Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the "how" of differentiation. We will review all the derivative rules learned last year in PreCalculus. In the next two chapters, we will review the "why." As a quick reference, here are those rules:

The Power Rule: $\frac{d}{dx}u^n = nu^{n\text{-}1}\frac{du}{dx}$

The Product Rule: $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

The Chain Rule: $\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \ \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \ \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = (\sec^2 u)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = (-\csc^2 u)\frac{du}{dx}$$

$$\frac{d}{dx}[e^u] = (e^u)\frac{du}{dx} \qquad \frac{d}{dx}[Ln\ u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}[a^u] = (a^u \cdot Ln\ a)\frac{du}{dx} \qquad \frac{d}{dx}[Log_a u] = \left(\frac{1}{u \cdot Ln\ a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}[\sin^{-1} u] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \qquad \frac{d}{dx}[\csc^{-1} u] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\cos^{-1} u] = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx} \qquad \frac{d}{dx}[\sec^{-1} u] = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$$

$$\frac{d}{dx}[\tan^{-1} u] = \frac{1}{u^2+1} \cdot \frac{du}{dx} \qquad \frac{d}{dx}[\cot^{-1} u] = \frac{-1}{u^2+1} \cdot \frac{du}{dx}$$

Here is a quick review of from last year:

Identities While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x; \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin 2x = 2\sin x \ \cos x; \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverses Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1} x = \begin{Bmatrix} calculator & \pm 2\pi n \\ -calculator & \pm 2\pi n \end{Bmatrix} \qquad \sin^{-1} x = \begin{Bmatrix} calculator & \pm 2\pi n \\ \pi - calculator & \pm 2\pi n \end{Bmatrix}$$

$$\tan^{-1} x = \begin{Bmatrix} calculator & \pm 2\pi n \\ \pi + calculator & \pm 2\pi n \end{Bmatrix} = calculator \ \pm \pi n$$

$$\log_a x + \log_a y = \log_a (xy)$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

1.1: The Power and Exponential Rules with the Chain Rule

In PreCalculus, we developed the idea of the Derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the Derivative will be Explored. In this Chapter, we are primarily interested in how to find the Derivative and what it is used for.

Derivative—Def'n: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

—Means: The function that yields the slope of the tangent line.

Numerical Derivative—Def'n: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

—Means: The numerical value of the slope of the tangent line at x = a.

Symbols for the Derivative

$\frac{dy}{dx}$ = "d-y-d-x" f'(x) = "f prime of x" y' = "y prime"

$\frac{d}{dx}$ = "d-d-x" D_x = "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from PreCalc: The derivative yields the slope of the tangent line. [But there is more to it than that.]

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in PreCalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule:	The Exponential Rules:
$\frac{d}{dx}x^n = n\,x^{n-1}$	$\frac{d}{dx}\left[e^x\right] = e^x$ $\frac{d}{dx}\left[a^x\right] = a^x \cdot Ln\,a$

The difference between these is where the variable is. The Power Rule applies when the variable is in the base, while the Exponential Rules apply when the variable is in the Exponent. The difference between the two Exponential rules is what the base is. e = 2.718281828459…, while a is any positive number other than 1.

Ex 1 Find a) $\frac{d}{dx}\left[x^5\right]$ and b) $\frac{d}{dx}\left[5^x\right]$

The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\frac{d}{dx}\left[x^5\right] = 5x^4$ b) $\frac{d}{dx}\left[5^x\right] = 5^x\ Ln\ 5$

There were a few other basic rules that we need to remember.

D_x [constant] is always 0

$D_x\,[cx^n] = (cn)x^{n-1}$

$D_x\,[f(x)+g(x)] = D_x\,[f(x)] + D_x\,[g(x)]$

These rules allow us to easily differentiate a polynomial--term by term.

Ex 2 $y = 3x^2 + 5x + 1$; find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}[3x^2 + 5x + 1]$$
$$= 3(2)x^{2-1} + 5(1)x^{1-1} + 1(0)$$
$$= 6x + 5$$

Ex 3 $f(x) = x^2 + 4x - 3 + e^x$; find $f'(x)$.

$$f'(x) = 2x + 4 + e^x$$

Ex 4 $y = \sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$; find $\frac{dy}{dx}$.

$$y = \sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$$
$$= x^{3/2} + 4x^{-1/2} - x^{3/4} + e^4$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} - 2x^{-3/2} - \frac{3}{4}x^{-1/4}$$

Note in Ex 4 that e^4 is a constant, therefore, its derivative is 0.

As we have seen, when the variable is in the Exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an Exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right]=\frac{d}{dx}\left[e^{x^2\ Ln\ (2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Composite Function--A function made of two other functions, one within the other. For Example, $y=\sqrt{16x-x^3}$, $y=\sin\ x^3$, $y=\cos^3 x$, and $y=(x^2+2x-5)^3$. The general symbol is *f(g(x))*.

Ex 5 Given $f(x)=\cos^{-1}x$, $g(x)=x^2-1$, and $h(x)=\sqrt{1+x^2}$, find (a) $f\left(g\left(\sqrt{2}\right)\right)$, (b) $h\left(g(1)\right)$, and (c) $f\left(h\left(g(1)\right)\right)$.

(a) $g\left(\sqrt{2}\right)=\left(\sqrt{2}\right)^2-1=1$, so $f\left(g\left(\sqrt{2}\right)\right)=f(1)=\cos^{-1}(1)=0$.

(b) $g(1)=0$, so $h(g(1))=h(0)=\sqrt{1+0^2}=1$.

(c) $g(1)=0$ and $h(g(1))=h(0)=\sqrt{1+0^2}=1$, so
$f\left(h\left(g(1)\right)\right)=f(1)=\cos^{-1}(1)=0$

So. How do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. You end up with two open dolls next to each other.

The Chain Rule: $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$

If you think of the inside function (the) as equaling u, we could write The Chain Rule like this: . This is the way that most of the derivatives are written with The Chain Rule.
The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other Rules, as it was in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:	The Exponential Rules:
$\frac{d}{dx}\left[u^n\right] = n\,u^{n\text{-}1}\cdot\frac{du}{dx}$	$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx}$ $\frac{d}{dx}\left[a^u\right]=\left(a^u\cdot Ln\,a\right)\frac{du}{dx}$

where u is a function of x.

Ex 6 $\frac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]$

$$\frac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]=10\left(4x^2-2x-1\right)^9\ (8x-2)$$
$$=20\left(4x^2-2x-1\right)^9\ (4x-1)$$

Ex 7 If $y=\sqrt{16x-x^3}$, find $\frac{dy}{dx}$.

$$y=\sqrt{16x-x^3}=\left(16x-x^3\right)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(16x-x^3\right)^{-1/2}\left(16-3x^2\right)$$

$$=\frac{\left(16-3x^2\right)}{2\left(16x-x^3\right)^{1/2}}$$

In this case, the $\sqrt{\ }$ is the f function and the polynomial $16x - x^3$ is the g. Each derivative is found by the Power Rule, but, as $16x - x^3$ is inside the $\sqrt{\ }$, it is inside the derivative of the $\sqrt{\ }$.

Ex 8 $\frac{d}{dx}\left[e^{4x^2}\right]$

$$\frac{d}{dx}\left[e^{4x^2}\right] = e^{4x^2} \cdot 8x \ = \ 8xe^{4x^2}$$

Of course, the Graphing Calculator can cut all this work short.

EX 9 $\frac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5+7}\right]$

$$\begin{aligned}\frac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5+7}\right] &= \frac{d}{dx}\left[\left(\left(x^2+1\right)^5+7\right)^{1/2}\right] \\ &= \frac{1}{2}\left(\left(x^2+1\right)^5+7\right)^{-1/2} 5\left(x^2+1\right)^4\left(2x\right) \\ &= \frac{5x\left(x^2+1\right)^4}{\left(\left(x^2+1\right)^5+7\right)^{1/2}}\end{aligned}$$

Ex 10 Given this table of values, find $\frac{d}{dx}\left[f\left(g(x)\right)\right]$ and $\frac{d}{dx}\left[g\left(f(x)\right)\right]$ at $x=1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$ and $\frac{d}{dx}\left[g\left(f(x)\right)\right]=g'\left(f(x)\right)f'(x)$.

At $x=1$,

$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(1)\right)g'(1)$$
$$=f'(2)(6)$$
$$=(5)(6)$$
$$=30$$

$$\frac{d}{dx}\left[g\left(f(x)\right)\right]=g'\left(f(1)\right)f'(1)$$
$$=g'(3)(4)$$
$$=(9)(4)$$
$$=36$$

Ex 11 If $g(2)=-5$ and $g'(2)=4$, find $f'(2)$ if $f(x)=e^{g(x)}+g\left(x^3-6\right)+\left(g(x)\right)^3$

Notice that while we do not actually know the function that *g* represents, we still can take its derivative, because we know the derivative of *g* is *g*'. Of course, the Chain Rule is still essential in this process.

$$f(x)=e^{g(x)}+g\left(x^3-6\right)+\left(g(x)\right)^3$$
$$f'(x)=e^{g(x)}\cdot g'(x)+g'\left(x^3-6\right)\cdot\left(3x^2\right)+3\left(g(x)\right)^2\cdot g'(x)$$

$$f'(2)=e^{g(2)}\cdot g'(2)+g'\left(2^3-6\right)\cdot\left(3(2)^2\right)+3\left(g(2)\right)^2\cdot g'(2)$$
$$f'(2)=e^{-5}\cdot 4+g'(2)\cdot(12)+3(-5)^2\cdot 4$$

$$f'(2)=\frac{4}{e^5}+4\cdot(12)+300=\frac{4}{e^5}+348$$

1.1 Homework

Differentiate.

1. $f(x) = x^2 + 3x - 4$

2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$

3. $y = x^{-2/3}$

4. $y = 5e^x + 3$

5. $v(r) = \frac{4}{3}\pi r^3$

6. $g(x) = x^2 + \frac{1}{x^2}$

7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

9. $z = \frac{A}{y^{10}} + Be^y$

10. $y = e^{x+1} + 1$

11. If $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$. Compare the graphs of $f(x)$ and $f'(x)$, and use them to explain why your answer is reasonable.

12. $\frac{d}{dx}\left[x^3 + 4x - \pi\right]^{-7}$

13. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$.

14. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$.

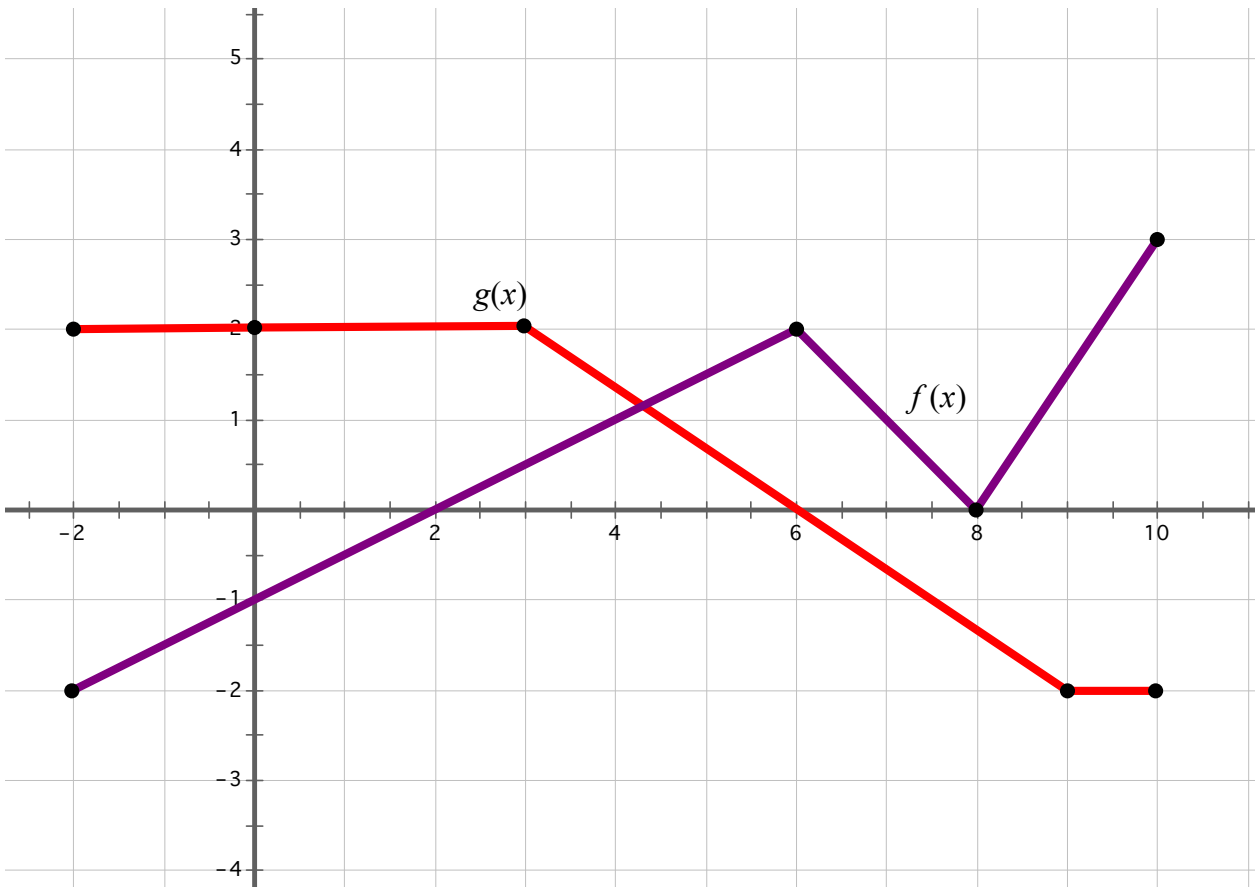
15. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a. $g'(2)$, where $g(x) = \left[f(x)\right]^3$

b. $h'(2)$, where $h(x) = f(x^3)$

For problems 16 – 19, the graphs of $f(x)$ and $g(x)$ are given below.



