}{2} + c$

---

4. $\int \left(\frac{x^5}{x^2+5} \right) dx =$

a) $\frac{1}{4}\left(x^2+5\right)^2 - 5\left(x^2+5\right) + \frac{25}{2} \ln\left(x^2+5\right) + c$

b) $\frac{1}{4}\left(x^2+5\right)^2 - 5\left(x^2+5\right) + \frac{25}{2} \tan^{-1}\left(x^2+5\right) + c$

c) $\frac{1}{4}\left(x^2+5\right)^2 + 5\left(x^2+5\right) + \frac{25}{2} \ln\left(x^2+5\right) + c$

d) $\frac{1}{4}\left(x^2+5\right)^2 + 5\left(x^2+5\right) + \frac{25}{2} \tan^{-1}\left(x^2+5\right) + c$

e) none of these

---

5. $\int \left(e^{2x}\sqrt{e^{x}+1} \right) dx$

a) $\frac{2}{5}\left(e^{x}+1\right)^{5/2}-3\left(e^{x}+1\right)^{3/2}+c$

b) $e^{2x}\left(e^{x}+1\right)^{3/2}+c$

c) $\frac{2}{5}e^{5x/2}-5e^{3x/2}+c$

d) $\frac{2}{5}\left(e^{x}+1\right)^{5/2}+3\left(e^{x}+1\right)^{3/2}+c$

e) $\frac{2}{5}\left(e^{x}+1\right)^{5/2}-\frac{2}{3}\left(e^{x}+1\right)^{3/2}+c$

---

6. $\int \left(\frac{x^{3}}{\sqrt{4-x^{2}}} \right) dx =$

a) $\frac{1}{3}\left(4-x^{2}\right)^{3/2}-4\left(4-x^{2}\right)^{1/2}+c$

b) $\frac{2}{3}\left(4-x^{2}\right)^{3/2}-2\left(4-x^{2}\right)^{1/2}+c$

c) $\frac{1}{3}\left(4-x^{2}\right)^{3/2}-4\sin^{-1}\frac{x}{2}+c$

d) $\frac{2}{3}\left(4-x^{2}\right)^{3/2}-2\sin^{-1}\frac{x}{2}+c$

e) $\frac{1}{3}\left(4-x^{2}\right)^{3/2}-\sin^{-1}\frac{x}{2}+c$

---

7. $\int \left(\frac{4-x}{\sqrt{4-x^{2}}} \right) dx =$

a) $4\sin^{-1}\frac{x}{2}+\left(4-x^{2}\right)^{1/2}+c$

b) $2\sin^{-1}\frac{x}{2}+\left(4-x^{2}\right)^{1/2}+c$

c) $4\left(4-x^{2}\right)^{1/2}+c$

d) $\sin^{-1}\frac{x}{2}-\left(4-x^{2}\right)^{1/2}+c$

e) $\frac{2}{3}\left(4-x^{2}\right)^{3/2}+c$

2.5: Powers of Trig Functions: Sine and Cosine

Another instance of Back Substitution involves the Trig Functions and the Pythagorean Identities. As we saw in previous sections, since

$$\frac{d}{dx}\left[\cos x\right]=\left(-\sin x\right) \text{ and } \frac{d}{dx}\left[\sin x\right]=\left(\cos x\right),$$

one of these functions can serve as the du when the other serves as u. But what about when there are higher exponents involved? In general, what about:

$$\int \sin^n x \cos^m x \, dx$$

OBJECTIVE
Use the Integration by Substitution to integrate integrands involving Sine and Cosine.

There are two cases of integration of this kind of integrand, depending on the powers m and n.

Case 1. The easier (and more common case on the AP test) is when either m or n is an odd number. One of whichever function has the odd power will be the du and the rest of those functions can convert to the other trig function by means of the Pythagorean Identities.

Remember:

$$\sin^2\theta + \cos^2\theta = 1$$
$$1-\cos^2\theta = \sin^2\theta$$
$$1-\sin^2\theta = \cos^2\theta$$

Ex 1 $\int \sin^4 x \ \cos^3 x \ dx$

Since $\cos x$ has the odd power, $u=\sin x$, $du=\cos x$, and $\cos^2 x = 1-\sin^2 x = 1-u^2$.

$$\begin{aligned}
\int \sin^4 x \cos^3 x \ dx &= \int \sin^4 x \ \cos^2 x \ \cos x \ dx \\
&= \int \sin^4 x \ \left(1-\sin^2 x\right) \ \cos x \ dx \\
&= \int u^4 \left(1-u^2\right) \ du \\
&= \int \left(u^4 - u^6\right) \ du \\
&= \frac{1}{5}u^5 - \frac{1}{7}u^7 + c
\end{aligned}$$

$$\boxed{= \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + c}$$

Ex 2 $\int \sin^5 x \ \cos^2 x \ dx$

Since $\sin x$ has the odd power, $u=\cos x$, $du=-\sin x$, and $\sin^2 x = 1-\cos^2 x = 1-u^2$.

$$\begin{aligned}
\int \sin^5 x \cos^2 x \ dx &= -\int \sin^4 x \ \cos^2 x \ \left(-\sin x\right) dx \\
&= -\int \left(\sin^2 x\right)^2 \cos^2 x \left(-\sin x\right) dx \\
&= -\int \left(1-\cos^2 x\right)^2 \cos^2 x \left(-\sin x\right) dx \\
&= -\int \left(1-u^2\right)^2 u^2 du \\
&= -\int \ \left(1-2u^2+u^4\right) u^2 du \\
&= -\int \ \left(u^2 - 2u^4 + u^6\right) du \\
&= -\left(\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7\right) + c
\end{aligned}$$

$$= -\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + c$$

If both powers are odd, either function can serve as u. Because of the negative sign, it is usually deemed easier to choose u = sin x.

Ex 3 $\int \tan x \; dx$

At first, this does not appear to be a sin/cos integral, but a basic substitution reveals it is:

$$\begin{aligned}\int \tan x \; dx &= \int \frac{\sin x}{\cos x} dx \\ &= -\int \frac{1}{u} du \\ &= -\ln|\cos x| + c \\ &= \ln|\cos x|^{-1} + c \\ &= \ln|\sec x| + c\end{aligned}$$

This gives us two more integral rules:

$$\int \tan u \; du = \ln|\sec u| + c \qquad \int \cot u \; du = \ln|\sin u| + c$$

The last two trig integrals, $\int \sec u \; du$ and $\int \csc u \; du$, seem to be of the same kind as these two, but proving these by sin and cos is difficult. We will give the rules here and prove them in later.

$$\int \sec u \; du = \ln|\sec u + \tan u| + c \qquad \int \csc u \; du = \ln|\csc u - \cot u| + c$$

Case 2. The more difficult situation is when both powers are even. In this case, variations on the half angle argument rules come into play.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \text{ and } \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

Ex 4 $\int \cos^2 x \, dx$

$$\begin{aligned}
\int \cos^2 x \, dx &= \int \left(\frac{1}{2}(1 + \cos 2x) \right) dx \\
&= \frac{1}{2} \cdot \frac{1}{2} \int (1 + \cos 2x) \; 2dx \\
&= \frac{1}{4} \int (1 + \cos u) du \\
&= \frac{1}{4} u + \frac{1}{4} \sin u + c \\
&= \frac{1}{4}(2x) + \frac{1}{4} \sin(2x) + c \\
&\boxed{= \frac{1}{2} x + \frac{1}{4} \sin 2x + c}
\end{aligned}$$

This example leads to two more integral equations that are helpful to know:

$$\int \cos^2 u \, du = \frac{1}{2} u + \frac{1}{4} \sin 2u + c \qquad \int \sin^2 u \, du = \frac{1}{2} u - \frac{1}{4} \sin 2u + c$$

Ex 5 $\int \sin^4 x \cos^2 x \; dx$

$$= \int \left(\frac{1}{2}(1-\cos 2x) \right)^2 \left(\frac{1}{2}(1+\cos 2x) \right) \; dx$$

$$= \frac{1}{8} \int \left(1 - 2\cos 2x + \cos^2 2x\right)(1+\cos 2x) \; dx$$

$$= \frac{1}{8} \int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) dx$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2dx \end{aligned}}$$

$$= \frac{1}{8} \cdot \frac{1}{2} \int \left(1 - \cos 2x - \cos^2 2x + \cos^3 2x\right) 2dx$$

$$= \frac{1}{16} \int \left(1 - \cos u - \cos^2 u + \cos^3 u\right) \; du$$

$$= \frac{1}{16} \int (1) du - \frac{1}{16} \int (\cos u) du - \frac{1}{16} \int \left(\cos^2 u\right) du + \frac{1}{16} \int \left(\cos^3 u\right) du$$

$$= \frac{1}{16} u - \frac{1}{16} \sin u - \frac{1}{16} \left(\frac{1}{2} u - \frac{1}{4} \sin 2u \right) + \frac{1}{16} \int \left(\cos^2 u\right) \; \cos u \, du + c$$

$$\boxed{\begin{aligned} v &= \sin u \\ dv &= \cos u dx \end{aligned}}$$

$$= \frac{1}{16} u - \frac{1}{16} \sin \; u - \frac{1}{16} \left(\frac{1}{2} u - \frac{1}{4} \sin \; 2u \right) + \frac{1}{16} \int \left(1 - \sin^2 \; u\right) \; \cos \; u \; du + c$$

$$= \frac{1}{16} u - \frac{1}{16} \sin u - \frac{1}{32} u + \frac{1}{64} \sin 2u + \frac{1}{16} \int \left(1 - v^2\right) dv + c$$

$$= \frac{1}{16} u - \frac{1}{16} \sin u - \frac{1}{32} u + \frac{1}{64} \sin 2u + \frac{1}{16} \left(v - \frac{1}{3} v^3 \right) + c$$

$$= \frac{1}{16}(2x) - \frac{1}{16}\sin(2x) - \frac{1}{32}(2x) + \frac{1}{64}\sin 2(2x) + \frac{1}{16}\left(\sin 2x - \frac{1}{3}\sin^3 2x \right) + c$$

$$= \frac{1}{8} x - \frac{1}{16} \sin 2x - \frac{1}{16} x + \frac{1}{64} \sin 4x + \frac{1}{16} \sin 2x - \frac{1}{48} \sin^3 2x + c$$

$$\boxed{= \frac{1}{16} x + \frac{1}{64} \sin 4x - \frac{1}{48} \sin^3 2x + c}$$

2.5 Free Response Homework

Perform the Anti-differentiation.

1. $\int \sin^3 x \ \cos^2 x \ dx$

2. $\int \sin^4 x \cos^5 x \ dx$

3. $\int \sin^2 x \ \cos^7 x \ dx$

4. $\int \sin^5 x \ \cos^6 x \ dx$

5. $\int \sin x \ \cos^5 x \ dx$

6. $\int \sin^5 x \ \cos^5 x \ dx$

7. $\int \sin^2 x \ \cos^2 x \ dx$

8. $\int \sin^2 x \ \cos^4 x \ dx$

2.5 Multiple Choice Homework

1. For $\int \sin^3 3x \cos^5 3x \ dx$, the correct u-substitution is

a) $u = \sin x$

b) $u = \cos x$

c) either $u = \sin x$ or $u = \cos x$

d) neither $u = \sin x$ nor $u = \sin x$

e) none of these

---

2. For $\int \sin^3 5x \cos^2 5x \ dx$, the correct u-substitution is

a) $u = \sin x$

b) $u = \cos x$

c) either $u = \sin x$ or $u = \cos x$

d) neither $u = \sin x$ nor $u = \sin x$

e) none of these

---

3. For $\int \sin^4 4x \cos^5 4x \, dx$, the correct u-substitution is

a) $u = \sin x$

b) $u = \cos x$

c) either $u = \sin x$ or $u = \cos x$

d) neither $u = \sin x$ nor $u = \sin x$

e) none of these

---

4. For $\int \sin^2 x \cos^4 x \, dx$, the correct u-substitution is

a) $u = \sin x$

b) $u = \cos x$

c) either $u = \sin x$ or $u = \cos x$

d) neither $u = \sin x$ nor $u = \sin x$

e) none of these

---

5. $\int \cos^2 2x \, dx =$

a) $\sin 4x + c$

b) $\frac{1}{2}x + \frac{1}{8}\sin 4x + c$

c) $\frac{1}{2}x - \frac{1}{8}\sin 4x + c$

d) $x + \frac{1}{4}\sin 4x + c$

e) $x + \frac{1}{8}\cos 4x + c$

---

6. $\int \cos^2 \frac{1}{2}x \; dx =$

a) $\sin 4x + c$

b) $\frac{1}{2}x + \frac{1}{4}\sin x + c$

c) $\frac{1}{2}x - \frac{1}{4}\sin x + c$

d) $\frac{1}{4}x + \frac{1}{2}\sin x + c$

e) $\frac{1}{4}x + \frac{1}{2}\cos x + c$

---

7. Where is the mistake in this process:

$$\int \sin^3 2x \cos^4 2x \; dx =$$

Step 1: $-\frac{1}{2}\int \sin^2 2x \cos^4 2x \; (-\sin 2x \, 2dx) =$

Step 2: $-\frac{1}{2}\int \left(1-u^2\right) u^4 du =$

Step 3: $-\frac{1}{2}\int \left(u^4 - u^6\right) du =$

Step 4: $-\frac{1}{2}\left(\frac{1}{5}u^5 - \frac{1}{7}u^7 + c\right) =$

Step 5: $-\frac{1}{10}\sin^5 4x + \frac{1}{14}\sin^7 4x + c$

a) Step 1
b) Step 2
c) Step 3
d) Step 4
e) There is no mistake.