16. $u'(2)$ if $u = f(g(x))$

17 $v'(4)$ if $v = g(f(x))$

18. $w'(6)$ if $w = g(g(x))$

19. $t'(8)$ if $t = f(f(x))$

1.1 Multiple Choice Homework

1. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given on the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	8	8	4	3
8	6	-12	2	4

Given that $h(x) = g(f(x))$, $h'(4) =$

a) 32 b) 10 c) -6 c) 24 e) 16

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2. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given on the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

Given that $h(x) = f(g(x))$, $h'(8) =$

a) -12 b) -2 c) -1 d) -8 e) 10

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3. If $f(x)=\cos^2(3-x)$, then $f'(0)=$

a) $-2\cos 3$ b) $-2\sin 3\cos 3$ c) $6\cos 3$

d) $2\sin 3\cos 3$ e) $6\sin 3\cos 3$

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4. If $y=\left(x^4+4\right)^2$, then $\frac{dy}{dx}=$

a) $2\left(x^4+4\right)$ b) $\left(4x^3\right)^2$ c) $2\left(4x^3+4\right)$

d) $4x^3\left(x^4+4\right)$ e) $8x^3\left(x^4+4\right)$

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5. If $h(x)=\left[f(x)\right]^2 g(x)$ and $g(x)=3$, then $h'(x)=$

a) $2f'(x)g'(x)$

b) $6f'(x)f(x)$

c) $g'(x)\left[f(x)\right]^2+2f(x)f'(x)g(x)$

d) $2f'(x)g(x)+g'(x)\left[f(x)\right]^2$

e) 0

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6. Which of the following statements must be true?

I. $\frac{d}{dx}\sqrt{e^x+3}=\frac{e^x}{2\sqrt{e^x+3}}$ II. $\frac{d}{dx}(\ln\cos x)=\tan x$

III. $\frac{d}{dx}\left(6x^3-\pi+\sqrt[3]{x^8}-\frac{2}{x^3}\right)=18x^2+\frac{8}{3}\sqrt[3]{x^5}+\frac{6}{x^4}$

a) I only b) II only c) III only

d) I and III only e) I, II, and III

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7. The function $f(x)=\tan(3^x)$ has one zero in the interval $[0, 1.4]$. The derivative at this point is

a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

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8. If $f(x)=e^{5x^2}+x^4$, then $f'(1)=$

a) e^5+1 b) $5e^4+4$ c) $5e^5+1$

d) $10e+4$ e) $10e^5+4$

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9. If h is the function defined by $h(x)=e^{5x}+x+3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

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10. If h is the function defined by $h(x)=x^2-5x+3$, what is the equation of the tangent line to the function when $h'(x)=-1$?

a) $y=-x-1$ b) $y=-x+24$ c) $y=-x-5$

d) $y=-7x+2$ e) $y=-7x$

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11. Let f be the function defined by $f(x)=4x^2-5x+3$. Which of the following is an equation of the line tangent to the graph of f at the point where $x=-2$

a) $y=-21x-13$ b) $y=-21x+29$ c) $y=-21x-71$

d) $y=-11x+7$ e) $y=-11x-51$

1.2: Trig, Trig Inverse, and Log Rules

Trigonometric--Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."

Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic--Defn: "The inverse of an Exponential function."

Means: there is a Log or Ln in the equation.

$$\frac{d}{dx}[\sin\ u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u\ \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin\ u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec\ u] = (\sec u\ \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = (\sec^2\ u)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = (-\csc^2\ u)\frac{du}{dx}$$

$$\frac{d}{dx}[Ln\ u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}[Log_a u] = \left(\frac{1}{u \cdot Ln\ a}\right)\frac{du}{dx}$$

Note that all these Rules are Expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1 $\frac{d}{dx}\left(\sin^3 x\right)$

$$\frac{d}{dx}\left(\sin^3\ x\right) = 3\sin^2\ x\ \cos\ x$$

Ex 2 $\frac{d}{dx}\left[\sin\ \left(x^3\right)\right]$

$$\frac{d}{dx}\left[\sin\ \left(x^3\right)\right] = \cos\ x^3\left(3x^2\ \right)$$

$$= 3x^2\ \cos\ x^3$$

Ex 3 $\frac{d}{dx}\left[Ln\ 4x^3\right]$

$$\frac{d}{dx}\left[Ln\ 4x^3\right] = \frac{1}{4x^3}\cdot 12x^2$$

$$= \frac{3}{x}$$

We could have also simplified algebraically before taking the derivative:

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are fuinctions.

Ex 4 $\frac{d}{dx}\left(\sec^5\ 3x^4\right)$

$$\frac{d}{dx}\left(\sec^5\ 3x^4\right) = 5\sec^4\ 3x^4\left(\sec\ 3x^4\ \tan\ 3x^4\right)\left(12x^3\right)$$

$$= 60x^3\sec^5\ 3x^4\tan\ 3x^4$$

Ex 5 $\frac{d}{dx} Ln\left(\cos\sqrt{x}\right)$

$$\begin{aligned}\frac{d}{dx} Ln\left(\cos\sqrt{x}\right) &= \frac{1}{\cos\ x^{1/2}} \cdot \left(-\sin\ x^{1/2}\right)\left(\frac{1}{2}x^{-1/2}\right) \\ &= -\tan\ x^{1/2}\left(\frac{1}{2}x^{-1/2}\right) \\ &= \frac{-\tan\ x^{1/2}}{2x^{1/2}}\end{aligned}$$

General inverses are not all that interesting. We are more interested in particular TRANSCENDENTAL inverse functions, like the Ln. Another particular kind of inverse function that bears more study is the Trig Inverse Function. Interestingly, as with the Log functions, the derivatives of these Transcendental Functions become Algebraic Functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|\ u\ |\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|\ u\ |\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{u^2+1} \cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{u^2+1} \cdot D_u$$

Ex 6 $\frac{d}{dx}\left[\tan^{-1} 3x^4\right]$

$$\begin{aligned}\frac{d}{dx}\left[\tan^{-1} 3x^4\right] &= \frac{1}{\left(3x^4\right)^2+1} \cdot \left(12x^3\right) \\ &= \frac{12x^3}{9x^8+1}\end{aligned}$$

Ex 7 $\frac{d}{dx}\left[\sec^{-1} x^2\right]$

$$\frac{d}{dx}\left[\sec^{-1} x^2\right] = \frac{1}{\left|x^2\right|\sqrt{\left(x^2\right)^2 - 1}} \cdot 2x$$
$$= \frac{2x}{\left(x^2\right)\sqrt{\left(x^2\right)^2 - 1}}$$
$$= \frac{2}{x\sqrt{x^4 - 1}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 8 If $f(x) = x^2 + 2x + 3$, $g(x) = f^{-1}(x)$, and $g(1) = 2$; find $g'(1)$.

$$f'(x) = 2x + 2 \rightarrow f'(g(x)) = 2\left[g(x)\right] + 2$$

$$\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'\left[f^{-1}(x)\right]} = \frac{1}{f'\left[g(x)\right]} = g'(x)$$

$$g'(1) = \frac{1}{f'\left[g(1)\right]} = 2\left[g(1)\right] + 2 = 6$$

1.2 Homework Set A

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin 4x$

2. $y = 4\sec x^5$

3. $y = a^3 + \cos^3 x$

4. $y = \cot^2(\sin\theta)$

5. $f(t) = \sqrt[3]{1+\tan t}$

6. $f(\theta) = \ln(\cos\theta)$

7. $y = \cos(a^3 + x^3)$

8. $y = \tan^2(3\theta)$

9. $f(x) = \cos(\ln x)$

10. $f(x) = \sqrt[5]{\ln x}$

11. $f(x) = \log_{10}(2+\sin x)$

12. $f(x) = \log_2(1-3x)$

13. $y = \sin^{-1}(e^x)$

14. $y = \tan^{-1}(\sqrt{x})$

15. $y = \sin^{-1}(2x+1)$

16. $y = \sin^{-1}(\sqrt{2}(x))$

17. $y = \csc^{-1}(x^2+1)$

18. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x$

The following table shows some values of $g(x)$, $g'(x)$, and $h(x)$, where $h(x) = g^{-1}(x)$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
1	2	3	$1/2$	$1/3$
2	5	-1	4	$1/5$
3	1	2	-2	$1/2$

19. Find $h'(1)$.

20. Find $g'(1)$

1.2 Multiple Choice Homework

1. If $y = \sin^{-1} e^{3\theta}$, then $\dfrac{dy}{d\theta} =$

a) $\dfrac{1}{\sqrt{1-e^{3\theta}}}$ b) $\dfrac{e^{3\theta}}{\sqrt{1-e^{6\theta}}}$ c) $\dfrac{e^{3\theta}}{\sqrt{1-e^{9\theta^2}}}$

d) $-3e^{3\theta}\cos^{-1} e^{3\theta}$ e) $\dfrac{3e^{3\theta}}{\sqrt{1-e^{6\theta}}}$

---

2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

a) $\sec^{-2}(\cos x)$ b) $-\sin x \sec^{-2}(\cos x)$ c) $-\csc x$

d) $\dfrac{-\cos x}{1-\sin^2 x}$ e) $\dfrac{-\sin x}{\cos(x^2)+1}$

---

3. If $h(x) = \ln(x^2)\tan^{-1}(x)$, then $h'(1) =$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{4}+1$ c) $\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2}+1$ e) $\dfrac{\pi}{2}+2$

---

4. If $f(t) = t\sqrt{1-t^2} + \cos^{-1} t$, then $f'(t) =$

a) $\dfrac{t-2}{2\sqrt{t^2-1}}$ b) $\dfrac{-2t^2}{\sqrt{1-t^2}}$ c) $\dfrac{-2t^2+2}{\sqrt{1-t^2}}$

d) $\dfrac{-1-t^2}{\sqrt{1-t^2}}$ e) $\dfrac{1-t^2}{\sqrt{1-t^2}}$

1.2 Homework Set B

1. Given the table of values below, find $g'(3)$ if $g(x)=f(h(x))\sin(h(x))$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	-7
3	$\frac{\pi}{4}$	2	$\frac{\pi}{3}$	1
$\frac{\pi}{3}$	2	3	$\frac{\pi}{4}$	-1

2. $\frac{d}{d\theta}\left[e^{\csc\theta}+\ln\left(\cot\theta^2\right)-\sec\theta\right]$

3. If $g(3)=\frac{\pi}{2}$, $g'(3)=\frac{\pi}{4}$, and $f(x)=x^3g(x)+g\left(-3\cos\left(\frac{\pi}{3}x\right)\right)-e^{\sin(g(x))}$, find $f'(3)$

4. If $z=\ln(\cot\theta)+\sec(\ln\theta)$, find $\frac{dz}{d\theta}$

5. $\frac{d}{dx}\left[\ln\left(\sec\left(x^3+5\ln x+7\right)^3\right)\right]$

6. If $z=\ln(\cos t)+\sec(e^t)+7\pi^2$, find $\frac{dz}{dt}$

7. If $z=\ln(\tan t)+\sin(e^t)+7\pi^2$, find $\frac{dz}{dt}$

8. If $z=\ln(\cos\theta)+\sin(\ln\theta)$, find $\frac{dz}{d\theta}$

9. Find $f'\left(\frac{\pi}{6}\right)$ when $f(x)=\cos^3(3x)$

10. Find $g'(2)$ when $g(x)=\ln(x^2-3)$

11. $\frac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$

12. $\frac{d}{dt}\left[\sin^5\left(\ln(7t+3)\right)\right]$

13. $\frac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$

14. $\frac{d}{dt}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$

15. $\frac{d}{dx}\left[\sec\left(5x\right)+\cot\left(e^x\right)-10\ln x\right]$

Find the first derivative for the following functions

16. $z\left(y\right)=\frac{A}{y^{10}}+Be^y$

17. $f\left(r\right)=\frac{A}{r}+Be^{\sin r}$

18. $y=\frac{x^3+3x^2-12}{x^2}$

19. $\frac{d}{dx}\left[\frac{d}{dx}\left[\sqrt{9x-27x^2+\frac{5}{x^3}}\right]\right]$

20. $\frac{d}{dx}\left[\frac{d}{dx}\left[9x-27x^2+\frac{5}{x^3}\right]\right]$

21. $\frac{d}{dx}\left[\ln\left(\tan\left(x^2+5e^x+7\right)^3\right)\right]$

22. $\frac{d}{dx}\left[\frac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$

Find the derivatives of the following functions.

23. $y=\cos^{-1}\left(e^{3z}\right)$

24. $y=\tan^{-1}\left(\sqrt{x^2-1}\right)$

26. $y=\sec^{-1}4x+\csc^{-1}4x$

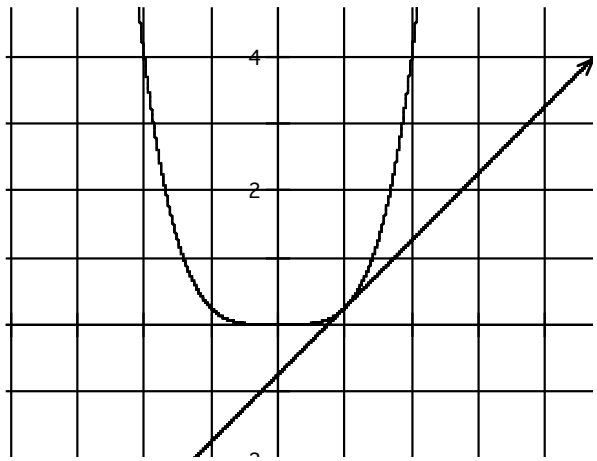
27. $f(x)=Ln\left(\tan^{-1}5x\right)$

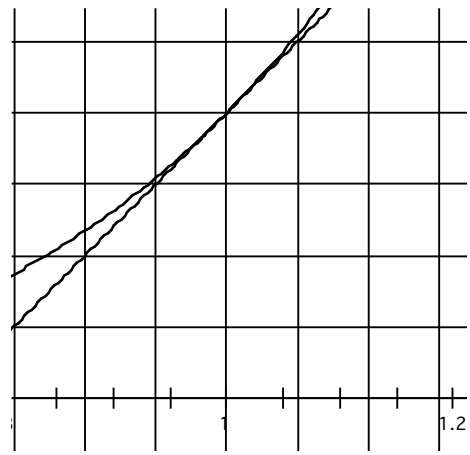
28. $g\left(w\right)=\sin^{-1}\left(5w\right)+\cos^{-1}\left(5w\right)$

29. $f(t)=\sec^{-1}\sqrt{9+t^2}$

1.3: Local Linearity, Euler's Method, and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, *y*-values on the line are very close to *y*-values on the function. This idea is called **local linearity**—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency. Consider the graphs of $y = .25x^4$ and its tangent line at $x = 1$, $y = x + .75$.





The closer you zoom in, the more the line and the curve become one. The *y*-values on the line are good approximations of the *y*-values on the curve. For a good animation of this concept, see

http://www.ima.umn.edu/~arnold/tangent/tangent.mpg

Since it is easier to find the *y*-value of a line arithmetically than for other functions—especially transcendental functions—the tangent line approximation is useful if you have no calculator.

OBJECTIVES
Use the equation of a tangent line to approximate function values.

Ex 1 Find the equations of the lines tangent and normal to

$$f(x) = x^4 - x^3 - 2x^2 + 1 \text{ at x = -1}$$

The slope of the tangent line will be f '(-1)

$$f'(x) = 4x^3 - 3x^2 - 4x$$
$$f'(-1) = -3$$

[Note that we could have gotten this more easily with the nDeriv function on our calculator.]

f(-1) = 1, so the tangent line will be

$$y - k = m(x - h)$$
$$y - 1 = -3(x + 1)$$
$$\text{or}$$
$$y = -3x - 2$$

The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope = 1/3. The normal line is

$$y - 1 = \frac{1}{3}(x + 1)$$

One of the uses of the tangent line is based on the idea of Local Linearity. This means that in small areas, algebraic curves act like lines—namely their tangent lines. Therefore, one can get an approximate y-value for points near the point of tangency by plugging x-values into the equation of the tangent line.

Ex 2 Use the tangent line equation found in Example 5 to get an approximate value of f (-0.9).

While we can find the Exact value of f(-0.9) with a calculator, we can get a quick approximation from the tangent line. If x=-.9 on the tangent line, then

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = .7$$

This last example is somewhat trite in that we could have just plugged -0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but it is doable. Consider the next example, though.

Ex 3 Find the tangent line equation to $f(x) = e^{2x}$ at $x = 0$ and use it to approximate value of $e^{.2}$.

Without a calculator, we could not find the exact value of $e^{.2}$. In fact, even the calculator only gives an approximate value.

$$f'(x) = 2e^{2x} \text{ and } f'(0) = 2e^{2x} = 2$$
$$f(0) = e^0 = 1$$

So the tangent line equation is $y - 1 = 2(x - 0)$ or $y = 2x + 1$

$$e^{.2} \approx 2(.2) + 1 = 1.2$$

Note that the value that you get from a calculator for $e^{0.2}$ is 1.221403… Our approximation of 1.2 seems very reasonable.

Though not as practically useful (in 2 dimensions) as the tangent lines, another context for the derivative is in finding the equation of the normal line.

Normal line--Defn: The line perpendicular to a curve.

Ex 4 Find the equation of the line normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at x = -1.

In Ex 1, we saw that the slope of the tangent line was $m_{\tan} = f'(-1) = -3$. Therefore, $m_{normal} = f'(-1) = \frac{1}{3}$, and the equation of the line is

$$y - 1 = \frac{1}{3}(x + 1) \quad \text{or} \quad y = \frac{1}{3}x - \frac{4}{3}$$

<u>Euler's Method</u>

Vocabulary:

Euler's method – an algorithm used to approximate numerical values to the solution of a differential equation.

OBJECTIVES:
Use Euler's Method to approximate a numerical solution to a differential equation at a given point.

Last year we learned a bit on approximations with tangent lines. Euler's method is just a better approximation method. It uses more than one tangent line to get the job done.

The process is similar to approximating with tangent lines. We use $\frac{dy}{dx}$ to find a tangent line, then use that tangent line to find an approximate value for y. We then use that y value and another x value to create another "tangent line". Of course, it isn't actually a tangent line because our y value wasn't actually on the curve. We then repeat the process until we get to the value we want to approximate.

Euler's Method:

1) Identify your starting point and step size.
2) Use $\frac{dy}{dx}$ to find the slope and make a tangent line.
3) Find an approximate y value by plugging in x + "1 step size" to the tangent line.
4) Use the approximate y value, and the next x step over to make a new tangent line.
5) Repeat steps 3 and 4 until you reach your final x value – the one you actually want an approximation for.

Ex 5 Use Euler's method with a step size of 0.5 to estimate $f(3)$, where $f(x)=\ln(x)$.

$f(1)=\ln(1)=0$ I chose to start at 1, because I know $\ln(1)=0$

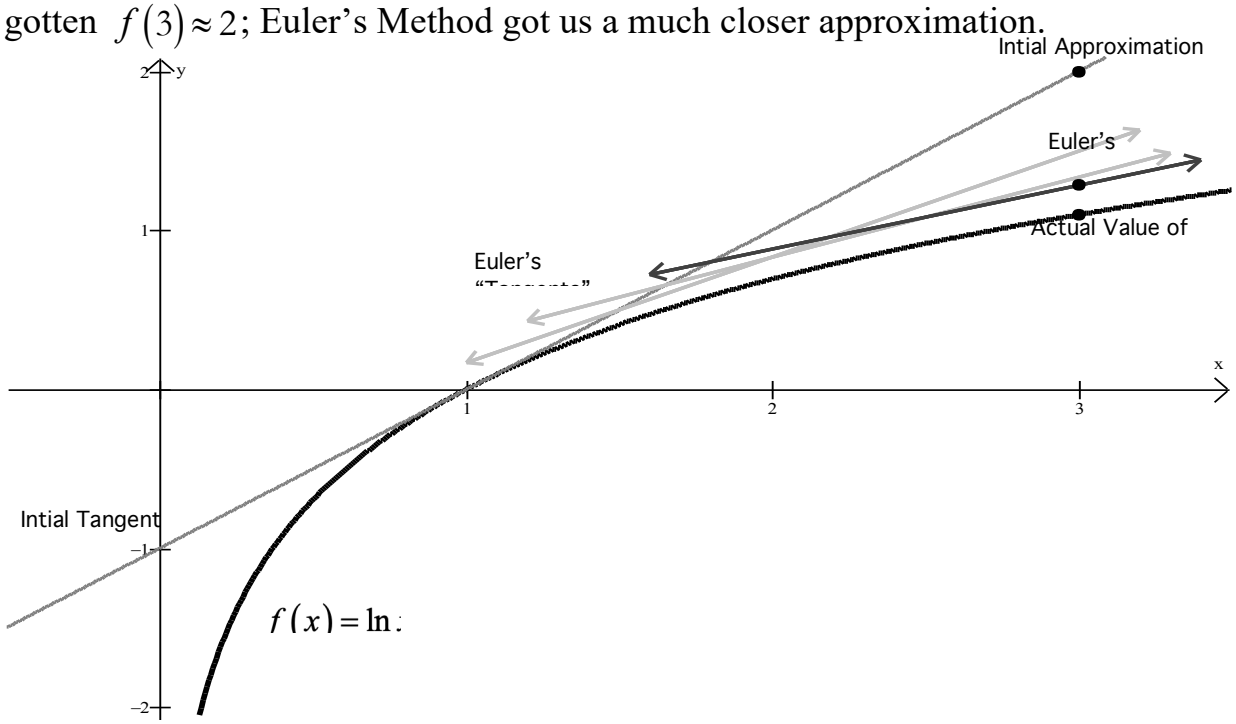
$f'(x)=\frac{1}{x}$ We start by taking the derivative.

Note that in the chart below, we are getting our "New *y*" from our tangent line. Our "New *x*" comes from the next step in the *x*. Our new slope comes from plugging in the new *x* into $f'(x)$.

Point	$f'(x)$	Tangent Line	New *x*	New *y*
(1, 0)	1	$y-0=1(x-1)$	1.5	0.5
(1.5, 0.5)	$\frac{2}{3}$	$y-0.5=\frac{2}{3}(x-1.5)$	2.0	$\frac{5}{6}$
$(2, \frac{5}{6})$	$\frac{1}{2}$	$y-\frac{5}{6}=\frac{1}{2}(x-2)$	2.5	$\frac{13}{12}$
$(2.5, \frac{13}{12})$	$\frac{2}{5}$	$y-\frac{13}{12}=\frac{2}{5}(x-2.5)$	3.0	$\frac{77}{60}$

So $f(3)\approx y(3)=\frac{77}{60}$ or $1.28\overline{3}$. By way of comparison, $\ln(3)\approx 1.0986$

If we had just used the initial tangent line to get an approximation, we would have gotten $f(3)\approx 2$; Euler's Method got us a much closer approximation.



We could also use this process to approximate a value for a curve when we only know its derivative and an initial value on the curve.

Ex 6

Use Euler's method with a step size of 0.5 to estimate $f(2.5)$ for the function whose derivative is given by $\frac{dy}{dx} = 2x + y$ with an initial value of $f(1) = 4$.

Point	$f'(x)$	Tangent Line	New x	New y
(1,4)	6	$y - 4 = 6(x - 1)$	1.5	7
(1.5,7)	10	$y - 7 = 10(x - 1.5)$	2.0	12
(2, 12)	16	$y - 12 = 16(x - 2)$	2.5	20

$f(2.5) \approx y(2.5) = 20$. We cannot get an actual value for this function, because we have no techniques that allow us to solve this differential equation.