2.6: Powers of Trig Functions: Secant and Tangent

As with sin and cos, Tan and Sec work together in a Pythagorean Identity and Csc and Cot work together in a Pythagorean Identity. So, we can consider integrals of these forms

$$\int \sec^n x \ \tan^m x \ dx \text{ or } \int \csc^n x \cot^m x \ dx$$

to be cases of Integration by Substitution.

Remember:

$$\tan^2\theta + 1 = \sec^2\theta \qquad 1+\cot^2\theta = \csc^2\theta$$
$$\sec^2\theta - 1 = \tan^2\theta \qquad \csc^2\theta - 1 = \cot^2\theta$$
$$\sec^2\theta - \tan^2\theta = 1 \qquad \csc^2\theta - \cot^2\theta = 1$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \ \cot u)\frac{du}{dx}$$
$$\frac{d}{dx}[\cot u] = (-\csc^2 u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \ \tan u)\frac{du}{dx}$$

OBJECTIVE Use the Integration by Substitution to integrate integrands involving Secant and Tangent or Cosecant and Cotangent.

There are three cases of integration involving these kinds of integrand, depending on the powers m and n.

Case 1. When the Secant's (or Cosecant's) power is even, then

$$u = \tan x$$
$$du = \sec^2 x \ dx$$
$$and$$
$$\tan^2 x + 1 = \sec^2 x \ \left(or \ u^2 + 1 = \sec^2 x\right)$$

Ex 1 $\int \sec^4 x \ \tan^2 x \ dx$

$$\begin{aligned}\int \sec^4 x \tan^2 x \ dx &= \int \sec^2 x \tan^2 x \sec^2 x \ dx \\ &= \int \left(1+\tan^2 x\right)\tan^2 x \sec^2 x \ dx \\ &= \int \left(1+u^2\right)u^2 du \\ &= \int \left(u^2+u^4\right) \ du \\ &= \frac{1}{3}u^3+\frac{1}{5}u^5+c\end{aligned}$$

$$\boxed{=\frac{1}{3}\tan^3 x+\frac{1}{5}\tan^5 x+c}$$

Case 2. When the Tangent's (or Cotangent's) power is odd, then

$$u = \sec x$$
$$du = \left(\sec x \tan x\right)dx$$
$$and$$
$$\tan^2 x = \sec^2 x - 1 \ \left(or \ \ u^2 - 1 \ = \tan^2 x\right)$$

Ex 2 $\int \csc^3 x \ \cot^3 x \ dx$

Because cot has an odd power, it will be part of *du* and

$$u = \csc x$$
$$du = \left(-\csc x \cot x\right)dx$$
$$\cot^2 x = u^2 - 1$$

$$\int \csc^3 x \cot^3 x \; dx = -\int \csc^2 x \;\; \cot^2 x \;\; (-\csc x \cot x) dx$$
$$= -\int u^2 \left(u^2 - 1\right) du$$
$$= -\int \left(u^4 - u^2\right) du$$
$$= -\frac{1}{5}u^5 + \frac{1}{3}u^3 + c$$

$$\boxed{= -\frac{1}{5}\csc^5 x + \frac{1}{3}\csc^3 x + c}$$

If both cases are present (that is, the tangent power is odd and the secant power is even), either function can serve as u.

Case 3. When the Tangent's (or Cotangent's) power is even AND the Secant's (or Cosecant's) power is odd, then

> **The problem is not an Integration-by-Substitution problem. It is an Integration-by-Parts problem. We will need to wait until Chapter 8 to do them.**

Summary of Integrals of Powers of Trig Functions

I. $\int \sin^n x \cos^m x \, dx$

a. The odd power determines *du*. The other function is *u*.

b. If both powers are even, use the half-angle formulas and simplify.

II. $\int \sec^n x \tan^m x \, dx$ or $\int \csc^n x \cot^m x \, dx$

a. If both powers are even, $u = \tan x \; and \; du = \sec^2 x \, dx$

b. If both powers are odd, $u = \sec x \; and \; du = \sec x \tan x \, dx$

c. If *n* is even and *m* is odd, either a. or b. will work

d. If *n* is odd and *m* is even, neither a. nor b. will work

III. Any other mix of trig functions

a. Convert all to $\sin x$ and $\cos x$ and use I. above

2.6 Free Response Homework

Perform the Anti-differentiation.

1. $\int \sec^2 x \tan^5 x \; dx$

2. $\int \sec^6 x \tan^4 x \; dx$

3. $\int \sec^5 x \tan^7 x \; dx$

4. $\int \sec^2 x \tan^6 x \; dx$

5. $\int \sec^6 x \tan^3 x \; dx$

6. $\int \csc^2 x \cot^5 x \; dx$

7. $\int \csc^3 x \cot^5 x \; dx$

8. $\int \csc^7 x \cot^5 x \; dx$

9. $\int \csc^4 x \cot^4 x \; dx$

10. $\int \csc^4 x \cot x \; dx$

2.6 Multiple Choice Homework

1. For $\int \csc^3 x \cot^5 x \; dx$, the correct u-substitution is

a) $u = \csc x$ b) $u = \cot x$

c) either $u = \csc x$ or $u = \cot x$ d) neither $u = \csc x$ nor $u = \cot x$

---

2. For $\int \csc^4 x \cot^4 x \, dx$, the correct u-substitution is

a) $u = \csc x$ b) $u = \cot x$

c) either $u = \csc x$ or $u = \cot x$ d) neither $u = \csc x$ nor $u = \cot x$

---

3. For $\int \sec^4 x \tan^5 x \, dx$, the correct u-substitution is

a) $u = \sec x$ b) $u = \tan x$

c) either $u = \sec x$ or $u = \tan x$ d) neither $u = \sec x$ nor $u = \tan x$

---

4. For $\int \sec^5 x \tan^4 x \, dx$, the correct u-substitution is

a) $u = \sec x$ b) $u = \tan x$

c) either $u = \sec x$ or $u = \tan x$ d) neither $u = \sec x$ nor $u = \tan x$

---

5. Which of the following statements are true?

I. $\int (\sec x) \, dx = \ln|\sec x + \tan x| + c$ II. $\int \tan x \, dx = \sec^2 x + c$

III. $\int x^2 \cot x^3 \, dx = \frac{1}{3} \ln\left|\sin x^3\right| + c$

a) I only b) II only c) III only

d) I and II only e) I and III only

---

6. Which of the following statements are **false**?

I. $\int \left(x^5 \sec x^6\right) dx = \frac{1}{6}\ln\left|\sec x^6 + \tan x^6\right| + c$

II. $\int \tan x \, dx = \sec^2 x + c$

III. $\int \csc x \, dx = \ln\left|\csc x - \cot x\right| + c$

a) I only b) II only c) III only

d) I and II only e) I and III only

---

7. State the step that has the first mistake in this process:

$$\int \sec^4 2x \tan^3 2x \; dx =$$

Step 1: $\frac{1}{2}\int \sec^2 2x \tan^3 2x \; \sec^2 2x \, 2dx =$

Step 2: $\frac{1}{2}\int \left(1-u^2\right)u^3 du =$

Step 3: $\frac{1}{2}\int \left(u^3 - u^5\right) du =$

Step 4: $\frac{1}{2}\left(\frac{1}{4}u^4 - \frac{1}{6}u^6 + c\right) =$

Step 5: $\frac{1}{8}\tan^4 2x - \frac{1}{12}\tan^6 2x + c$

a) Step 1 b) Step 2 c) Step 3 d) Step 4

e) There is no mistake.

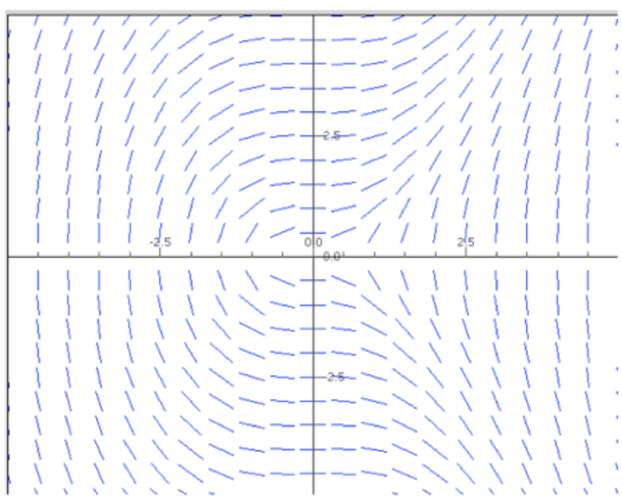
2.7 Intro to AP: Slope Fields

This section is very much an AP-driven section. The underlying philosophy is that math should be understood and explained algebraically, graphically, numerically and verbally. The topic of differential equations fits nicely into this paradigm in that the visual is a graphical representation and the connection between the equation and the slopes is a numerical process.

Vocabulary:

Slope field – Given any function *f*, a slope field is drawn by taking evenly spaced points on the Cartesian coordinate system (usually points having integer coordinates) and, at each point, drawing a small line with the slope of the function.

Here is an example of a slope field:



The line segments represent the slopes of the lines tangent to the solution curve at the specific points where each is drawn.

Objectives:

Given a differential equation, sketch its slope field.
Given a slope field, sketch a particular solution curve.
Given a slope field, determine the family of functions to which the solution curves belong.
Given a slope field, determine the differential equation that it represents.

There are four ways that the AP Exam usually approaches Slope Fields:

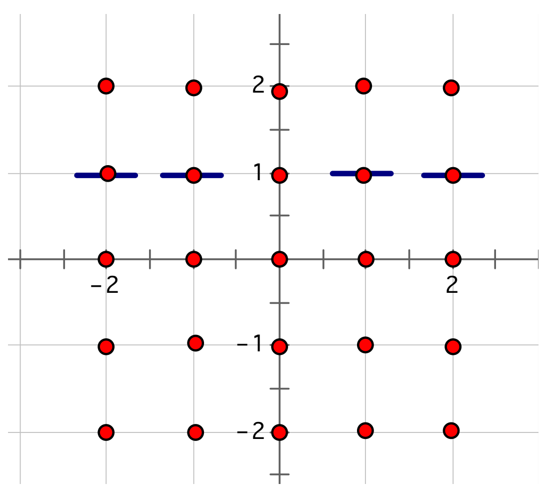
1. Draw a Slope Field (free response)
2. Sketch the solution to a Slope Field (free response)
3. Identify the solution equation to a Slope Field (multiple choice)
4. Identify the differential equation for a Slope Field (multiple choice)

Two of these questions (#2 and #3) are graphically oriented and two (#1 and #4) are numerically based.

1. Slope Fields Numerically (FRQs)

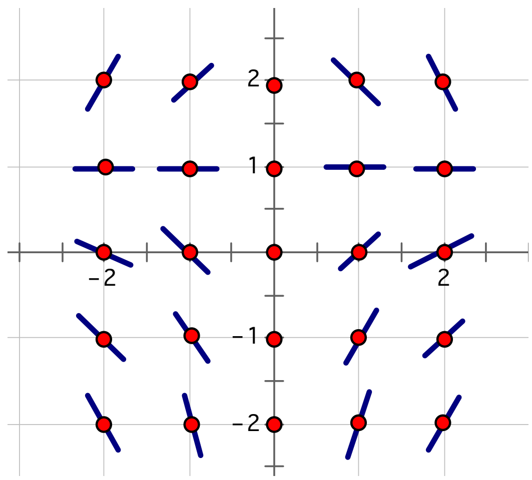
Ex 1 Sketch the slope field for $\frac{dy}{dx}=\frac{1-y}{x}$ at the points indicated.

Note that, wherever $y=1$, $\frac{dy}{dx}=0$. So, the segments at $y=1$ will be horizontal.
Also, where $x=0$, $\frac{dy}{dx}=\text{dne}$. So, there are not segments on the x-axis.



We can then plug the numerical values of the points into $\frac{dy}{dx}=\frac{1-y}{x}$ to determine the slant of the line segments.

$$(-2,2)\to\frac{dy}{dx}=\frac{1}{2} \qquad (2,2)\to\frac{dy}{dx}=-\frac{1}{2}$$
$$(-2,0)\to\frac{dy}{dx}=-\frac{1}{2} \qquad (2,0)\to\frac{dy}{dx}=\frac{1}{2}$$
$$(-2,-2)\to\frac{dy}{dx}=-\frac{3}{2} \qquad (2,-2)\to\frac{dy}{dx}=\frac{3}{2}$$
$$etc.$$

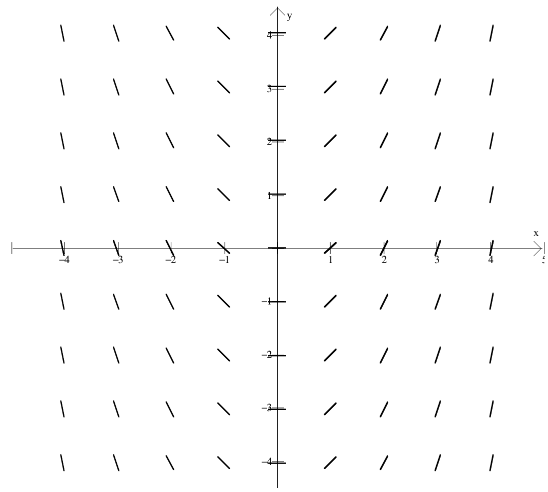


<u>Steps to Sketching a Slope Field:</u>

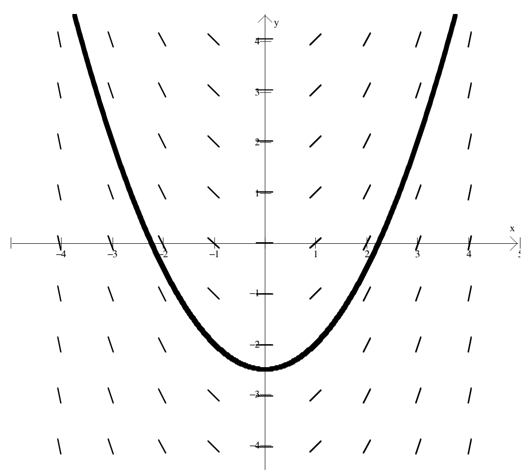
1. Determine the grid of points for which you need to sketch (many times the points are given).
2. Pick your first point. Note its x and y coordinate. Plug these numbers into the differential equation. This is the slope at that point.
3. Find that point on the graph. Make a little line (or dash) at that point whose slope represents the slope that you found in Step 2.
4. Repeat this process for all the points needed.

2. Slope Fields Graphically (FRQs):

Ex 2 Given the slope field of $\frac{dy}{dx} = x$ below, sketch the particular solution given the initial condition of $(3, 2)$.

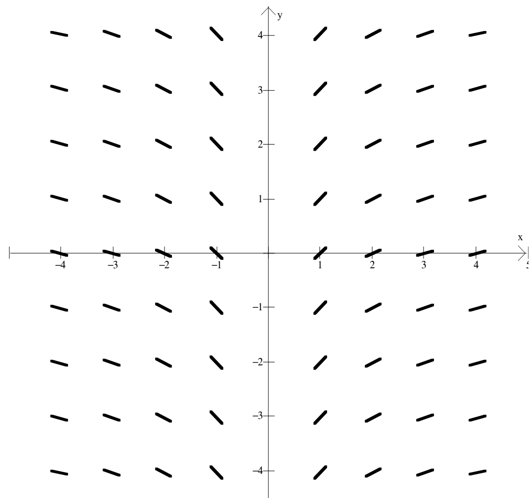


Starting at $(3, 2)$ and following the slope segments, the graph appears thus:

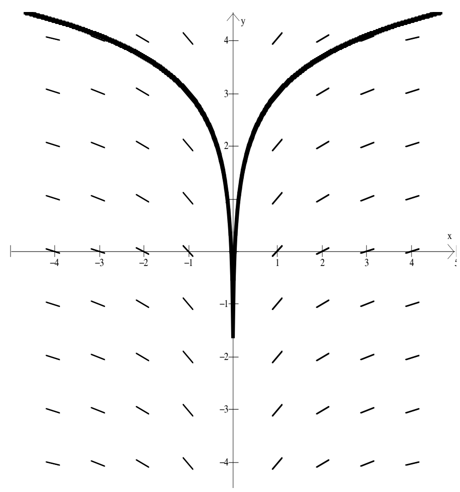


Ex 3 Given the slope field for $x\frac{dy}{dx} = 1$ below, sketch the particular solution given the initial condition of $(1, 3)$.

.



Starting at $(1, 3)$ and following the slope segments, the graph appears thus:



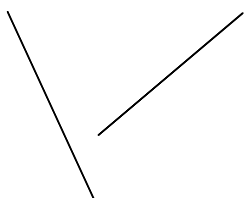
Note that there appears to be a vertical asymptote at $y = 0$.

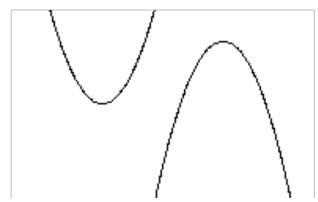
3. Slope Fields Graphically (MCQs)

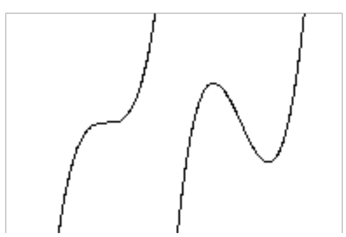
For these problems, one would sketch a solution and decide from among the options based on what was learned about families of functions in PreCalulus.

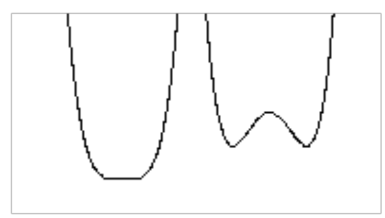
I. *Polynomial*s

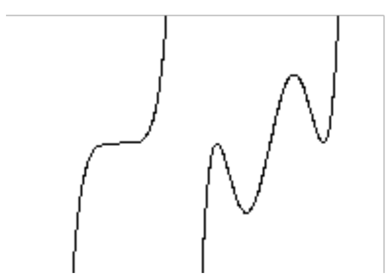
- Defn: "An expression containing no other operations than addition, subtraction, and multiplication performed on the variable."
- Means: any equation of the form $y = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where n is a non-negative integer.
- **Most Important Traits: Zeros (*x*-intercepts) and Extreme Points.**





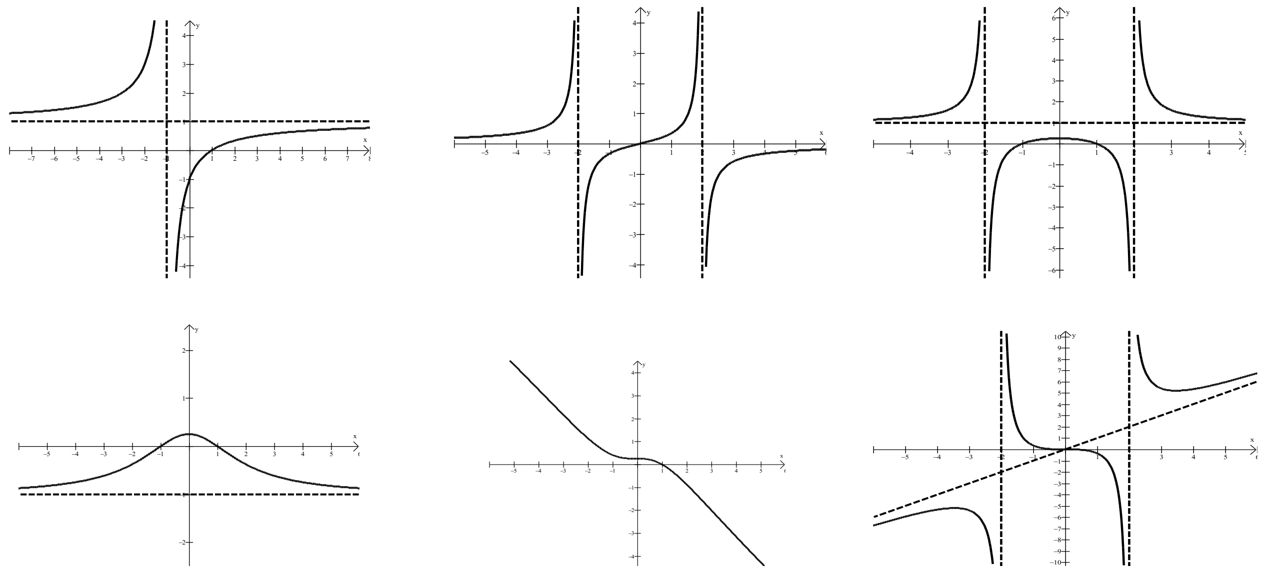






II. *Rationals*

- Defn: "An expression that can be written as the ratio of one polynomial to another."
- Means: an equation with an x in the denominator.
- **Most Important Traits: Zeros vs. VAs vs. POEs and End Behavior.**

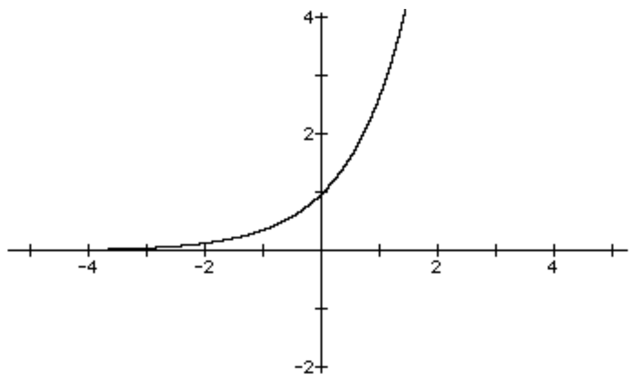


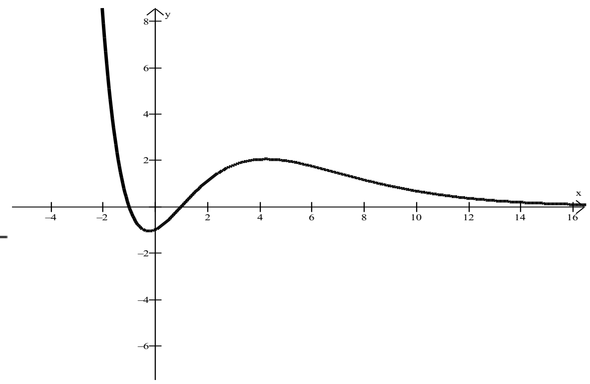
III. *Radicals* (*Irrationals*)

- Defn: "An expression whose general equation contains a root of a variable and possibly addition, subtraction, multiplication, and/or division."
- Means: An equation with an x in a radical.
- **Most Important Traits: Domain and Extreme Points.**

IV. *Exponentials*

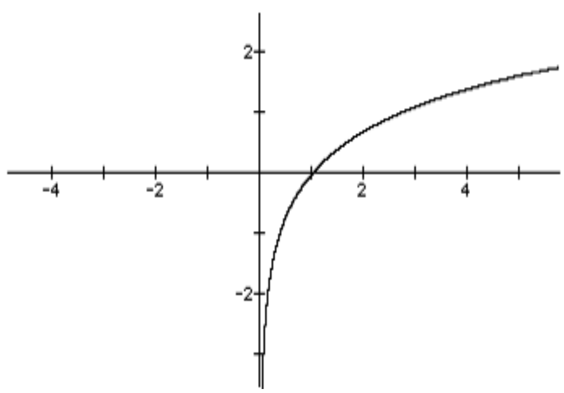
- Defn: "A function whose general equation is of the form $y = a \cdot b^x$."
- Means: there is an x in the exponent.
- **Most Important Traits: Extreme Points and End Behavior.**

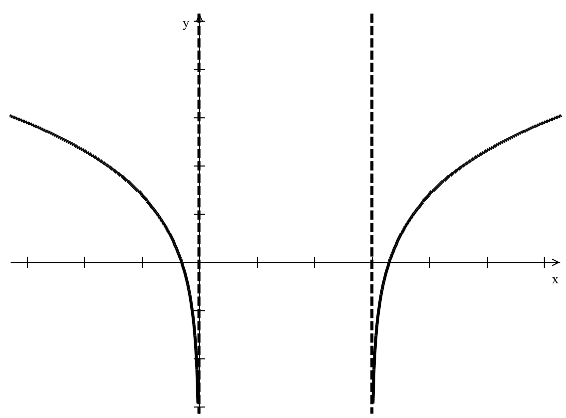




V. *Logarithmic Functions*

- Defn: "The inverse of an exponential function."
- Means: there is a Log or Ln in the equation.
- Most Important Traits: Domain, VAs, and Extreme Points.



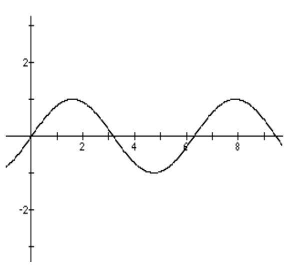


VI. *Trigonometric Functions*

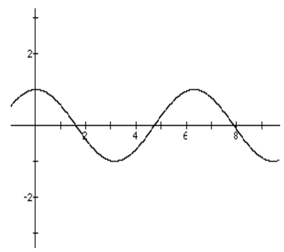
Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."

Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

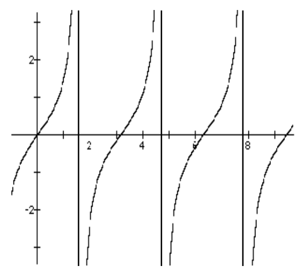
- **Most Important Traits: VAs, Axis Points, and Extreme Points.**



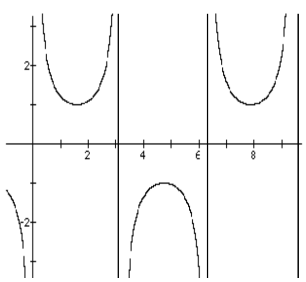
$y = \sin x$



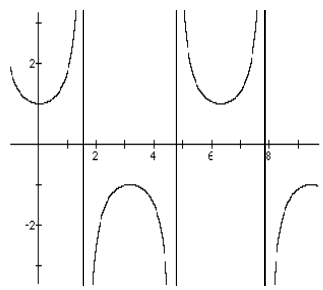
$y = \cos x$



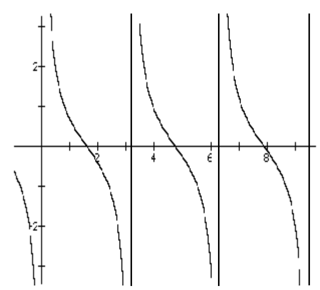
$y = \tan x$



$y = \csc x$



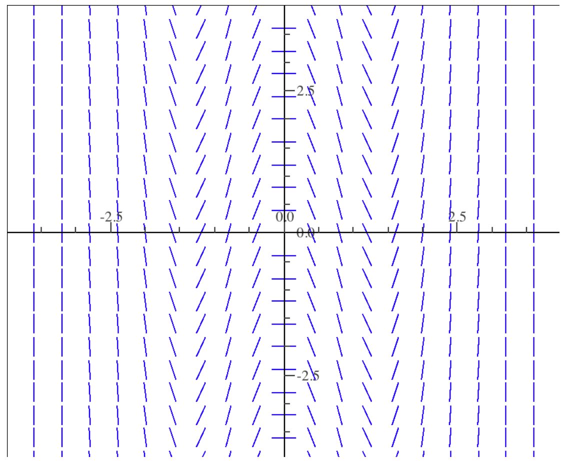
$y = \sec x$



$y = \cot x$

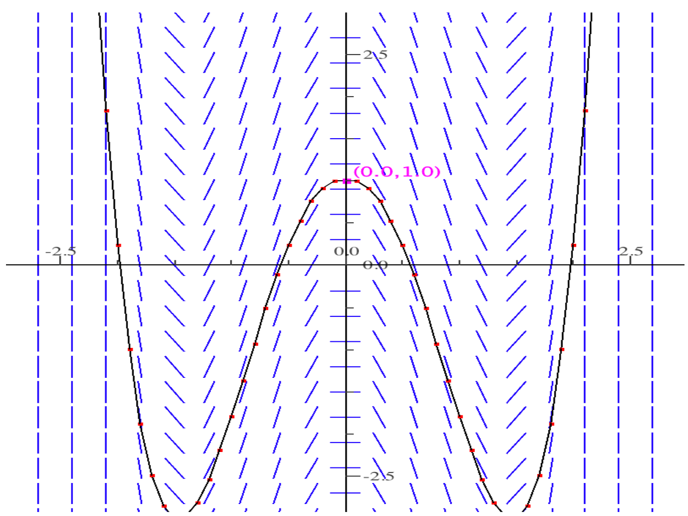
Finding the solution equation entails looking at the pattern of the slopes and matching it against the graphs we know.

Ex 4 Which of the following equations might be the solution to the slope field shown in the figure below?



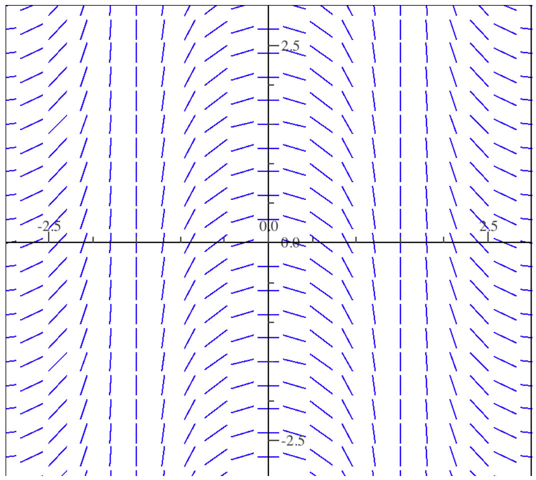
a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:



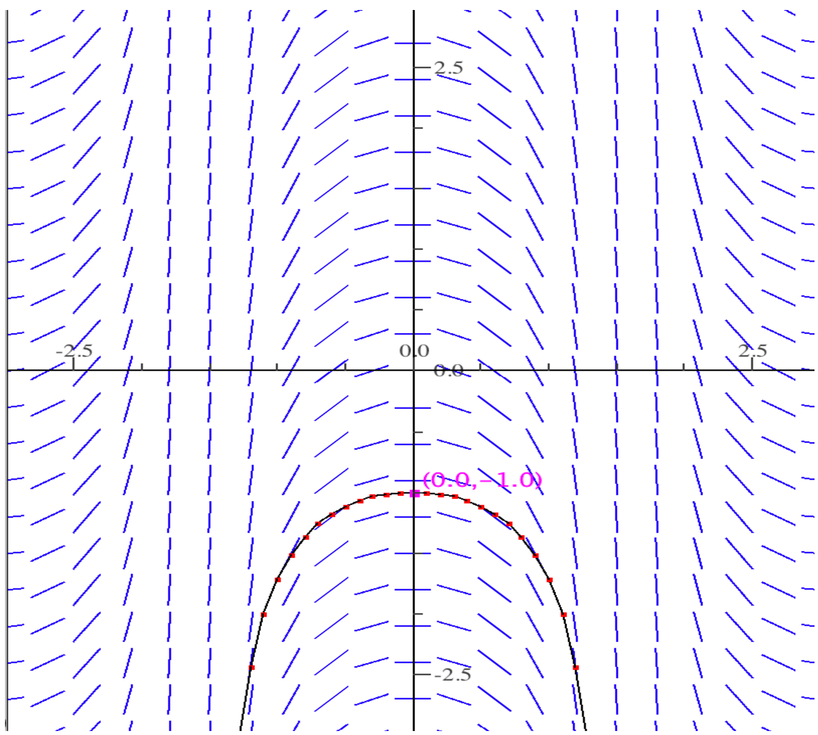
This is a quartic function, so the answer is a) $y = x^4 - 4x^2$

Ex 5   Which of the following equations might be the solution to the slope field shown in the figure below?



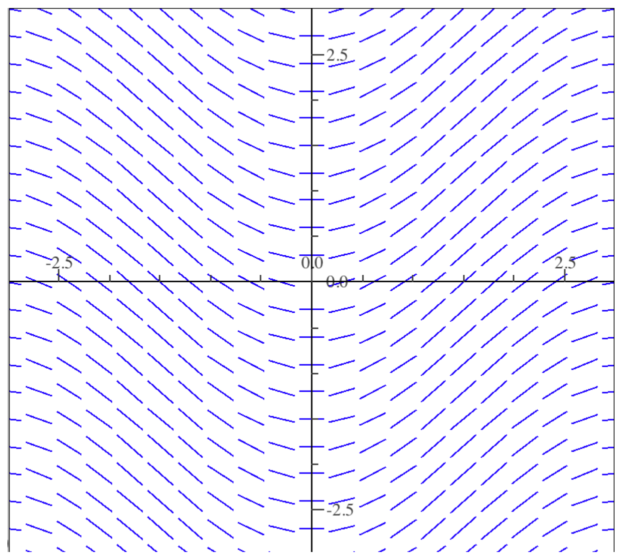
a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:



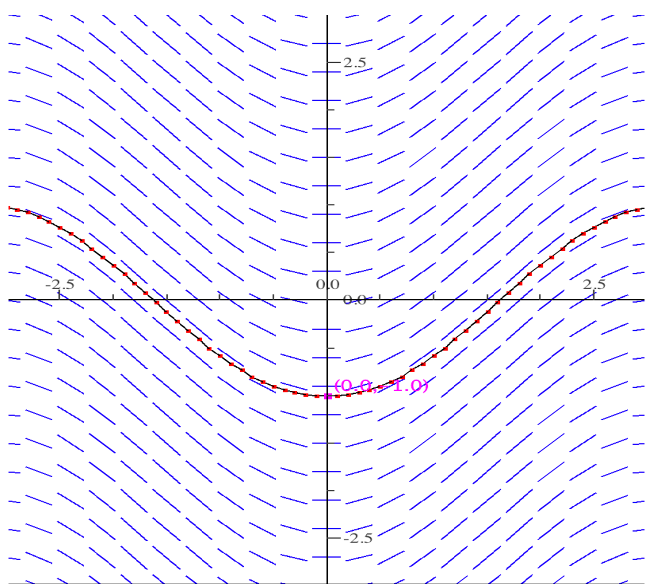
This is a secant curve, so the answer is d) $y = -\sec x$

Ex 6 Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:



This is a negative cosine wave, so the answer is c) $y = -\cos x$

4. Slope Fields Numerically (MCQs)

Let's summarize what we know about slopes of lines in terms of numbers:

1. Horizontal lines have $\frac{dy}{dx} = 0$

2. Vertical lines have $\frac{dy}{dx} = dne$

3. Lines with positive slopes go up from left to right

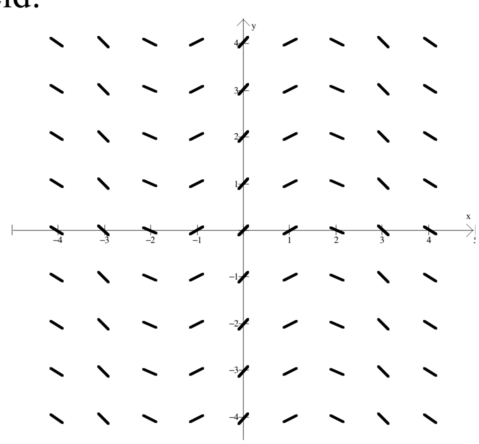
4. Lines with negative slopes go down from left to right

Two other facts are apparent from viewing a slope field and its differential equation:

5. If all Dashes in each column are // to each other, then $\frac{dy}{dx}$ has no *y*.

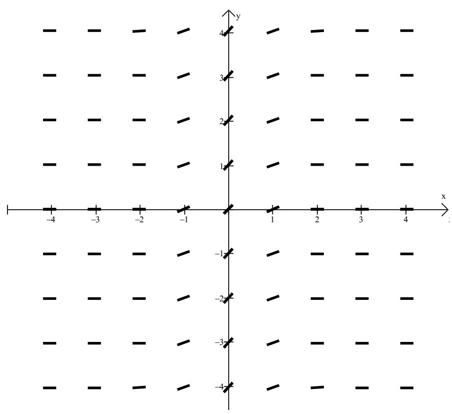
6. If all Dashes in each row are // to each other, then $\frac{dy}{dx}$ has no *x*.

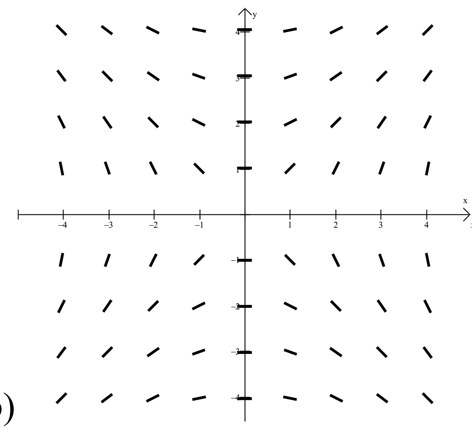
So, consider this slope field:

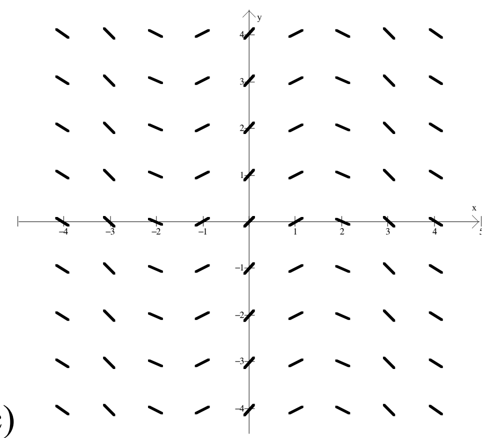


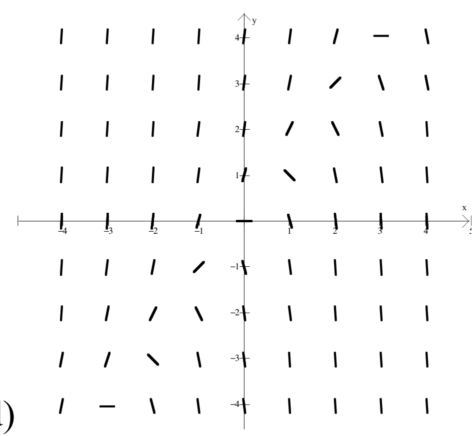
The differential equations that yields this would not have a *y* in the equation because of the segments being parallel in. each column.

Ex 7 Which of the following slope fields matches $\frac{dy}{dx} = 3y - 4x$.

a) 

b) 

c) 

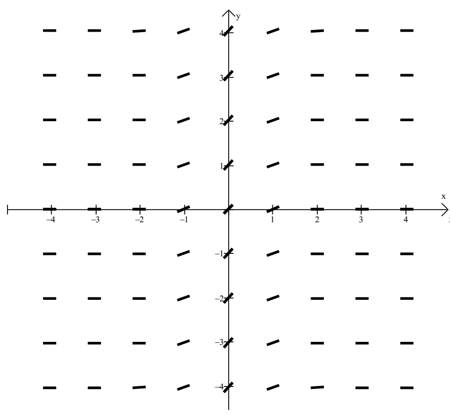
d) 

a) and c) have all slopes in each column parallel to one another, therefore, there is no y in those equations. Neither can be the $\frac{dy}{dx} = 3y - 4x$.