Ex 7

Is the approximation in Example 2 an overestimate or an underestimate? Why?
To determine this, we need to look at the concavity of the curve – this requires the 2nd derivative.

$$\frac{dy}{dx} = 2x + y$$

$$\frac{d^2y}{dx^2} = 2 + \frac{dy}{dx}$$ Don't forget implicit differentiation.

$$\frac{d^2y}{dx^2} = 2 + 2x + y$$

$$\left.\frac{d^2y}{dx^2}\right|_{(1,4)} = 2 + 2(1) + 4 = 8$$ Plug in our initial value.

Since the second derivative is positive, our curve is concave up at this point, which means our tangent line is under the curve, so we should have an underestimate.

Generally:

- Your approximation will be an overestimate if the curve is concave down (since your "tangent lines" will be above the curve).
- Your approximation will be an underestimate if the curve is concave up (since your "tangent lines" will be below the curve).

Ex 8 Use Euler's method taking four steps to approximate $f(2)$ for $\frac{dy}{dx} = 3y - x$, given $f(0) = 1$. Is this an overestimate or an underestimate?

Point	$\frac{dy}{dx}$	Tangent Line	New x	New y
(0, 1)				
(0.5,)				
(1,)				
(1.5,)				

Since $\frac{d^2y}{dx^2} = 3\frac{dy}{dx} - 1 = 9y - 3x - 1$,

$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = 9(1) - 3(0) - 1 = 8$, so this will be an underestimate, because y is concave up at (0,1).

1.3 Homework Set A

1. Find the equation of the tangent line to $f(x) = x^5 - 5x + 1 \ at \ x = -2$ and use it to get an approximate value of f (-1.9).

2. Find all points on the graph of $y = 2\sin x + \sin^2 x$ where the tangent line is horizontal.

3. Find an equation of the line tangent to the curve $y = x^4 + 2e^x$ at the point $(0, 2)$.

4. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

5. Find the equation of the tangent line to $y = x + \cos x$ at the point $(0,1)$.

6. Find the equation of the tangent line to $y = \sec x - 2\cos x$ at the point $\left(\pi/3, 1\right)$.

7. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

8. Use Euler's Method with 2 equal step sizes to find an approximation for $f(0)$, given that $f(-1) = 2$ and $\frac{dy}{dx} = 6x^2 - x^2 y$

9. Use Euler's Method with 4 equal step sizes to find an approximation for $f(1.4)$, given that $f(x) = \ln(2x - 1)$ and $f(1) = 0$.

10. Use Euler's Method with 3 equal step sizes to find an approximation for $f(2.6)$, given that $f(2) = -2$ and $\frac{dy}{dx} = 2x + y$

11. Given the differential equation, $\frac{dy}{dx} = \frac{y}{8}(6 - y)$ where $y = f(t)$ and $f(0) = 8$, use Euler's Method with two steps of equal size to approximate $f(1)$.

1.3 Multiple Choice Homework

1. Let f be the function given by $f(x) = 2e^{4x^2}$. For what value of x is the slope of the line tangent to the graph of f at $(x, f(x))$ equal to 3?

a) 0.168 b) 0.274 c) 0.318 d) 0.342 e) 0.551

---

2. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

a) $y = -3x - 2$

b) $y = -3x + 4$

c) $y = -x + 0.905$

d) $y = -x + 0.271$

e) $y = -x - 0.271$

---

3. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent parallel to the line $2x - 4y = 3$

a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ b) $\left(\frac{1}{2}, \frac{1}{8}\right)$ c) $\left(\frac{1}{2}, -\frac{1}{4}\right)$

d) $\left(1, -\frac{1}{2}\right)$ e) $(2, 2)$

---

4. A normal line to the graph of a function f at the point $(x, f(x))$ is defined to be the line perpendicular to the tangent line at that point. An equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

a) $y + 12x = 38$
b) $y - 4x = 10$
c) $y + 2x = 4$
d) $y + 2x = 8$
e) $y - 2x = -4$

---

5. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = \frac{4x}{y}$ with the initial condition $f(0) = 1$. What is the best approximation for $f(1)$ if Euler's method is used, starting at $x = 0$ with a step size of 0.5 ?

a) 1 b) 2 c) $\sqrt{5}$ d) 2.5 e) 3

---

6. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = x - y^2$ with the initial condition $f(0) = 1$. What is the best approximation for $f(2)$ if Euler's method is used, starting at $x = 0$ with a step size of 1.0 ?

a) -1 b) 0 c) 1 d) 2 e) 3

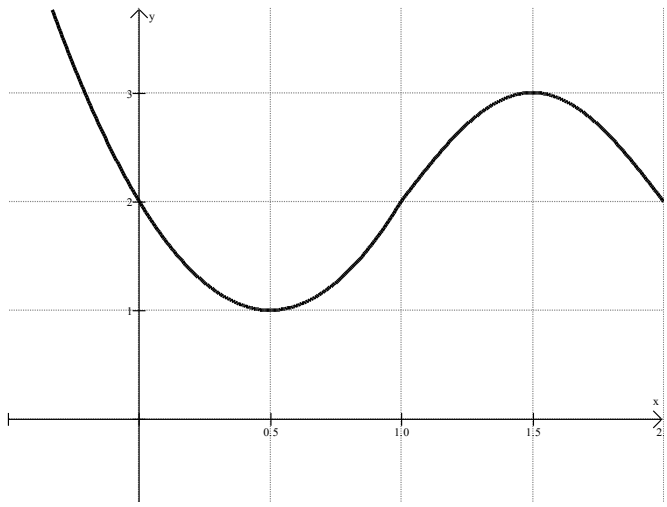
---

7. Let $y = f(x)$ be the solution to the differential equation $\frac{dy}{dx} = y - x$ with the initial condition $f(1) = 2$. What is the best approximation for $f(2)$ if Euler's method is used, starting at $x = 1$ with a step size of 0.5?

a) 1 b) 2 c) 3 d) 4.5 e) 6

---

8. The graph of $y = f'(x)$ is given below. Use this information and the fact that $f(0) = 3$ to find an approximate value for $f(1)$ using Euler's method with 2 equal step sizes.



a) 2.5 b) 3.5 c) 4 d) 4.5 e) 5

---

x	$f'(x)$
-1.0	2
-0.5	4
0	3
0.5	1
1.0	0
1.5	-3
2.0	-6

9. The table above gives selected values for the derivative of a function g on the interval $-1 \le x \le 2$. If $g(-1) = -2$ and Euler's method with a step-size of 1.5 is used to approximate $g(2)$, what is the resulting approximation?

a) -6.5 b) -1.5 c) 1.5 d) 2.5 e) 3

1.4: Product and Quotient Rules

Remember:

The Product Rule: $f'(x) = U \cdot \frac{dv}{dx} + V \cdot \frac{du}{dx}$

The Quotient Rule: $f'(x) = \frac{V \cdot \frac{du}{dx} - U \cdot \frac{dv}{dx}}{V^2}$

OBJECTIVE

Find the Derivative of a product or quotient of two functions.

<u>Product Rule</u>

Ex 1 $\frac{d}{dx}\left(x^2 \sin\ x\right)$

$$\frac{d}{dx}\left(x^2 \sin\ x\right) = x^2 \cos x + (\sin x)(2x)$$
$$= x^2 \cos x + 2x \sin x$$

Ex 2 $\frac{d}{dx}\left(5^x \cos x\right)$

$$\frac{d}{dx}\left(5^x \cos\ x\right) = 5^x\left(-\sin x\right) + \cos\ x\left(5^x \ln 5\right)$$
$$= 5^x\left(\left(\ln 5\right)\cos x - \sin x\right)$$

The Product Rule is pretty straight forward. The tricky part is simplifying the Algebra.

Ex 3 If $f(x) = x^2 e^{-x/2}$, find $f'(x)$.

$$u = x^2 \qquad v = e^{-x/2}$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = e^{-x/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-x/2}$$

$$f'(x) = x^2\left(-\frac{1}{2}e^{-x/2}\right) + e^{-x/2}(2x)$$

$$= xe^{-x/2}\left(-\frac{1}{2}x + 2\right)$$

Ex 4 $\frac{d}{dx}\left[x\sqrt{1-x^2}\right]$

$$u = x \qquad v = \sqrt{1-x^2} = \left(1-x^2\right)^{1/2}$$

$$D_u = 1 \qquad D_v = \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = \frac{-x}{\left(1-x^2\right)^{1/2}}$$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = (x)\frac{-x}{\left(1-x^2\right)^{1/2}} + \left(1-x^2\right)^{1/2}(1)$$

$$= \frac{-x^2 + \left(1-x^2\right)}{\left(1-x^2\right)^{1/2}}$$

$$= \frac{1-2x^2}{\left(1-x^2\right)^{1/2}}$$

Ex 5 $\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]$

$$u=(2x-3)^8 \qquad v=\left(3x^2-1\right)^7$$

$$\frac{du}{dx}=8(2x-3)^7(2) \qquad \frac{dv}{dx}=7\left(3x^2-1\right)^6(6x)$$

$$=16(2x-3)^7 \qquad =42x\left(3x^2-1\right)^6$$

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]=(2x-3)^8\,42x\left(3x^2-1\right)^8+\left(3x^2-1\right)^7 16(2x-3)^7$$

This, then, is factorable:

$$\begin{aligned}\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]&=42x(2x-3)^8\left(3x^2-1\right)^6+16\left(3x^2-1\right)^7(2x-3)^7\\&=2(2x-3)^7\left(3x^2-1\right)^6\left[21x(2x-3)+8\left(3x^2-1\right)\right]\\&=2(2x-3)^7\left(3x^2-1\right)^6\left[42x^2-63x+24x^2-8\right]\\&=2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2-63x-8\right)\end{aligned}$$

Remember that, in section 1.1, we said we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the Exponent. We can now address that situation.

Ex 6 $\frac{d}{dx}\left[(\cos\ x)^{x^2}\right]$

$$\begin{aligned}\frac{d}{dx}\left[(\cos\ x)^{x^2}\right]&=\frac{d}{dx}\left(e^{x^2\ln\cos\ x}\right)\\&=e^{x^2\ln\cos\ x}\left(x^2\frac{1}{\cos\ x}(-\sin\ x)+(\ln\cos\ x)2x\right)\\&=(\cos\ x)^{x^2}\left(2x\ln\cos\ x-x^2\tan x\right)\end{aligned}$$

Quotient Rule

Ex 7 $\frac{d}{dx}\left(\frac{6x}{x^2+4}\right)$

$$u = 6x, \text{ so } \frac{du}{dx} = 6$$
$$v = x^2+4, \text{ so } \frac{dv}{dx} = 2x$$

$$f'(x) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2} = \frac{\left(x^2+4\right)(6)-(6x)(2x)}{\left(x^2+4\right)^2}$$
$$= \frac{6x^2+24-12x^2}{\left(x^2+4\right)^2}$$
$$= \frac{24-6x^2}{\left(x^2+4\right)^2}$$

Ex 8 $\frac{d}{dx}\left(\frac{x^2+\ 2x-3}{x-4}\right)$

$$u = x^2+2x-3, \text{ so } \frac{du}{dx} = 2x+2$$
$$v = x-4, \text{ so } \frac{dv}{dx} = 1$$

$$f'(x) = \frac{v\cdot\frac{du}{dx} - u\cdot\frac{dv}{dx}}{v^2} = \frac{(x-4)\cdot(2x+2)-(x^2+2x-3)\cdot 1}{(x-4)^2}$$
$$= \frac{2x^2-6x-8-x^2-2x+3}{(x-4)^2}$$
$$= \frac{x^2-8x-5}{(x-4)^2}$$

Ex 9 $\dfrac{d}{dx}\left(\dfrac{x^2-4x+3}{2x^2-5x-3}\right)$

Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right) &= \frac{d}{dx}\left(\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right)\\ &= \frac{d}{dx}\left(\frac{x-1}{2x+1}\right)\\ &= \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)^2}\\ &= \frac{3}{(2x+1)^2}\end{aligned}$$

Ex 10 $\dfrac{d}{dx}\left(\dfrac{\cot\ 3x}{x^2+1}\right)$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cot\ 3x}{x^2+1}\right) &= \frac{\left(x^2+1\right)\left(-\csc^2 3x\right)(3)-\left(\cot\ 3x\right)2x}{\left(x^2+1\right)^2}\\ &= \frac{-3x^2\csc^2 3x-3\csc^2 3x-2x\ \cot\ 3x}{\left(x^2+1\right)^2}\end{aligned}$$

As with the Product Rule, the difficulty with the Quotient Rule arises from the Algebra needed to simplify our answer.