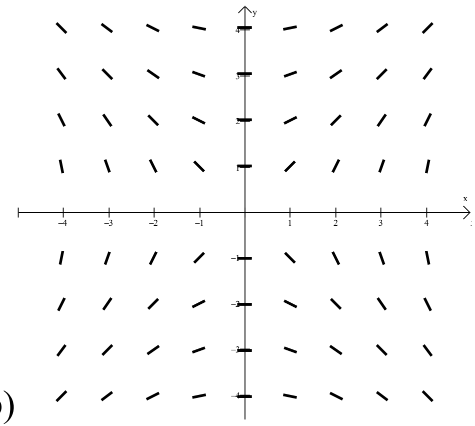
b) appears to have horizontal slopes at $x = 0$, but $\frac{dy}{dx} = 3y - 4x$ does not always equal 0 at $x = 0$. b) cannot be the slope field for $\frac{dy}{dx} = 3y - 4x$.

Therefore, by process of elimination, the correct answer is d).

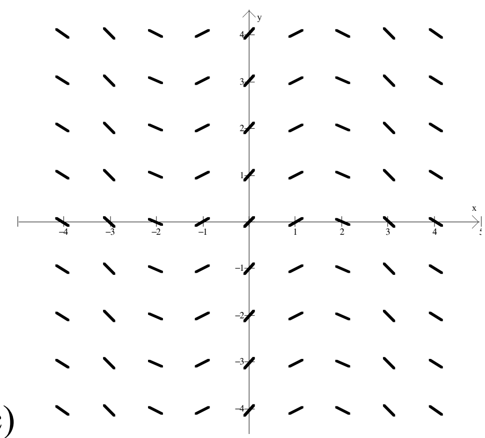
Ex 8 Which of the following slope fields matches $\frac{dy}{dx} = e^{-x^2}$.



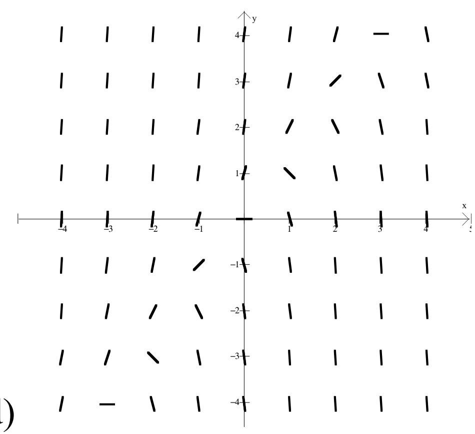
a)



b)



c)



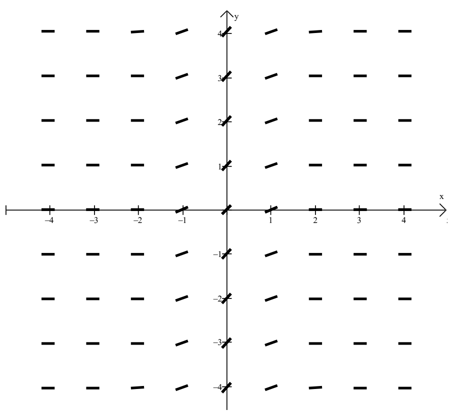
d)

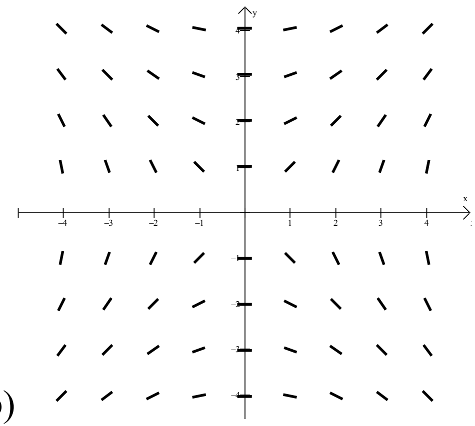
$\frac{dy}{dx} = e^{-x^2}$ has no y in the equation, therefore, the segments in each column must be parallel to each other. The answer must be either a) or c).

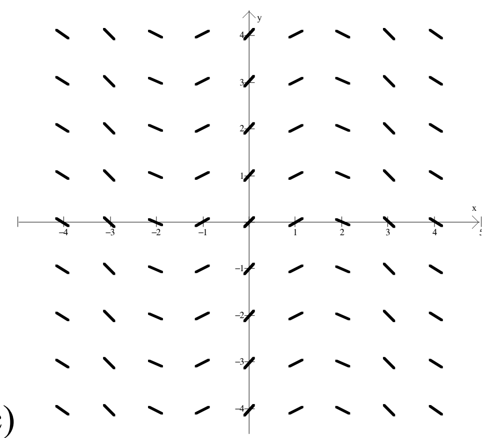
In c), the slopes at $x = 4$ are negative, but $\frac{dy}{dx} = e^{-4^2}$ is positive.

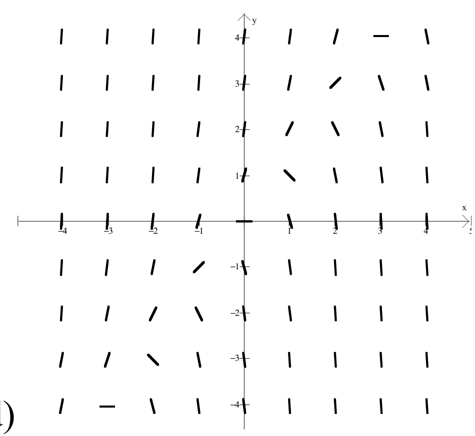
Again, by process of elimination, the correct answer is a).

Ex 9 Which of the following slope fields matches $\frac{dy}{dx} = \frac{x}{y}$?

a) 

b) 

c) 

d) 

$\frac{dy}{dx} = \frac{x}{y}$ has both *x* and *y* in it, therefore there cannot be parallel slopes in columns or rows. A) and c) must be wrong.

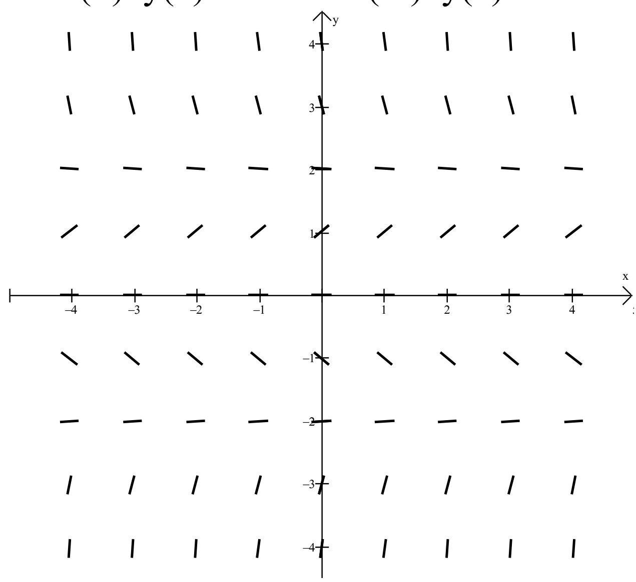
$\frac{dy}{dx} = \frac{x}{y} = 0$ when $x = 0$, so the answer must be B

2.7 Free Response Homework

1. A slope field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$ is shown.

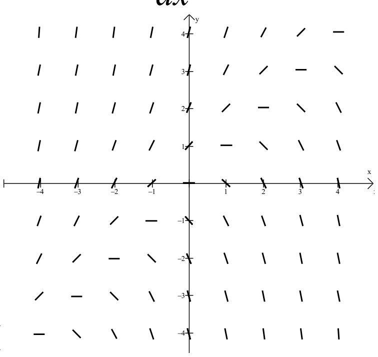
(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) y(0) = 1 (ii) y(0) = -1 (iii) y(0) = -3 (iv) y(0) = 3

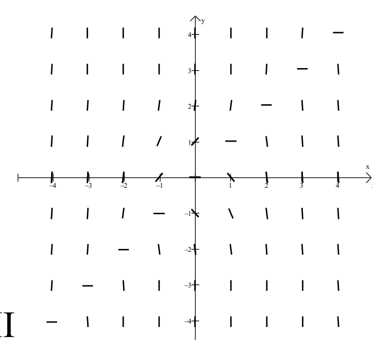


Match the differential equation with its slope field (labeled I-IV). Give reasons for your answer.

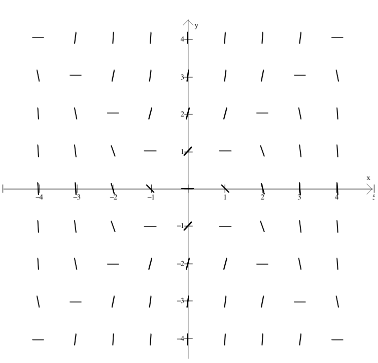
2. $\frac{dy}{dx} = y - 1$ 3. $\frac{dy}{dx} = y - x$ 4. $\frac{dy}{dx} = y^2 - x^2$ 5. $\frac{dy}{dx} = y^3 - x^3$



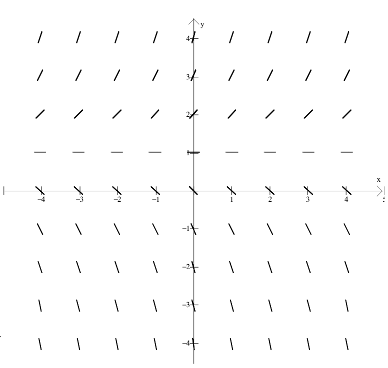
I



II



III



IV

6. Use the slope field labeled I (for exercises 2-5) to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0) = 1$ (b) $y(0) = 0$ (c) $y(0) = -1$

7. Sketch a slope field for the differential equation. Then use it to sketch three solution curves.

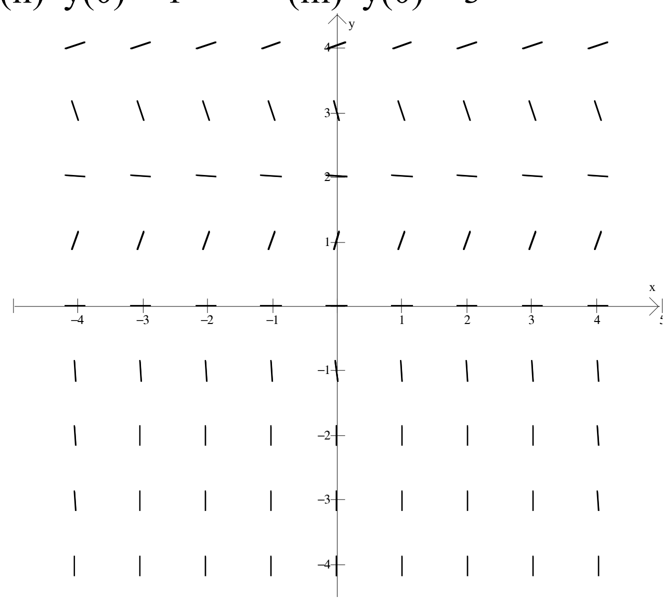
$y' = 1 + y$

8. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.

$y' = y - 2x$; $(1,0)$

9(a). A slope field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.

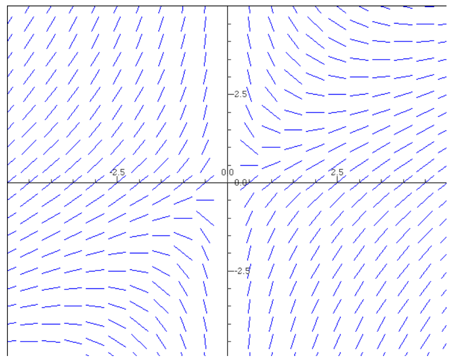
(i) $y(0) = -0.3$ (ii) $y(0) = 1$ (iii) $y(0) = 3$ (iv) $y(0) = 4.3$



9(b). If the initial condition is $y(0) = c$, for what values of c is $\lim_{t\to\infty} y(t)$ finite?