Ex 11 If $y=\dfrac{4x}{\sqrt{x^2+4}}$, find $\dfrac{dy}{dx}$.

$$u=4x \qquad\qquad v=\left(x^2+4\right)^{1/2}$$

$$\frac{du}{dx}=4 \qquad\qquad \frac{dv}{dx}=\frac{1}{2}\left(x^2+4\right)^{-1/2}(2x)=\frac{x}{\left(x^2+4\right)^{1/2}}$$

$$\begin{aligned}
\frac{dy}{dx}&=\frac{\left(x^2+4\right)^{1/2}(4)-(4x)\left(\dfrac{x}{\left(x^2+4\right)^{1/2}}\right)}{\left(x^2+4\right)^{1}}\\
&=\frac{\dfrac{4\left(x^2+4\right)}{\left(x^2+4\right)^{1/2}}-(4x)\left(\dfrac{x}{\left(x^2+4\right)^{1/2}}\right)}{\left(x^2+4\right)^{1}}\\
&=\frac{4x^2+16-4x^2}{\left(x^2+4\right)^{3/2}}\\
&=\frac{16}{\left(x^2+4\right)^{3/2}}
\end{aligned}$$

Ex 12 Find the equation of the tangent to $f(x)=\dfrac{x}{\sqrt{x^2+9}}$ at $x=-\sqrt{7}$.

As we recall, for the equation of a line, we need a point and a slope:

The point: $f\left(-\sqrt{7}\right)=\dfrac{-\sqrt{7}}{\sqrt{7+9}}=-\dfrac{\sqrt{7}}{4}$

The slope is the derivative at the given x-value:

$$\frac{dy}{dx}=\frac{\left(x^2+9\right)^{1/2}(1)-(x)\left(\dfrac{x}{\left(x^2+9\right)^{1/2}}\right)}{\left(x^2+9\right)^1}$$

Rather than simpifying the algebra, find the slope by substituting $x=-\sqrt{7}$:

$$m_{\tan}=\frac{\left(7+9\right)^{1/2}(1)-\left(-\sqrt{7}\right)\left(\dfrac{-\sqrt{7}}{\left(7+9\right)^{1/2}}\right)}{\left(7+9\right)}=\frac{4-\dfrac{7}{4}}{16}=\frac{9}{64}$$

The tangent line equation:

$$y+\frac{\sqrt{7}}{4}=\frac{9}{64}\left(x+\sqrt{7}\right)$$

1.4 Homework Set A

Find the derivative of the following functions.

1. $y = t^3 \cos t$

2. $g(x) = (1+4x)^5 (3+x-x^2)^8$

3. $y = \dfrac{\tan x - 1}{\sec x}$

4. $y = (2x-5)^4 (8x^2-5)^{-3}$

5. $y = xe^{-x^2}$

6. $y = \dfrac{\sin x}{x^2}$

7. $y = e^{x \cos x}$

8. $y = \dfrac{r}{\sqrt{r^2+1}}$

9. $y = x \sin \dfrac{1}{x}$

10. $y = e^{-5x} \cos 3x$

11. $f(x) = x\sqrt{\ln x}$

12. $y = \ln\left(e^{-x} + xe^{-x}\right)$

13. Find the equation of the line tangent to $y = x^2 e^{-x}$ at the point $\left(1, 1/e\right)$.

14. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

15. $h(t) = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$, find $h'(t)$

16. $y = x^2\sqrt{5-x^2}$, find $y'(1)$

17. $f(x) = \left[x \sin(2x) + \tan^4\left(x^7\right)\right]^5$, find $f'(x)$.

18. Find the equation of the lines tangent and normal to $y = x \sin\left(\dfrac{\pi}{2} \ln x\right)$ when $x = e$

19. Find the equation of the line tangent to $y = e^{x \sin(4x)} + 2$ when $x = 0$

20. Find the equation of the lines tangent and normal to $y = x\sin\left(\frac{1}{x}\right)$ when $x = \frac{4}{\pi}$.

21. $H(x) = \left(1 + x^2\right)\tan^{-1}(x)$

22. $y = x\cos^{-1}x - \sqrt{1 - x^2}$

23. If $f(x) = e^x - x^2 \arctan x$, find $f'(x)$.

24. $y = \cos^{-1}\ x + x\sqrt{1 - x^2}$

25. $y = \frac{\sec^{-1}\ x}{x}$

26. $y = Ln\ \left(x^2 + 4\right) - x\ \tan^{-1}\left(\frac{x}{2}\right)$

<u>1.4 Multiple Choice Homework</u>

1. If $y = x^2\cos 2x$, then $\frac{dy}{dx} =$

a) $-2x\sin 2x$ b) $-4x\sin 2x$ c) $2x\left(\cos 2x - \sin 2x\right)$

d) $2x\left(\cos 2x - x\sin 2x\right)$ e) $2x\left(\cos 2x + \sin 2x\right)$

---

2. If $x(t) = 2t\cos t^2$, find $x'(t)$.

a) $x(t) = -4t^2\sin t^2$ b) $x(t) = -4t^2\sin t^2 + 2\cos t^2$

c) $x(t) = \sin t^2 + 3$ d) $x(t) = -\sin t^2 + 4$

e) $x(t) = \sin t^2 + 2$

---

3. If $f(x) = x\tan x$, then $f'\left(\frac{\pi}{4}\right) =$

a) $1-\frac{\pi}{2}$ b) $1+\frac{\pi}{2}$ c) $1+\frac{\pi}{4}$

d) $1-\frac{\pi}{4}$ e) $\frac{\pi}{2}-1$

---

4. If f is a function that is differentiable throughout its domain and is defined by $f(x) = \frac{1+e^x}{\sin\left(x^2\right)}$, then the value of $f'(0)=$

a) -1 b) 0 c) 1 d) e e) nonexistent

---

5. If $y = \frac{5x-4}{4x-5}$, then $\frac{dy}{dx} =$

a) $\frac{-9}{(4x-5)^2}$ b) $\frac{9}{(4x-5)^2}$ c) $\frac{40x-41}{(4x-5)^2}$

d) $\frac{40x+41}{(4x-5)^2}$ e) $\frac{5}{4}$

---

6. If $y = \frac{3-2x}{3x+2}$, then $\frac{dy}{dx} =$

a) $\frac{12x+2}{(3x+2)^2}$ b) $\frac{12x-2}{(3x+2)^2}$ c) $\frac{13}{(3x+2)^2}$

d) $\frac{-13}{(3x+2)^2}$ e) $-\frac{2}{3}$

---

7. If $y = \dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx} =$

a) $\dfrac{-6x}{\left(4+x^2\right)^2}$

b) $\dfrac{3x}{\left(4+x^2\right)^2}$

c) $\dfrac{6x}{\left(4+x^2\right)^2}$

d) $\dfrac{-3x}{\left(4+x^2\right)^2}$

e) $\dfrac{3}{2x}$

---

8. An equation of the line normal to the graph of $y = \dfrac{3x+4}{4x-3}$ at $(1, 7)$ is

a) $25x + y = 32$

b) $25x - y = 18$

c) $7x - y = 0$

d) $x - 25y = -174$

e) $x + 25y = 176$

---

9. Let $f(x)$ and $g(x)$ be differentiable functions. The table below gives the values of $f(x)$ and $g(x)$, and their derivatives, at several values of x.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	-6
2	1	8	-5	7
3	7	-2	7	9

If $h(x)=\dfrac{f(x)}{g(x)}$, what is the value of $h'(2)$?

a) -4 b) -63 c) 51 d) $-\dfrac{47}{64}$ e) $-\dfrac{33}{64}$

---

10. Which of the following statements must be true?

I. $\dfrac{d}{dx}(x\tan x)=x\tan x+x\sec^2 x$ II. $\dfrac{d}{dx}\left(\dfrac{3}{4+x^2}\right)=\dfrac{-6x}{\left(4+x^2\right)^2}$

III. $\dfrac{d}{dx}\sqrt{1-x}=\dfrac{1}{2\sqrt{1-x}}$

a) I only b) II only c) III only

d) I and II only e) I, II, and III

1.4 Homework Set B

1. Find the first derivative for the following function: $x(t)=e^{t^2}\sin\left(t^2-5t^4\right)$

2. Find the first derivative for the following function: $x(t)=e^{5t}\tan\left(3t^4\right)$

3. Find the first derivative for the following function: $y=\dfrac{x^2+2x-15}{x-3}$

4. Find the first derivative for the following function: $x(t)=e^t\left(t^2-5t^4\right)$

5. $\dfrac{d}{dx}\left[\dfrac{e^x+7x^2+5}{\sin x^3}\right]$

6. $\dfrac{d}{dy}\left[e^{(\sin y)}\ln\left(\cot\left(e^y\right)\right)\right]$

7. $\dfrac{d}{dx}\left[x^2\sin x^2+\dfrac{x+1}{\ln x}\right]$

8. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=\left(h(x)\right)^4+x\ln\left(h(x)\right)$

9. Find $g'(z)$ if $g(z)=\left(\dfrac{e^{5z}}{1+\ln z}\right)^{118}$

10. $\dfrac{d}{dx}\left[x^2\cos x^2+\dfrac{e^x}{x}\right]$

11. $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

12. $\dfrac{d}{dx}\left[\dfrac{\cos\left(x^2-3\right)}{e^{-5x}}\right]$

13. $\dfrac{d}{dx}\left[x^5\ln(5x+4)+\dfrac{x}{\ln x}\right]$

14. Find $g'(t)$ if $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$

15. $\dfrac{d}{dx}\left[e^{x^2}\cos x\right]$

16. $\dfrac{d}{dx}\left[\dfrac{1+\tan x}{\ln 4x}\right]$

17. $\dfrac{d}{dt}\left[\sin t\tan t\right]$

18. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{\csc x}\right]$

19. $\dfrac{d}{dx}\left[e^{5x^4}\ln(\sin x)\right]$

20. $\dfrac{d}{dp}\left[5p\sin p+e^{2p}-\ln\left(3p^2+1\right)+\dfrac{p}{p^2+1}\right]$

21. $\frac{d}{dx}\left[\tan\left(e^x\right)\left(x^4-5x^3+x\right)\right]$

22. $\frac{d}{dx}\left[\frac{5x+2}{\ln\left(3x+7\right)}\right]$

23. $\frac{d}{dx}\left[\frac{x-4}{x^2-9x+20}\right]$

24. $\frac{d}{dx}\left[\frac{d}{dx}\left[\sin^2\left(4x+2\right)\right]\right]$

25. $\frac{d}{dc}\left[\frac{c^5-12c^3-19c}{3c^3}\right]$

3. $y=4\sin^{-1}\left(\frac{1}{2}x\right)+x\sqrt{4-x^2}$

4. $y=\cos^{-1}\left(\frac{x-1}{x+1}\right)$

6. $f(t)=c\sin^{-1}\frac{t}{c}-\sqrt{c^2-t^2}$

7. $f(x)=x^2\arccos x$

11. $y=Ln\ \left(u^2+1\right)-u\cot^{-1}(u)$

12. $y=\tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$

1.5: Higher Order Derivatives

What we have been calling the Derivative is actually the First Derivative. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. In this section, we will Explore the process of finding the higher order derivatives.

Second Derivative--Defn: The derivative of the derivative.

Just as with the First Derivative, there are several symbols for the 2nd Derivative:

<u>Higher Order Derivative Symbols</u>

Liebnitz: $\frac{d^2y}{dx^2}$= d squared y, d x squared; $\frac{d^3y}{dx^3}$;… $\frac{d^ny}{dx^n}$

Function: $f''(x)$= f double prime of x; $f'''(x)$**;** $f^{IV}(x)$;… $f^n(x)$

Combination: y'' = y double prime

OBJECTIVE

Find higher order derivatives.

Ex 1 $\frac{d^2}{dx^2}\left[x^4-7x^3-3x^2+2x-5\right]$

$$\frac{d^2}{dx^2}\left[x^4-7x^3-3x^2+2x-5\right]=\frac{d}{dx}\left[\frac{d}{dx}\left[x^4-7x^3-3x^2+2x-5\right]\right]$$

$$=\frac{d}{dx}\left[4x^3-21x^2-6x+2\right]$$

$$=12x^2-42x-6$$

Ex 2 Find $\frac{d^3y}{dx^3}$ if $y = \sin 3x$

$$
\begin{aligned}
y &= \sin 3x \\
\frac{dy}{dx} &= \cos 3x \cdot 3 = 3\cos 3x \\
\frac{d^2y}{dx^2} &= 3(-\sin 3x)\cdot 3 = -9 \sin 3x \\
\frac{d^3y}{dx^3} &= -9 \cos 3x \cdot 3 = -27 \cos 3x
\end{aligned}
$$

More complicated functions, in particular Composite Functions, have a complicated process. When the Chain Rule is applied, the answer becomes a product or quotient. Therefore, the 2nd Derivative will require the Product or Quotient Rules as well as, possibly, the Chain Rule again.