2.7 Multiple Choice Homework

1. Which of the following differential equations corresponds to the slope field shown in the figure below?

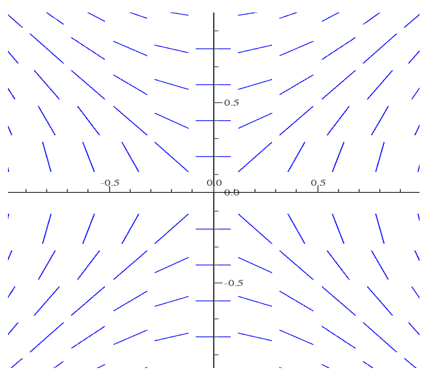


a) $\frac{dy}{dx} = -\frac{y^2}{x}$ b) $\frac{dy}{dx} = 1 - \frac{y}{x}$ c) $\frac{dy}{dx} = -y^3$

d) $\frac{dy}{dx} = x - \frac{1}{2}x^3$ e) $\frac{dy}{dx} = x + y$

---

2. Which of the following differential equations corresponds to the slope field shown in the figure below?

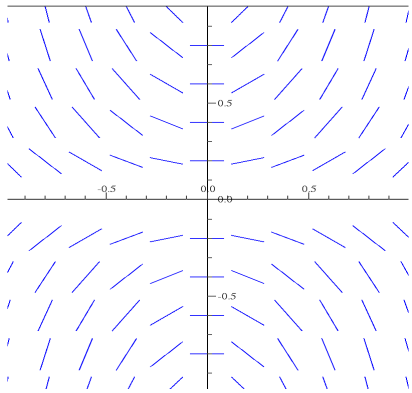


a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $\frac{dy}{dx} = 5xy$ c) $\frac{dy}{dx} = \frac{1}{10}xy$

d) $\frac{dy}{dx} = \frac{y}{x}$ e) $\frac{dy}{dx} = \frac{x}{y}$

---

3. Which of the following differential equations corresponds to the slope field shown in the figure below?

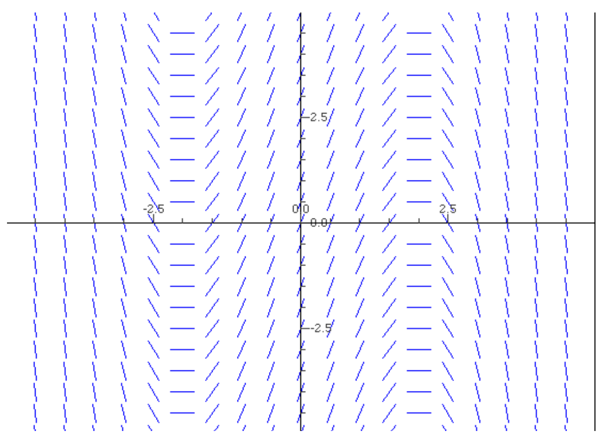


a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $\frac{dy}{dx} = 5xy$ c) $\frac{dy}{dx} = \frac{1}{10}xy$

d) $\frac{dy}{dx} = \frac{y}{x}$ e) $\frac{dy}{dx} = \frac{x}{y}$

---

4. Which of the following equations might be the solution to the slope field shown in the figure below?

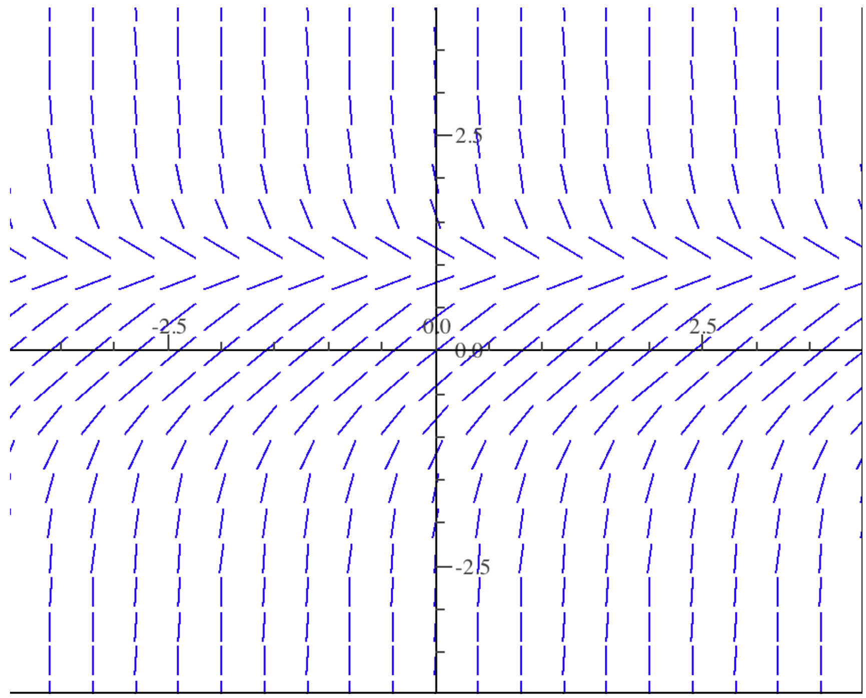


a) $y = 12x - x^3$ b) $y = -\cos x$ c) $y = \sec x$

d) $x = -y^2$ e) $x = -y^3$

---

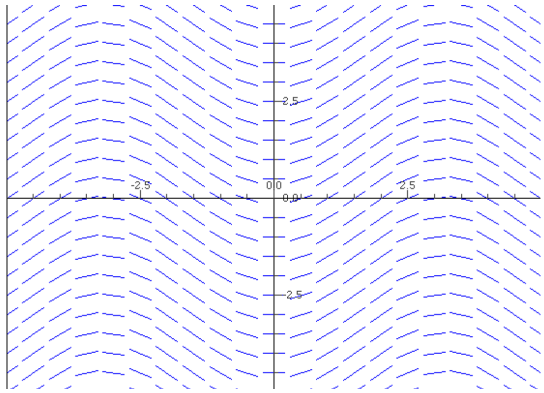
5. Which of the following differential equations corresponds to the slope field shown in the figure below?



a) $\frac{dy}{dx} = 1 - y^3$ b) $\frac{dy}{dx} = y^2 - 1$ c) $\frac{dy}{dx} = -\frac{x^2}{y^2}$

d) $\frac{dy}{dx} = x^2 y$ e) $\frac{dy}{dx} = x + y$

6. Which of the following equations might be the solution to the slope field shown in the figure below?



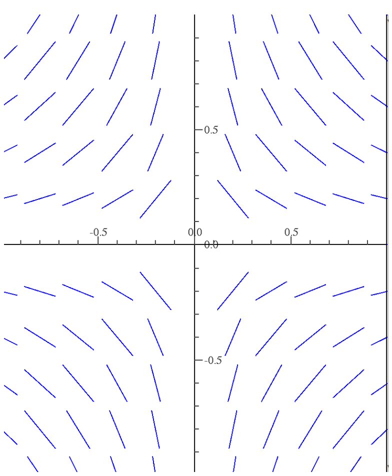
a) $y = 4x - x^3$ b) $y = -\cos x$ c) $y = \sec x$

d) $x = -y^2$ e) $x = -y^3$

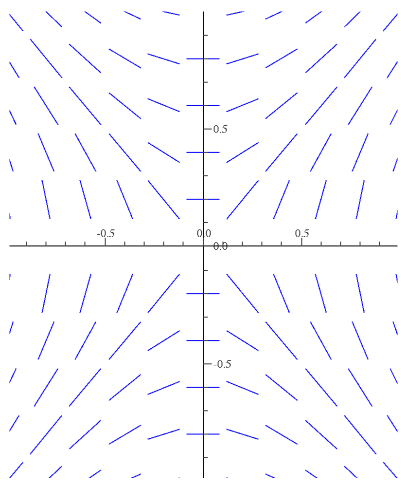
---

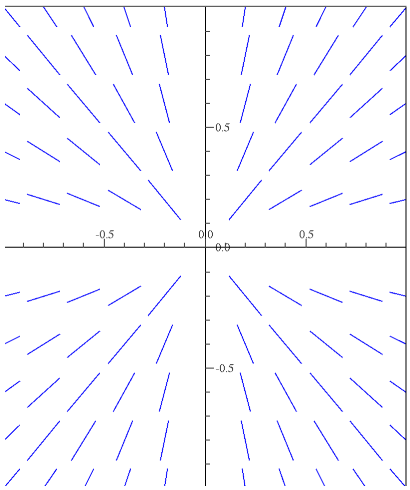
7. Which of the slope field shown below corresponds to $\dfrac{dy}{dx} = -\dfrac{y}{x}$?

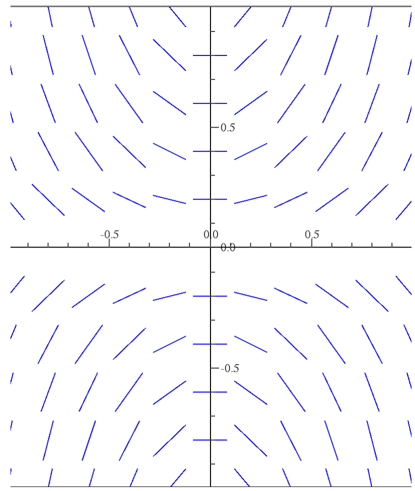
a)

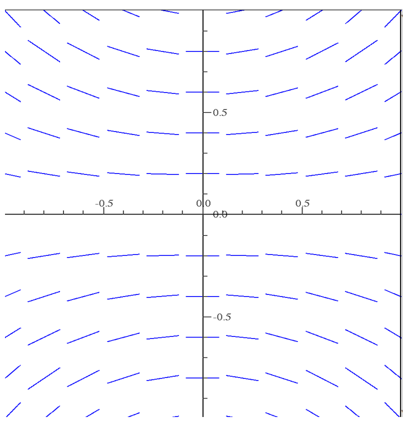


b)

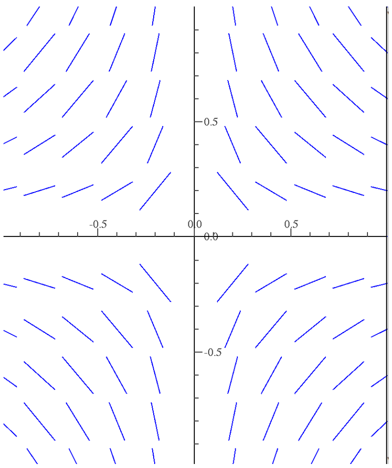


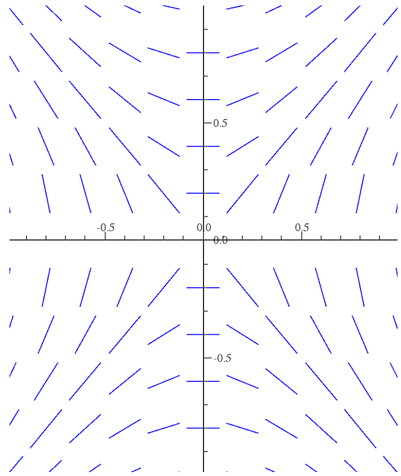
c) 

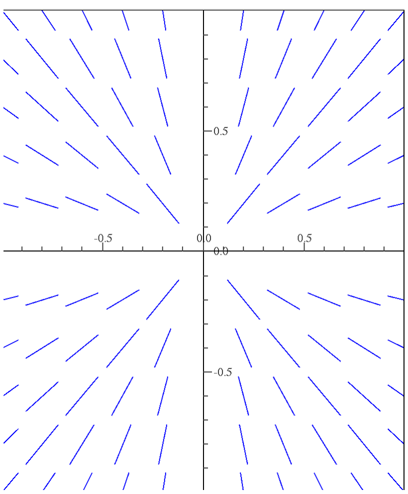
d) 

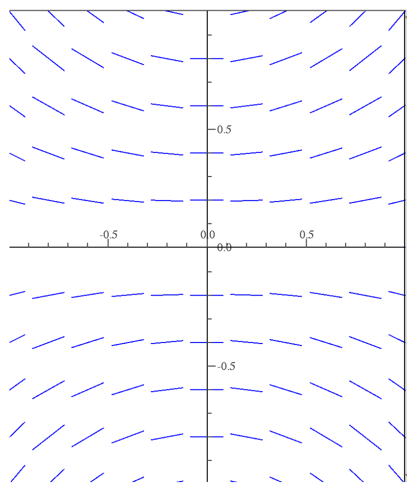
e) 

8. Which of the slope field shown below corresponds to $\dfrac{dy}{dx} = yx$?

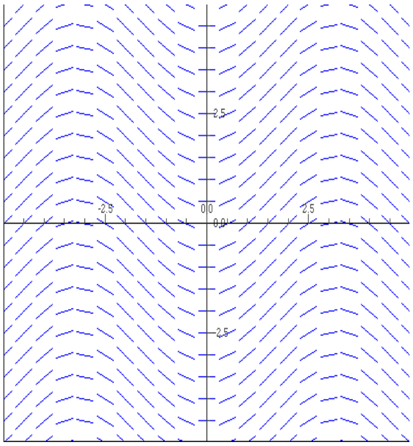
a) 

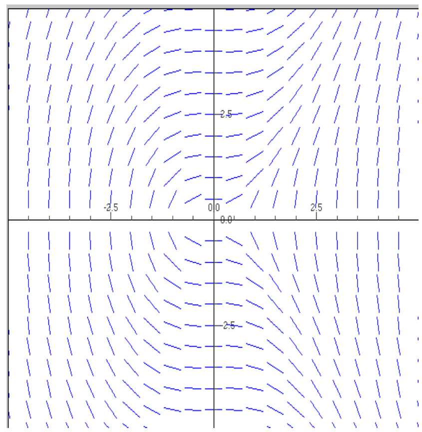
b) 

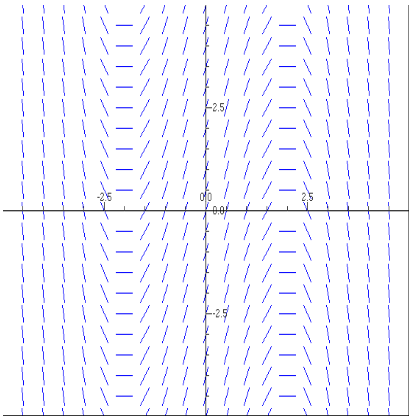
c) 

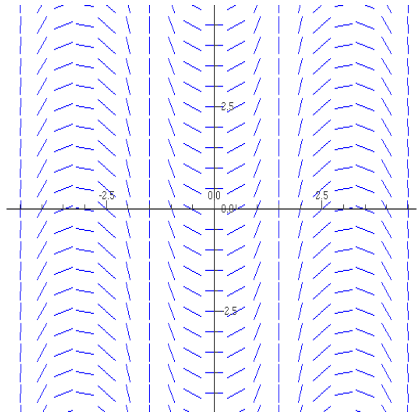
d) 

9. Which of the slope field shown below corresponds to $|y| = e^{x^3}$?

a)


b)


c)


d)