Ex 3 $y = e^{3x^2}$, find y".

$$
\begin{aligned}
\frac{dy}{dx} &= e^{3x^2}\cdot 6x = 6xe^{3x^2} \\
\frac{d^2y}{dx^2} &= 6x\left(e^{3x^2}\cdot 6x\right) + e^{3x^2}\cdot 6 \\
&= 36x^2e^{3x^2} + 6e^{3x^2} \\
&= 6e^{3x^2}\left(6x^2+1\right)
\end{aligned}
$$

Ex 4 $y = \sin^3 x$, find y"

$$
\begin{aligned}
y' &= 3\sin^2 x \cdot \cos x \\
y'' &= 3\sin^2 x(-\sin x) + \cos x(6\sin x \cdot \cos x) \\
&= 3\sin x\left(2\cos^2 x - \sin^2 x\right)
\end{aligned}
$$

Ex 5 $f(x) = Ln\left(x^2 + 3x - 1\right)$, find f''(x).

$$f'(x) = \frac{1}{x^2 + 3x - 1}(2x + 3) = \frac{2x + 3}{x^2 + 3x - 1}$$

$$f''(x) = \frac{\left(x^2 + 3x - 1\right)(2) - (2x + 3)(2x + 3)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \frac{\left(2x^2 + 6x - 2\right) - \left(4x^2 + 12x + 9\right)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \frac{-2x^2 - 6x - 11}{\left(x^2 + 3x - 1\right)^2}$$

Ex 6 $g(x) = \sqrt{4x^2 + 1}$, find g''(x).

$$g'(x) = \frac{1}{2}\left(4x^2 + 1\right)^{-1/2}(8x) = \frac{4x}{\left(4x^2 + 1\right)^{1/2}}$$

$$g''(x) = \frac{\left(4x^2 + 1\right)^{1/2}(4) - (4x)\left[\frac{1}{2}\left(4x^2 + 1\right)^{-1/2}(8x)\right]}{\left[\left(4x^2 + 1\right)^{1/2}\right]^2}$$

$$= \frac{\left(4x^2 + 1\right)^{1/2}(4) - \frac{16x^2}{\left(4x^2 + 1\right)^{1/2}}}{\left(4x^2 + 1\right)}$$

$$= \frac{\left(4x^2 + 1\right)(4) - 16x^2}{\left(4x^2 + 1\right)^{3/2}}$$

$$= \frac{4}{\left(4x^2 + 1\right)^{3/2}}$$

1.5 Homework Set A

In #1-8, find the second derivative of the function.

1. $f(x) = x^5 + 6x^2 - 7x$

2. $h(x) = \sqrt{x^2 + 1}$

3. $y = \left(x^3 + 1\right)^{2/3}$

4. $H(t) = \tan 3t$

5. $g(t) = t^3 e^{5t}$

6. $y = e^{3x^2}$

7. $y = \sin^3 x$

8. $f(t) = t \cos t$

9. $\dfrac{d^2}{dx^2}\left[5x^4 + 9x^3 - 4x^2 + x - 8\right]$

10. $\dfrac{d^2}{dx^2}\left[4x^7 - 3x^5 + 3x^3 + 6x - 1\right]$

11. $y = \cos x^2$, find y''

12. $y = \tan^2 x$, find y''

13. $y = \sec 3x$, find $\dfrac{d^2y}{dx^2}$

14. $y = xe^{2x}$, find $\dfrac{d^2y}{dx^2}$

15. $f(x) = \ln\left(x^2 + 3\right)$, find $f''(x)$

16. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g''(x)$

17. $h(x) = \sqrt{x^2 + 5}$, find $h''(x)$

18. $F(x) = \sqrt{3x^2 - 2x + 1}$, find $F''(x)$

19. $y = \dfrac{x^2 - 3}{x^2 - 10}$, find $\dfrac{d^2y}{dx^2}$

20. $y = \dfrac{3x + 3}{x^3 + 1}$, find $\dfrac{d^2y}{dx^2}$

21. $y = x^3 + x^2 - 7x - 15$

22. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

23. $y = \dfrac{-4x}{x^2 + 4}$

24. $y = \dfrac{x^2 - 1}{x^2 - 4}$

25. $y = x\sqrt{8 - x^2}$

26. $y = \dfrac{1}{2}x + \sin x$

27. $y = xe^{-x}$

28. $y = e^{-x^2}$

29. $y = \dfrac{x}{x^2 - 9}$

30. $y = 2x - x^{2/3}$

1.5 Multiple Choice Homework

1. If f and g are twice differentiable and if $h(x) = g\left(f(x)\right)$, then $h''(x) =$

a) $g''\left(f(x)\right)$

b) $g''\left(f(x)\right)f''(x)$

c) $g''\left(f(x)\right)\left[f'(x)\right]^2$

d) $g'\left(f(x)\right)\left[f'(x)\right]^2 + f'(x)\left(f''(x)\right)$

e) $g'\left(f(x)\right)f''(x) + \left[f'(x)\right]^2 g''\left(f(x)\right)$

---

2. Find $\dfrac{d^2y}{dx^2}$ if $y = \dfrac{x+2}{x-3}$

a) $\dfrac{-2}{(x-3)^2}$ b) 0 c) $\dfrac{10}{(x-3)^3}$ d) $\dfrac{2}{(x-3)^2}$ e). None of these

---

3. If $y = \ln(\cos x)$ and $0 \le x \le \pi$, then $\dfrac{d^2y}{dx^2}$ is

a) $-\tan x$

b) $-\sec^2 x$

c) $\tan x$

d) $\sec^2 x$

e) $\sec x \tan x$

---

4. If $y = \ln\left(x^2 + 4\right)$, then $\dfrac{d^2y}{dx^2}$ is

a) $\dfrac{1}{x^2+4}$ b) $\dfrac{2x}{x^2+4}$ c) $\dfrac{-2x^2+8}{x^2+4}$

d) $\dfrac{2x}{\left(x^2+4\right)^2}$ e) $\dfrac{-2x^2+8}{\left(x^2+4\right)^2}$

---

5. If $y = e^{x^2}$, then $\dfrac{d^2y}{dx^2} =$

a) e^{x^2} b) $2e^{x^2}\left(2x^2+1\right)$ c) $2xe^{x^2}$

d) $4x^2e^{x^2}$ e) $2e^{x^2}\left(2x^2-1\right)$

1.6: Implicit Differentiation

One of the more useful aspects of the chain rule that we reviewed earlier is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of y with respect to x – that is, $\frac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra.

Ex 1 Find the $\frac{dy}{dx}$ for the function $y = 5x^4 - 22x$

$$\frac{d}{dx}\left[y = 5x^4 - 22x\right]$$

$$\frac{dy}{dx} = 20x^3 - 22$$

Now, this is an obvious and easy example, but notice what we have on the left side of the equation – in the process of taking the derivative, the y became a $\frac{dy}{dx}$. That is because the derivative of y is $\frac{dy}{dx}$!

OBJECTIVES
Take derivatives of relations implicitly. Use implicit differentiation to find higher order derivatives.

Ex 2 Find if $x^2 + y^2 = 25$

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right] \rightarrow \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

We can now isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

But even with this function, we could have solved for y and then found $\frac{dy}{dx}$.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \sqrt{25 - x^2}$$
$$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$$

Notice that this is the Exact same answer as we found with implicit differentiation. You could substitute y for $\sqrt{25 - x^2}$ in the denominator and come up with the same derivative, $\frac{dy}{dx} = -\frac{x}{y}$.

The other thing that one might notice is that this is the differential equation solved previously – and the solution to the differential was a circle.

Ex 3 Find the derivative of $\ln(y) = 3x^2 + 5x + 7$

Notice there are two different ways of doing this problem. First, we could simply solve for y and then take the derivative.

$$\ln(y) = 3x^2 + 5x + 7$$
$$y = e^{3x^2+5x+7}$$ Now take the derivative
$$\frac{dy}{dx} = \left(e^{3x^2+5x+7}\right)(6x+5)$$

Implicit Differentiation allows us the luxury of taking the derivative without the first algebra step because of the chain rule

$$\frac{d}{dx}\left[\ln(y) = 3x^2 + 5x + 7\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = 6x + 5$$

Notice when we took the derivative, we had to use the chain rule;

$$\frac{d}{dx}\left[y\right]=\frac{dy}{dx}$$

$$\frac{dy}{dx}=y\left(6x+5\right)$$

You might not immediately recognize that the two answers are the same, but since $y=e^{3x^2+5x+7}$, a simple substitution can show you that they are actually the same.

In terms of functions, this may not be very interesting or important, because it is often simple to isolate y. But consider a non-function, like this circle.

Ex 4 Find $\frac{dy}{dx}$ for the hyperbola $x^2-3xy+3y^2=2$

It would be very difficult to solve for y here, so implicit differentiation is really our only option.

$$\frac{d}{dx}\left[x^2-3xy+3y^2=2\right]$$

$$2x-3x\frac{dy}{dx}-3y+6y\frac{dy}{dx}=0$$

$$2x-3y=3x\frac{dy}{dx}-6y\frac{dy}{dx}$$

$$2x-3y=\left(3x-6y\right)\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{2x-3y}{3x-6y}$$

Of course if we want to find a second derivative, we can use implicit differentiation a second time.

Ex 5 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the hyperbola $x^2-3y^2+4x-12y-2=0$

$$\frac{d}{dx}\left[x^2-3y^2+4x-12y-2=0\right]$$
$$2x-6y\frac{dy}{dx}+4-12\frac{dy}{dx}=0$$
$$2x+4=6y\frac{dy}{dx}+12\frac{dy}{dx}$$
$$2x+4=\left(6y+12\right)\frac{dy}{dx}$$
$$\frac{dy}{dx}=\frac{2x+4}{6y+12}$$
$$\frac{dy}{dx}=\frac{x+2}{3y+6}$$

Now we just take the derivative again to find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[\frac{dy}{dx}=\frac{x+2}{3y+6}\right]$$
$$\frac{d^2y}{dx^2}=\frac{\left(3y+6\right)-\left(x+2\right)3\frac{dy}{dx}}{\left(3y+6\right)^2}$$

Since we already know $\frac{dy}{dx}$, we can substitute

$$\frac{d^2y}{dx^2}=\frac{\left(3y+6\right)-\left(x+2\right)\left(3\right)\left(\frac{x+2}{3y+6}\right)}{\left(3y+6\right)^2}$$
$$=\frac{\left(3y+6\right)^2-3\left(x+2\right)^2}{\left(3y+6\right)^3}$$

Ex 6 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $\sin(y) = 2\cos(3x)$

$$\frac{d}{dx}\left[\sin(y) = 2\cos(3x)\right]$$

$$\cos(y)\frac{dy}{dx} = -6\sin(3x)$$

$$\frac{dy}{dx} = \frac{-6\sin(3x)}{\cos(y)}$$

$$\frac{d^2y}{dx^2} = \frac{-18\cos(y)\cos(3x) - 6\sin(3x)\sin(y)\left(\frac{dy}{dx}\right)}{\cos^2(y)}$$

$$\frac{d^2y}{dx^2} = \frac{-18\cos^2(y)\cos(3x) + 36\sin^2(3x)\sin(y)}{\cos^3(y)}$$

So $\frac{dy}{dx} = \frac{-6\sin(3x)}{\cos(y)}$ and $\frac{d^2y}{dx^2} = \frac{-18\cos^2(y)\cos(3x) + 36\sin^2(3x)\sin(y)}{\cos^3(y)}$

Ex 7 AB 2004 #4

Ex 8 BC 2006 #5

Be Careful! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don't make careless mistakes.

Another issue that arises is the need to use both the Product Rule and the Quotient Rule. Make sure you look for these when you are working through a problem.

Ex 9 Find $\frac{dy}{dx}$ for $e^{x^2}+xy^2-16=\frac{\tan y}{3x}$

$$\frac{d}{dx}\left[e^{x^2}+xy^2-16=\frac{\tan y}{3x}\right]$$

$$2xe^{x^2}+2xy\frac{dy}{dx}+y^2=\frac{3x\sec^2 y\frac{dy}{dx}-3\tan x}{9x^2}$$

$$2xe^{x^2}+2xy\frac{dy}{dx}+y^2=\frac{x\sec^2 y\frac{dy}{dx}-\tan x}{3x^2}$$

$$6x^3e^{x^2}+6x^3y\frac{dy}{dx}+3x^2y^2=x\sec^2 y\frac{dy}{dx}-\tan x$$

$$6x^3e^{x^2}+3x^2y^2+\tan x=x\sec^2 y\frac{dy}{dx}-6x^3y\frac{dy}{dx}$$

$$6x^3e^{x^2}+3x^2y^2+\tan x=\frac{dy}{dx}\left(x\sec^2 y-6x^3y\right)$$

$$\frac{dy}{dx}=\frac{6x^3e^{x^2}+3x^2y^2+\tan x}{x\sec^2 y-6x^3y}$$

There is an easier way to do this problem. It is called Logarithmic Differentiation, but it is not in the AP Calculus syllabus. It can be found in an Appendix at the end of this book.

1.6 Homework Set A

Find $\frac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for y and differentiating. Show that $\frac{dy}{dx}$ is the same in both cases.

1. $xy + 2x + 3x^2 = 4$

2. $\frac{1}{x} + \frac{1}{y} = 1$

3. $\sqrt{x} + \sqrt{y} = 4$

Find $\frac{dy}{dx}$ for each of these equations by implicit differentiation.

4. $x^2 + y^2 = 1$

5. $x^3 + 10x^2y + 7y^2 = 60$

6. $x^2y^2 + x\sin(y) = 4$

7. $4\cos(x)\sin(y) = 1$

8. $e^{x^2y} = x + y$

9. $\tan(x - y) = \frac{y}{1+x^2}$

Find the equation of the line tangent to each of the following relations at the given point.

10. $x^2 - y^2 - 6y - 3 = 0$ at $(\sqrt{3}, 0)$

11. $9x^2 + 4y^2 + 36x - 8y - 32 = 0$ at $(0, -2)$

12. $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $(-1, -3)$

13. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y\sin x} + y^3 - 3$ through the point $\left(\frac{\pi}{2}, 0\right)$.

14. Find the equation of the line tangent to $x^3 + \frac{y}{x} + y^2 = 7$, through the point (1,2).

15. Find the equation of the line tangent to $x^2 + 3xy + y^2 = 11$, through the point (1,2).

16. Find $\frac{d^2y}{dx^2}$ if $xy + y^2 = 1$

17. Find $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$

18. AP Packet: AB 2000 #5, AB08B #6, AB 15 #6

1.6. Multiple Choice Homework

1. Use Implicit Differentiation to find the points on $x^3 - y^2 + x^2 = 0$ has vertical tangent lines.

a) $(0, 0)$ only
b) $(-1, 0)$ only
c) $\left(1, \sqrt{2}\right)$ only
d) $(-1, 0)$ and $(0, 0)$
e) The tangent line is never vertical

---

2. If $x^2 + xy = 10$, then when $x = 2$, $\frac{dy}{dx} =$

a) $-\frac{7}{2}$ b) -2 c) $\frac{2}{7}$ d) $\frac{3}{2}$ e) $\frac{7}{2}$

---

3. What is the slope of the line tangent to the curve $y^2 + x = -2xy - 5$ at the point (2,1)?

a) $-\frac{4}{3}$ b) $-\frac{3}{4}$ c) $-\frac{1}{2}$ d) $-\frac{1}{4}$ e) 0

---

4. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x:

a) $\frac{6x-4y}{4x+4}$ b) $\frac{9x^2-4}{4x+8y}$ c) $\frac{9x^2-4}{4+8y}$

d) $\frac{9x^2-4y}{4x+8y}$ e) $\frac{9x^2-4y}{4+8y}$

---

5. Given $x + xy + 2y^2 = 6$, then $\left.\frac{dy}{dx}\right|_{(2,1)} =$

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) $-\frac{1}{5}$ e) $-\frac{3}{4}$

---

6. Consider the closed curve in the x-y plane given by $2x^2 + 5x + y^2 + y = 8$. Which of the following is correct?

a) $\frac{dy}{dx} = -\frac{4x+5}{8x+2y+1}$ b) $\frac{dy}{dx} = \frac{4x+5}{2y+1}$

c) $\frac{dy}{dx} = -\frac{4x+5}{8x+2y}$ d) $\frac{dy}{dx} = \frac{4x+5}{8x+2y}$

e) $\frac{dy}{dx} = -\frac{4x+5}{2y+1}$

---

7. The slope of the line tangent to $xy - y^3 + 6 = 0$ at (1,2) is

a) 0 b) $-\frac{1}{12}$ c) $\frac{2}{11}$ d) $\frac{1}{6}$ e) $\frac{1}{4}$

---

8. Find the equation of the line tangent to the curve $\sec\left(x^2\right) + xy^3 = 2 - y$ at $x = 0$.

a) $y = -x$ b) $y - 1 = -x$ c) $y - 2 = -x$

d) $y - 1 = x$ e) $y - 2 = x$

---

9. If $\sin^{-1} x = \ln y$, then $\frac{dy}{dx} =$

a) $\frac{y}{\sqrt{1-x^2}}$ b) $\frac{xy}{\sqrt{1-x^2}}$ c) $\frac{y}{1+x^2}$ d) $e^{\sin^{-1} x}$ e) $\frac{e^{\sin^{-1} x}}{1+x^2}$

---

10. If $x^2 y + yx^2 = 6$, then, at $(1, 3)$, $\frac{d^2 y}{dx^2} =$

a) -18 b) -6 c) 6 d) 12 e) 18

---

11. If $y = x + \sin(xy)$, then $\frac{dy}{dx} =$

a) $1 + \cos(xy)$ b) $1 + y\cos(xy)$ c) $\frac{1}{1 - \cos(xy)}$

d) $\frac{1}{1 - x\cos(xy)}$ e) $\frac{1 + y\cos(xy)}{1 - x\cos(xy)}$

---

12. If $\sin(xy) = x^2$, then $\frac{dy}{dx} =$

a) $2x\sec(xy)$ b) $\frac{\sec(xy)}{x^2}$ c) $2x\sec(xy) - y$

d) $\frac{2x\sec(xy)}{y}$ e) $\frac{2x\sec(xy) - y}{x}$

1.7 Logarithmic Differentiation

With implicit differentiation and the chain rule, we learned some powerful tools for differentiating functions and relations. The product and quotient rules also allowed us to take derivatives of certain functions that would otherwise be impossible to differentiate. Sometimes, however, with very complEx functions, it becomes easier to manipulate an equation so that it is easier to take the derivative. This is where logarithmic differentiation comes in.

OBJECTIVES

Determine when it is appropriate to use logarithmic differentiation.
Use logarithmic differentiation to take the derivatives of complicated functions.

Before we begin, it would be helpful to look at a few rules that we should remember from algebra and precalculus concerning logarithms.

$a^x a^y = a^{x+y}$	$\log_a x + \log_a y = \log_a(xy)$
$\dfrac{a^x}{a^y} = a^{x-y}$	$\log_a x - \log_a y = \log_a \dfrac{x}{y}$
$\left(a^x\right)^y = a^{xy}$	$\log_a x^n = n \log_a x$

Since logarithms are Exponents Expressed in a different form, all of the above rules are derived from the rules for Exponents, and you can see the corresponding Exponential rule. Because of our algebraic rules, we can do whatever we want to both sides of an equation. In algebra, we usually used this to solve for a variable. In Calculus, we can use this principle to make many derivative problems significantly easier.

Ex 1 Find the derivative of $y = \left(x^2 + 7x - 3\right)\left(\sin\left(x\right)\right)$

What we would traditionally use to take the derivative of this function is the product rule.

$$\frac{d}{dx}\left[y=\left(x^2+7x-3\right)\left(\sin\left(x\right)\right)\right]$$
$$\frac{dy}{dx}=\left(x^2+7x-3\right)\left(\cos\left(x\right)\right)+\left(2x+7\right)\left(\sin\left(x\right)\right)$$

Obviously, this is a straightforward problem that can be easily done using the product rule. If, however, I took the natural log of both sides of the equation, I can achieve the same results, and never use the product rule. (Remember, we will almost Exclusively use the natural log because it works so well within the framework of Calculus)

$$\ln\left(y\right)=\ln\left[\left(x^2+7x-3\right)\left(\sin\left(x\right)\right)\right]$$
$$\ln\left(y\right)=\ln\left(x^2+7x-3\right)+\ln\left(\sin\left(x\right)\right)$$ This is simplifying using log rules.
$$\frac{d}{dx}\left[\ln\left(y\right)=\ln\left(x^2+7x-3\right)+\ln\left(\sin\left(x\right)\right)\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right)=\frac{2x+7}{x^2+7x-3}+\frac{\cos\left(x\right)}{\sin\left(x\right)}$$
$$\frac{dy}{dx}=\left(\frac{2x+7}{x^2+7x-3}+\frac{\cos\left(x\right)}{\sin\left(x\right)}\right)\left(y\right)$$ Now just substitute y back in and simplify.
$$\frac{dy}{dx}=\left(\frac{2x+7}{x^2+7x-3}+\frac{\cos\left(x\right)}{\sin\left(x\right)}\right)\left(x^2+7x-3\right)\left(\sin\left(x\right)\right)$$
$$\frac{dy}{dx}=\left(2x+7\right)\left(\sin\left(x\right)\right)+\left(\cos\left(x\right)\right)\left(x^2+7x-3\right)$$

Clearly, we got the same answer that we got from the product rule, but with significantly more effort. Logarithmic differentiation is a tool we can use, but we have to use it judiciously, we don't want to make problems more difficult than they have to be.

Where logarithmic differentiation is really useful is in functions that are Excessively painful to work with (or impossible to take the derivative of any other way) because of multiple operations.

Ex 2 Find $\frac{dy}{dx}$ for $y=\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan\left(5x+2\right)}$

We could take the derivative by applying the chain rule, quotient rule, and product rule, but that would be a time-consuming and tedious process. It's much easier to take the log of both sides, simplify and then take the derivative.

$$\ln y=\ln\left[\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan\left(5x+2\right)}\right]$$

$$\ln y=\ln\left(x^2+5\right)+\ln\left(\sin\left(3x^3\right)\right)-\ln\left(\tan\left(5x+2\right)\right)$$

$$\frac{d}{dx}\left[\ln y=\ln\left(x^2+5\right)+\ln\left(\sin\left(3x^3\right)\right)-\ln\left(\tan\left(5x+2\right)\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}$$

$$\frac{dy}{dx}=\left(\frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}\right)y$$

$$\frac{dy}{dx}=\left(\frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}\right)\left(\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan\left(5x+2\right)}\right)$$

Now that may seem long and messy, but try it any other way, and you might end up taking a lot more time, with a lot more algebra and a lot more potential spots to make mistakes.

Ex 3 Find $f'(\pi)$ for $f(z)=z^{\cos z}$

$$\ln\left[f(z)\right]=\ln\left[z^{\cos z}\right]$$

$$\frac{d}{dz}\left(\ln\left[f(z)\right]=\left(\cos z\right)\left(\ln z\right)\right)$$

$$\frac{f\ '(z)}{f(z)} = \frac{\cos z}{z} - (\ln z)(\sin z)$$

$$f\ '(z) = \left(\frac{\cos z}{z} - (\ln z)(\sin z)\right) f(z)$$

$$f\ '(z) = \left(\frac{\cos z}{z} - (\ln z)(\sin z)\right)\left(z^{\cos z}\right)$$

$$f\ '(\pi) = \left(\frac{\cos \pi}{\pi} - (\ln \pi)(\sin \pi)\right)\left(\pi^{\cos \pi}\right) = \frac{-1}{\pi^2}$$

You could have also done this problem using the change of base property that you learned in PreCalculus and you would get the same answer in about the same number of steps.

Ex 4 again: Find for

Since ,

Again, there are often more than one way to do a specific problem, and part of what we do as mathematicians is decide on the simplest **correct** method to solving a problem.

The issue many people have when learning more difficult mathematical concepts is that they try to oversimplify a problem and end up getting it wrong as a result.

As Albert Einstein once said, “Make things as simple as possible but not simpler.”

1.7 Homework

Find the derivatives of the following functions. Use logarithmic differentiation **when appropriate.**

1. $y=(2x+1)^4(x^3-3)^5$
2. $z=(y^3-3)e^{(2y+1)}$
3. $y=\dfrac{\sin^2 x\ \tan^4 x}{(x^2+5)^2}$
4. $g(t)=t\ \ln(t)$
5. $y=\ \ln^x(x)$
6. $p(v)=v^{e^v}$

7. Use logarithmic differentiation to prove the product rule.

8. Find $\dfrac{dt}{du}$ if $t^u=u^t$.

9. For the function, $f(x)=x^{\ln x}$, find an equation for a tangent line at $x = e$, and use that to approximate the value for f (2.7). Find the percent difference between this and the actual value of f (2.7).