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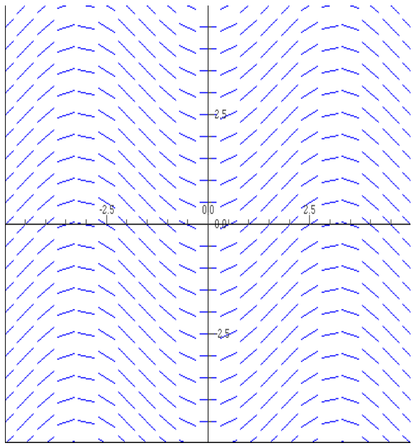
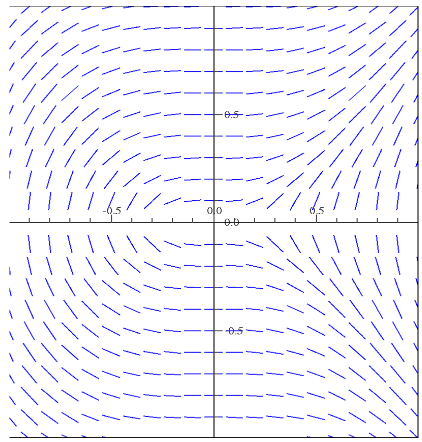
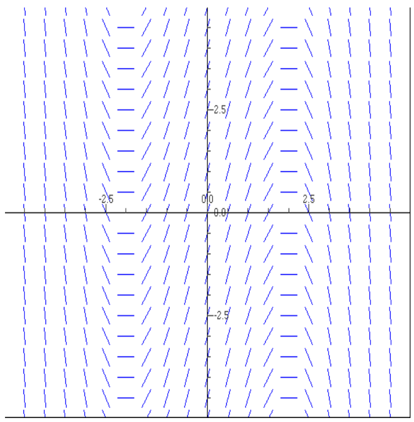
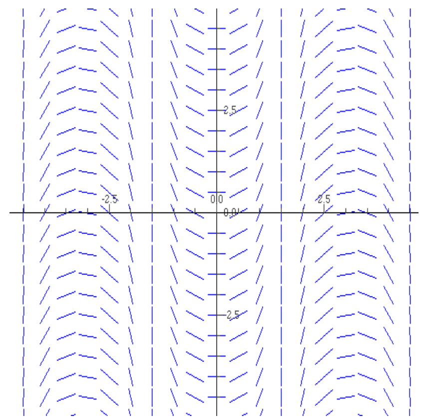
10. Which of the slope field shown below corresponds to $y = \sec x$?

a) 

b) 

c) 

d) 

AP Calculus BC Name___________________________

Anti-Derivative Practice Test

1. Which of the following statements are true?

I. $\int\left(\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\right)dx=\frac{1}{5}\left(x^4+2x^2-5\right)^{5/4}+c$

II. $\int\left(x^5\sin x^6\right)dx=-\frac{1}{6}\cos x^6+c$

III. $\int\csc x\,dx=\ln\left|\csc x+\cot x\right|+c$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

2. $\int\frac{x-2}{x-1}dx=$

a) $-\ln\left|x-1\right|+c$ b) $x+\ln\left|x-1\right|+c$ c) $x-\ln\left|x-1\right|+c$

d) $x-\sqrt{x-1}+c$ e) $x+\sqrt{x-1}+c$

---

3. If $\frac{dy}{dx} = \sin x \cos^3 x$ and if $y = 1$ when $x = \pi$, what is the value of y when $x =$ 0?

a) –3
b) –2
c) 1
d) 2
e) 3

---

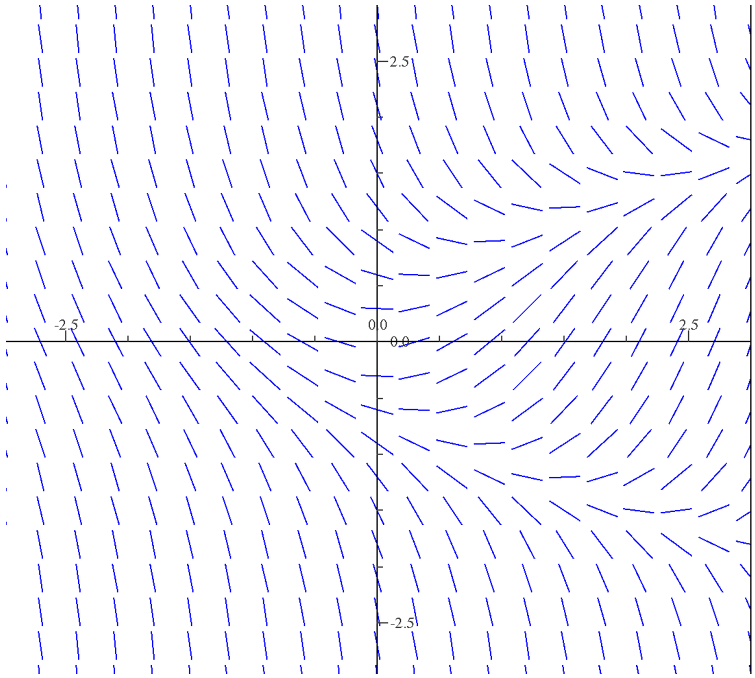
4. $\int x\sqrt{1-x^2}\,dx$

a) $\frac{\left(1-x^2\right)^{3/2}}{3}+c$ b) $-\left(1-x^2\right)^{3/2}+c$ c) $\frac{x^2\left(1-x^2\right)^{3/2}}{3}+c$

d) $\frac{-x^2\left(1-x^2\right)^{3/2}}{3}+c$ e) $\frac{-\left(1-x^2\right)^{3/2}}{3}+c$

---

5. Which of the following differential equations corresponds to the slope field shown in the figure below?

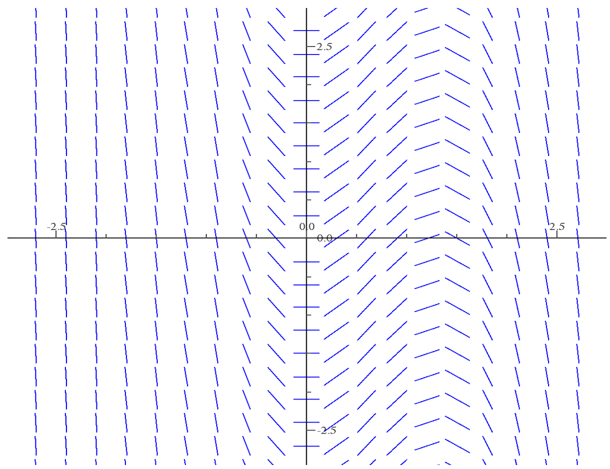


a) $\dfrac{dy}{dx} = x - y^2$ b) $\dfrac{dy}{dx} = 1 - \dfrac{y}{x}$ c) $\dfrac{dy}{dx} = -y^3$

d) $\dfrac{dy}{dx} = x - \dfrac{1}{2}x^3$ e) $\dfrac{dy}{dx} = x + y$

---

6. Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = 4x - x^3$ b) $y = x^3 - 4x$ c) $y = 4x^4 - x^6$

d) $y = x^3 - 15x^5$ e) $y = \sec x$

---

7. Identify is the first mistake (if any) in this process:

$$\frac{dy}{dx} = xy + x$$

Step 1: $\frac{1}{y+1} dy = x dx$

Step 2: $\ln|y+1| = x^2 + c$

Step 3: $|y+1| = e^{x^2} + c$

Step 4: $y = e^{x^2} + c$

a) Step 1 b) Step 2 c) Step 3

d) Step 4 e) There is no mistake.

---

8. $\int \left(\frac{t^3 - 4t - 3}{5t^{2/3}} \right) dt$

9. $\int \frac{x^2}{\left(x^3 - 1\right)^{3/2}} \, dx$

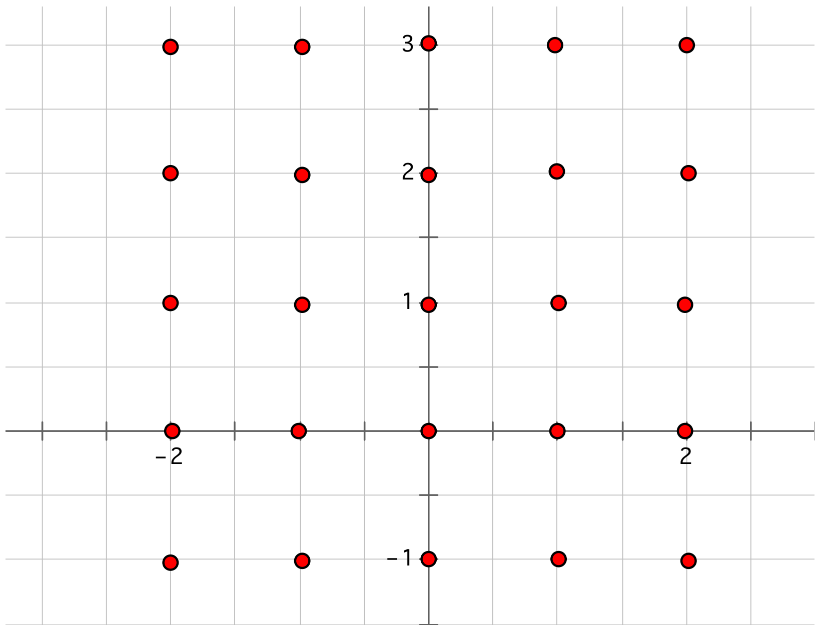
10. $\int \left(3x^5 + \frac{\csc^2 x}{e^{\cot x}} - x^3 \csc\left(x^4\right) \right) dx$

11. $\int \left(x\sqrt{-3x^2 + 17} \right) dx$

12. The acceleration of a particle is described by $a(t) = 48t^2 - 18t + 6$. Find the distance equation for *x(t)* if $v(1) = 1$ and $x(1) = 3$.

13. Given the differential equation, $\frac{dy}{dx} = \frac{y-2}{x+1}$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.



b. If the solution curve passes through the point (0, 0), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve of $\frac{dy}{dx} = (y-2)(x+1)$ given that $y(0) = 5$

Chapter 2 Answer Key

2.1 Free Response Answers

1. $2x^3 - x^2 + 3x + c$

2. $\frac{1}{4}x^4 + x^3 - x^2 + 4x + c$

3. $2\sqrt[3]{x^2} + c$

4. $\frac{8}{5}x^5 - x^4 + 3x^3 + x^2 + x + c$

5. $\frac{2}{3}x^6 + \frac{5}{4}x^4 + C$

6. $4x^3 + \frac{29}{2}x^2 - 8x + c$

7. $\frac{2}{3}x^{3/2} - 12x^{1/2} + c$

8. $\frac{1}{2}x^2 + 2x^{1/2} + 3\ln|x| + c$

9. $\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

10. $\frac{16}{3}x^3 - 12x^2 + 9x + c$

11. $\frac{2}{3}x^{3/2} + \frac{6}{5}x^{5/2} - 12x^{1/2} + c$

12. $2x^2 - 2x^{-1/2} - 3x^{-1} + c$

13. $\frac{1}{3}x^3 + \frac{5}{2}x^2 + 6x + c.$

14. $\frac{1}{2}x^2 - 4x + 7\ln|x| + c$

15. $\frac{1}{10}x^5 - \frac{7}{6}x^3 + 2x - \frac{9}{2}\ln|x| + c$

16. $\frac{1}{3}x^2 + x^2 + x + c$

17. $\frac{1}{5}y^5 + \frac{10}{3}y^3 + 5y + c$

18. $2t^6 + \frac{3}{4}t^4 + \frac{28}{3}t^3 + 7t + c$

19. $f(x) = x^3 - 3x^2 + 3x + 2$

20. $\frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{1}{2}x^2 + 3x - \frac{37}{12}$

21. $f(x) = \frac{3}{2}x^3 - \frac{10}{3}x^2 - 2x + \frac{35}{3}$

22. $f(x) = 3x^4 - 2x^3 + 4x^2 - 13x + 11$

23. $x(t) = \frac{1}{12}t^4 - \frac{1}{3}t^3 + 2t^2 + 2t + 4$

2.1 Multiple Choice Answer Key

1.	D	2.	A	3.	A	4.	D	5.	A	6.	D

2.2 Free Response Answer Key

1. $\frac{1}{20}(5x+3)^4+c$

2. $\frac{1}{100}(x^4+5)^{25}+c$

3. $\frac{1}{7}x^7+\frac{1}{2}x^4+x+c$

4. $-\frac{3}{5}(2-x)^{5/3}+c$

5. $\frac{1}{6}(2x^2+3)^{3/2}+c$

6. $\frac{1}{-10(5x+2)^2}+c$

7. $\frac{1}{2}\sqrt{1+x^4}+c$

8. $\frac{3}{4}(x^2+2x+3)^{2/3}+c$

9. $\frac{1}{6}x^6+\frac{1}{3}\cos 3x+\frac{1}{2}e^{x^2}+c$

10. $\frac{1}{3}\tan x^3+\frac{1}{4}\ln^4 x+c$

11. $\frac{1}{5}\sin x^5+c$

12. $-\frac{1}{7}\cos(7x+1)+c$

13. $\frac{1}{3}\tan(3x-1)+c$

14. $-2\cos\sqrt{x}+c$

15. $\frac{1}{5}\tan^5 x+c$

16. $\frac{1}{2}Ln^2\ x+c$

17. $\frac{1}{6}e^{6x}+c$

18. $-\frac{1}{4}\csc^2 2x+c$

19. $\frac{1}{4}Ln^2\ (x^2+1)+c$

20. $2e^{\sqrt{x}}+c$

21. $-\frac{2}{3}\cot^{3/2} x + c$

22. $-\frac{1}{2}\sin^2 \frac{1}{x} + c$

23. $\frac{1}{2}\tan^{-1} x^2 + c$

24. $x + c$

2.2 Multiple Choice Answer Key

1.	C	2.	E	3.	D	4.	A	5.	E	6.	B
7.	A	8.	D	9.	C	10.	A	11.	D	12.	C
13.	D	14.	E	15.	A	16.	D	17.	A	18.	E
19.	C										

2.3 Free Response Answer Key

1. $y = kx$

2. $y = \frac{2}{-x^2 + C}$

3. $y = k\sqrt{x^2 + 1}$

4. $y = \frac{1}{C - 3\tan^{-1} x}$

5. $y = \left(\frac{2}{3}\left(x^3 - 3\right)^{3/2} + C\right)^{1/3}$

6. $y = \pm\frac{1}{5}\sqrt{\ln\left|5\tan x + C\right|}$

7. $y = \left(\frac{1}{2}x^4 - 2\right)^{1/3}$

7. $y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

8. $y = \sec^{-1}\left(\frac{x^3}{3} + x + C\right)$

9. $y = \pm\frac{1}{\sqrt{c - 4x^2}}$

10. $y = \frac{1}{c - \sin x}$

11. $v = -1 + ke^{2t + \frac{1}{2}t^2}$

12. $y = \pm\sqrt{\left(\frac{3}{2}t^2 + c\right)^{2/3} - 1}$

13. $\theta = \left(\frac{2}{3}r + r^{3/2} + c\right)^{2/3}$

14. $y = \frac{-2}{5x^2 + 1}$

15. $y = \sqrt{x^2 + 1}$

16. $y = \left(\frac{1}{2}x^4 - 2\right)^{1/3}$

17. $y = 2 + e^{\frac{-x^3}{3} - x + \frac{4}{3}}$

18. $y = \tan(x - 1)$

19. $y = -\sqrt{2e^{x^2} - 1}$

20. $u = -\sqrt{t^2 + \tan t + 25}$

21. $y = 5e^{\frac{1}{2}x^2 + \cos x}$

22. $y = \cos^{-1}(\cos x - 1)$

23. $y = \ln\left(\frac{1}{\sin x + C}\right)$

24. $y = 7e^{x^4}$

25a) $\frac{d^2y}{dx^2} = -2y^2 + (6 - 2x)^2 2y^3$

25b) Maximum

25c) $y = \frac{1}{x^2 - 6x - 21}$

26a) $\frac{d^2y}{dx^2} = y\left[x^2 + 2x + 2\right]$

26b) Minimum

26c) $y = 2e^{x^2 + x}$

27a) $\frac{d^2y}{dx^2} = \frac{(y+2)^2(6x) - 9x^4}{(y+2)^3}$

27b) Neither

27c) $y = x^{3/2} - 1$

28a) $\frac{d^2y}{dx^2} = \left((x-1)^2+1\right)(y+2)$ 28b) Minimum

28c) $y = -2 + e^{\frac{1}{2}x^2 - x + 0.5}$

2.3 Multiple Choice Answer Key

1. B 2. C 3. C 4. B 5. E

2.4 Free Response Answer Key

1. $-\frac{8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + c$

2. $\frac{2}{15}\left(x^3+4\right)^{5/2} - \frac{8}{9}\left(x^3+4\right)^{3/2} + c$

3. $\frac{1}{4}(2x+3) + \frac{7}{4}\ln|2x+3| + c$

4. $\frac{1}{28}\left(x^2+1\right)^{14} - \frac{1}{26}\left(x^2+1\right)^{13} + c$

5. $\frac{5}{3}(3+\ln x)^3 - \frac{1}{4}(3+\ln x)^4 + c$

6. $-\frac{16}{3}\left(4-\sqrt{x}\right)^{3/2} + \frac{4}{5}\left(4-\sqrt{x}\right)^{5/2} + c$

7. $\frac{1}{10}\left(x^2+4\right)^5 - \left(x^2+4\right)^4 + \frac{8}{3}\left(x^2+4\right)^3 + c$

8. $\frac{2}{7}(x+3)^{7/2} - \frac{8}{5}(x+3)^{5/2} + \frac{8}{3}(x+3)^{3/2} + c$

9. $\frac{1}{28}(2t+4)^7-\frac{1}{4}(2t+4)^6+c$

10. $\frac{1}{45}(3z-1)^5-\frac{2}{9}(3z-1)^4+c$

11. $\frac{2}{9}(y^3+4)^{3/2}-\frac{10}{3}(y^3+4)^{1/2}+c$

12. $=\frac{1}{4}(w^2+4)^2-4(w^2+4)+16\ln(w^2+4)+c$

13. $(x^2-1)^{1/2}-2(x^2-1)^{-1/2}-\frac{1}{3}(x^2-1)^{-3/2}+c$

14. $\frac{1}{4}(x^4+1)^{1/2}+8(x^4+1)^{-1/2}+c$

15. $\frac{3}{7}(x-1)^{7/3}-\frac{9}{4}(x-1)^{4/3}+c$

16. $-\frac{26}{3}(4-x)^{3/2}+\frac{4}{5}(4-x)^{1/2}+c$

17. $\frac{2}{5}(e^x+1)^{5/2}-\frac{2}{3}(e^x+1)^{3/2}+c$

2.4 Multiple Choice Answer Key

1.	D	2.	C	3.	B	4.	A
5.	E	6.	A	7.	A		

2.5 Free Response Answer Key

1. $-\frac{1}{3}\cos^3 x+\frac{1}{5}\cos^5 x+c$

2. $\frac{1}{5}\sin^5 x-\frac{2}{7}\sin^7 x+\frac{1}{9}\sin^9 x+c$

3. $\frac{1}{3}\sin^3 x-\frac{3}{5}\sin^5 x+\frac{3}{7}\sin^7 x-\frac{1}{9}\sin^9 x+c$

4. $-\frac{1}{7}\cos^7 x+\frac{2}{9}\cos^9 x-\frac{1}{11}\cos^{11} x+c$

5. $-\frac{1}{6}\cos^6 x+c$

6. $\frac{1}{6}\sin^6 x-\frac{1}{4}\sin^8 x+\frac{1}{10}\sin^{10} x+c$

7. $\frac{1}{8}x-\frac{1}{32}\sin 4x+c$

8. $\frac{1}{16}x+\frac{1}{64}\sin 4x+\frac{1}{48}\sin^3 2x+c$

2.5 Multiple Choice Answer Key

1. C 2. B 3. A 4. D

5. B 6. B 7. E

2.6 Free Response Answer Key

1. $\frac{1}{6}\tan^6 x+c$

2. $\frac{1}{9}\tan^9 x+\frac{2}{7}\tan^7 x+\frac{1}{5}\tan^5 x+c$

3. $\frac{1}{11}\sec^{11} x-\frac{1}{3}\sec^9 x+\frac{3}{7}\sec^7 x-\frac{1}{5}\sec^5 x+c$

4. $\frac{1}{7}\tan^7 x+c$

5. $\frac{1}{8}\tan^8 x+\frac{1}{3}\tan^6 x+\frac{1}{4}\tan^4 x+c$

6. $-\frac{1}{6}\cot^6 x+c$

7. $-\frac{1}{7}\csc^7 x+\frac{2}{5}\csc^5 x-\frac{1}{3}\csc^3 x+c$

8. $-\frac{1}{11}\csc^{11} x+\frac{2}{9}\csc^9 x-\frac{1}{7}\csc^7 x+c$

9. $-\frac{1}{5}\cot^5 x - \frac{1}{7}\cot^7 x + c$

10. $-\frac{1}{2}\cot^2 x - \frac{1}{4}\cot^4 x + c$

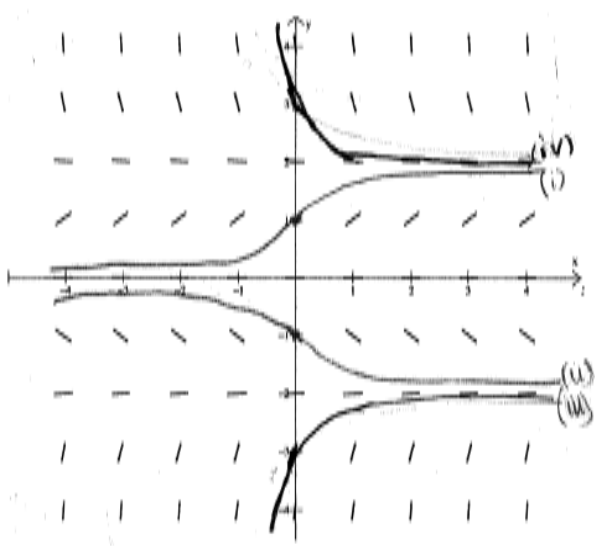
2.6 Multiple Choice Answer Key

1. A 2. B 3. C 4. D

5. E 6. B 7. B

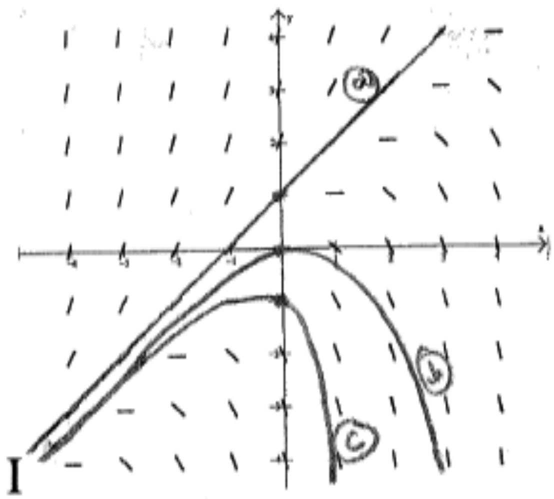
2.7 Free Response Answer Key

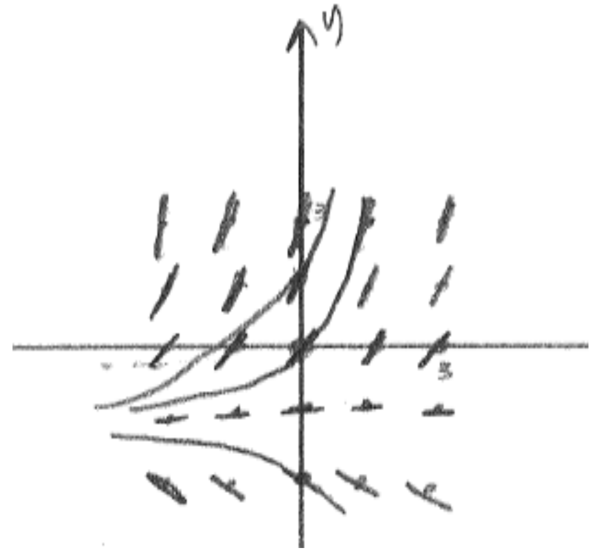
1.



2. IV 3. I 4. III 5. II

6. 7.





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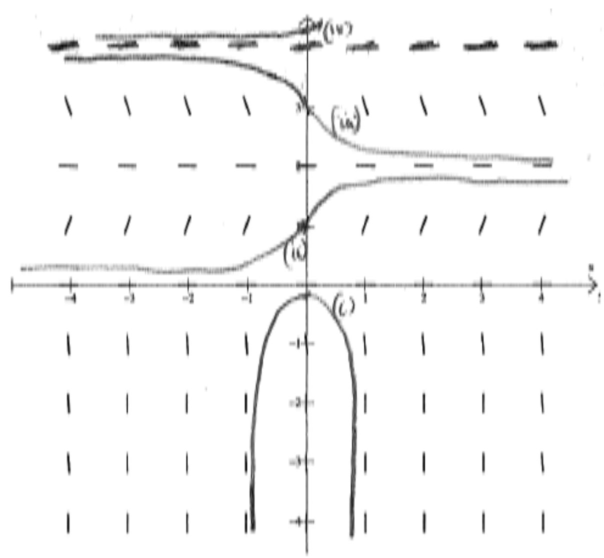


2.7 Multiple Choice Answer Key

1.	B	2.	E	3.	B	4.	A	5.	E	6.	B
7.	A	8.	D	9.	B	10.	D				

Chapter 2 Practice Test Key

1. D 2. B 3. C 4. E 5. A 6. A

7. B

8. $\frac{3}{50}t^{10/3}-\frac{12}{25}t^{5/3}-\frac{9}{5}t^{1/3}+c$

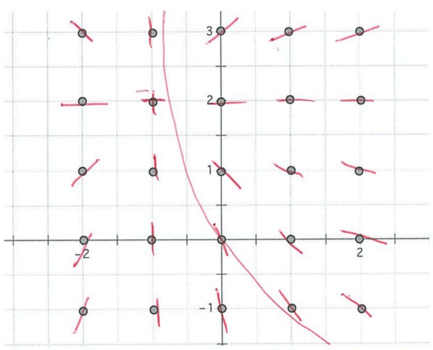
9. $=-\frac{2}{3}\left(x^3-1\right)^{-1/2}+c$

10. $\frac{1}{6}x^6+e^{-\cot x}-\frac{1}{4}\ln\left|\csc x^4-\cot x^4\right|+c$

11. $-\frac{1}{9}\left(-3x^2+17\right)^{3/2}+c$

12. $x(t)=4t^4-3t^3+3t^2-12t+8$

13a & b.



13c. $y=2+3e^{\frac{1}{2}x^2+x}$