1.7 Homework Set B

1. Use logarithmic differentiation to find $\dfrac{dq}{dt}$ if $q=\dfrac{e^{t^4-15}\sin^5 3t}{(\ln t)^{10}}$

2. Use logarithmic differentiation to find $\dfrac{dy}{dx}$ when

$y=e^{150x-19}\ln(\sin x)^{100}\sqrt{x^2-1}$

3. Use logarithmic differentiation to find $\dfrac{dq}{dt}$ if $q=\dfrac{e^{t^4-15}\csc^5 3t}{\ln t^{10}}$

4. Find $\frac{dy}{dx}$ for the function $y = \left(e^{17x^4}\right)\left(\sin^7 x\right)\left(5x-17\right)^{12}\left(\cot 5x\right)$

AP Calculus BC Name_____

Derivative Review Practice Test

1. If $y = \ln(\sin x)$ and $0 \le x \le \pi$, then $\frac{dy}{dx}$ is

a. $-\tan x$ b. $-\cot x$ c. $\tan x$

d. $\cot x$ e. $\csc x$

---

2. If $y = \sin^{-1}\left(e^{2x}\right)$, then $\frac{dy}{dx}$ is

a. $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ b. $\frac{e^{2x}}{\sqrt{1-e^{4x}}}$ c. $\frac{2e^{2x}}{\sqrt{1+e^{4x}}}$

d. $\frac{e^{2x}}{1+e^{4x}}$ e. $\frac{2e^{2x}}{\sqrt{e^{4x}-1}}$

---

3. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given on the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

Given that $h(x) = g\left(g(x)\right)$, $h'(8) =$

a) 1 b) 2 c) 3 d) 4 e) 8

---

4. If $g(x)=\tan^2\left(e^x\right)$, then $g'(x)$ is

a. $2\tan\left(e^x\right)\sec^2\left(e^x\right)$ b. $2e^x\tan\left(e^x\right)\sec^2\left(e^x\right)$

c. $2\tan^2\left(e^x\right)\sec\left(e^x\right)$ d. $e^x\sec^2\left(e^x\right)$ e. $2e^x\tan\left(e^x\right)$

---

5. Let $f(x)$ be the function with $f(2)=4$ and $f'(x)=\sqrt{x^3+1}$. Using the tangent line approximation to the graph of $f(x)$ at x=2, estimate $f(2.2)$.

a. 4.0 b. 4.2 c. 4.4 d. 4.6 e. 4.8

---

6. Which of the following statements must be true?

I. $\frac{d}{dx}\sqrt{e^x+3}=\frac{e^x}{2\sqrt{e^x+3}}$ II. $\frac{d}{dx}\left(\ln\cos x\right)=\tan x$

III. $\frac{d}{dx}\left(6x^3-\pi+\sqrt[3]{x^8}-\frac{2}{x^3}\right)=18x^2+\frac{8}{3}\sqrt[3]{x^5}+\frac{6}{x^4}$

(a) I only (b) II only (c) III only

(d) I and III only (e) I, II, and III

---

7. The value of the derivative of $y=\frac{\left(x^2-3\right)^3}{\left(5x-9\right)^2}$ at $x = 2$ is

a. -4 b. -2 c. 0 d. 2 e. 4

---

8. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x:

a) $\dfrac{6x-4y}{4x+4}$ b) $\dfrac{9x^2-4}{4x+8y}$ c) $\dfrac{9x^2-4}{4+8y}$

d) $\dfrac{9x^2-4y}{4x+8y}$ e) $\dfrac{9x^2-4y}{4+8y}$

---

9. $\dfrac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \dfrac{1}{\sqrt[7]{x^4}} + \dfrac{1}{5x}\right]$

10. $D_x\left[e^{3x^2}\cos 4x\right]$

11. $f(x) = e^{\sin 4x}$; find the exact value of $f''\left(\dfrac{\pi}{4}\right)$.

12. A fourth differentiable function is defined for all real numbers and satisfies each of the following:

$$g(2)=5,\ g'(2)=-2,\text{ and } g''(2)=3$$

and the function f is given by $f(x)=e^{k(x-1)}+g(2x)$, where k is a constant.

a. Find $f(1),\ f'(1),\ f''(1)$

b. Show that the fourth derivative of f is $k^4e^{k(x-1)}+16g^{IV}(2x)$

1.1 Free Response Answers

1. $f'(x) = 2x + 3$

2. $f'(t) = t^3$

3. $y' = -\frac{2}{3}x^{-5/3}$

4. $y' = 5e^x$

5. $v'(r) = 4\pi r^2$

6. $g'(x) = 2x - \frac{2}{x^3}$

7. $\frac{dy}{dx} = \frac{3}{2}x^{1/2} + 2x^{-1/2} - \frac{3}{2}x^{-3/2}$

8. $\frac{du}{dt} = \frac{2}{3}t^{-1/3} + 3t^{1/2}$

9. $\frac{dz}{dy} = \frac{-10A}{y^{11}} + Be^y$

10. $\frac{dy}{dx} = e^{x+1}$

11. $f'(x) = 45x^{14} - 15x^2.$

12. $-7\left(x^3 + 4x - \pi\right)^{-8}\left(3x^2 + 4\right)$

13. $\frac{dy}{dx} = \frac{e^{\sqrt{x}}}{2\sqrt{x}}$

14. $f'(x) = \frac{2 + 3x^2}{4\left(1 + 2x + x^3\right)^{3/4}}$

15a. 21 b. -36

16. $u'(2) = 0$

17. $v'(4) = 0$

18. $w'(6) = -\frac{4}{3}$

19. $t'(8) = dne$

1.1 Multiple Choice Answers

1.	A	2.	D	3.	D	4.	E	5.	B	6.	D
7.	C	8.	E	9.	D	10.	A	11.	A		

1.2 Homework

1. $\frac{dy}{dx} = 4\cos 4x$

2. $y' = 20x^4 \sec x^5 \tan x^5$

3. $\frac{dy}{dx} = -3\cos^2 x \sin x$

$\frac{dy}{dx} = -2\cos\theta \cot(\sin\theta)\csc^2(\sin\theta)$

4.

5. $f'(t) = \frac{\sec^2 t}{3\left(1+\tan t\right)^{2/3}}$

6. $f'(\theta) = -\tan\theta$

7. $y = -3x^2 \sin\left(a^3 + x^3\right)$

8. $\frac{dy}{dx} = 6\tan\left(3\theta\right)\sec^2\left(3\theta\right)$

9. $f'(x) = \frac{-\sin\left(\ln x\right)}{x}$

10. $f'(x) = \frac{1}{5x\ln^{4/5} x}$

11. $f'(x) = \frac{\cos x}{\ln 10\left(2+\sin x\right)}$

12. $f'(x) = \frac{-3}{\ln 2\left(1-3x\right)}$

13. $\frac{dy}{dx} = \frac{e^x}{\sqrt{1-e^{2x}}}$

14. $\frac{dy}{dx} = \frac{1}{2\left(x^{1/2} + x^{3/2}\right)}$

15. $\frac{dy}{dx} = \frac{1}{\left(-x^2 - x\right)^{1/2}}$

16. $\frac{dy}{dx} = \frac{\sqrt{2}}{\sqrt{1-2x^2}}$

17. $\frac{dy}{dx} = \frac{-2x}{\left(x^2+1\right)|x|\sqrt{x^2+2}}$

18. 0

19. 2

20. 5

<u>1.2 Multiple Choice Answers</u>

1. E 2. E 3. C 4. B

1.3 Homework

1. $y+21=75(x+2);\ f(-1.9)\approx -13.5$

2. $\left(\frac{\pi}{2}\pm 2\pi n,\ 3\right),\left(-\frac{\pi}{2}\pm 2\pi n,\ 1\right)$

3. $y-2=2(x-0)$

4. $(-2,\ 21),(1,\ -6)$

5. $y-1=1(x-0)$

6. $y-1=3\sqrt{3}\left(x-\pi/3\right)$

7. $y-2=\frac{2}{\pi}\left(x-\frac{\pi}{2}\right)$

8. $f(0)\approx 4.25$

9. $f(1.4)\approx .635$

10. $f(2.6)\approx -.288$

11. $f(1)\approx 6.563$

<u>1.3 Multiple Choice Answers</u>

1. A 2. E 3. B 4. E 5. 6,

7. 8. 9.

1.4 Homework

1. $y'=t^2(3\cos t-t\sin t)$

2. $g'(x)=4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)$

3. $y'=\frac{\tan x+1}{\sec x}$

4. $y' = 8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$

5. $y' = e^{-x^2}(1-2x^2)$

6. $y' = \dfrac{x\cos x - 2\sin x}{x^3}$

7. $y' = e^{x\cos x}(\cos x - x\sin x)$

8. $y' = \dfrac{1}{(r^2+1)^{3/2}}$

9. $\dfrac{dy}{dx} = -\dfrac{1}{x}\cos x + \sin\dfrac{1}{x}$

10. $\dfrac{dy}{dx} = e^{-5x}(5\cos 3x - 3\sin 3x)$

11. $f'(x) = \dfrac{1+2\ln x}{2\sqrt{\ln x}}$

12. $\dfrac{dy}{dx} = \dfrac{-x}{1+x}$

13. $y - \dfrac{1}{e} = \dfrac{1}{e}(x-1)$

14. $f'(e) = 0$.

15. $h'(t) = \dfrac{68t(1+t^2)^{16}}{(1-t^2)^{18}}$

16. $y'(1) = \dfrac{7}{2}$

17. $f'(x)=5\left[x\sin(2x)+\tan^4\left(x^7\right)\right]^4\left[\sin(2x)+2x\cos 2x+28x^6\tan^3\left(x^7\right)\sec^2\left(x^7\right)\right]$

18. Tan: $y-e=1(x-e)$ Normal: $y-e=-1(x-e)$

19. y=3

20. Tangent: $y-2\pi\sqrt{2}=\dfrac{\sqrt{2}(-\pi+4)}{8}\left(x-\dfrac{4}{\pi}\right)$

Normal: $y-2\pi\sqrt{2}=\dfrac{4\sqrt{2}}{\pi-4}\left(x-\dfrac{4}{\pi}\right)$

21. $H'(x)=1+2x\tan^{-1}x$

22. $\dfrac{dy}{dx}=\cos^{-1}x$

23. $f'(x)=e^x-\dfrac{x^2}{1+x^2}-2x\arctan x.$

24. $\dfrac{dy}{dx}=\dfrac{-2x^2}{\sqrt{1-x^2}}$

25. $\dfrac{dy}{dx}=\dfrac{1}{x^2\sqrt{x^2-1}}-\dfrac{\sec^{-1}x}{x^2}$

26. $\dfrac{dy}{dx}=-\tan^{-1}\left(\dfrac{x}{2}\right)$

1.4 Multiple Choice Answers

1. D 2. B 3. B 4. E 5. A 6. D

7. A 8. D 9. D 10. B

1.5 Free Response Homework

1. $f'(x)=5x^4+12x-7$ *and* $f''(x)=20x^3+12$

2. $h'(x)=\dfrac{x}{\sqrt{x^2+1}}$ *and* $h''(x)=\dfrac{1+x^2}{\left(x^2+1\right)^{3/2}}$

3. $\frac{dy}{dx}=\frac{2x^2}{\left(x^3+1\right)^{1/3}}$ *and* $\frac{d^2y}{dx^2}=\frac{2x\left(x^3+2\right)}{\left(x^3+1\right)^{4/3}}$

4. $H'(t)=3\sec^2 3t$ *and* $H''(t)=18\sec^2 3t\tan 3t$

5. $g'(t)=t^2e^{5t}\left(5t+3\right)$ *and* $g''(t)=te^{5t}\left(25t^2+30t+6\right)$

6. $y''=6e^{3x^2}\left(6x^2+1\right)$.

7. $y''=3\sin x\left(2\cos^2 x-\sin^2 x\right)$.

8. $f'\left(t\right)=t\cdot-\sin t+\cos t;\ f''\left(t\right)=-t\cos t-2\sin t$

9. $60x^2+54x-8$

10. $168x^5-60x^3+18x$

11. $y''=-2\ \left(\sin x^2+2x^2\cos x^2\right)$

12. $y''=2\sec^2 x\ \left(2\tan^2 x+\sec^2 x\right)$

13. $\frac{d^2y}{dx^2}=3\sec 3x\left(3\sec^2 3x+\tan^2 3x\right)$

14. $\frac{d^2y}{dx^2}=4e^{2x}\left(x+1\right)$

15. $f''\left(x\right)=\frac{-2\left(x^2-3\right)}{\left(x^2+3\right)^2}$

16. $g''\left(x\right)=\frac{-2}{\left(x-2\right)^2}$

17. $h''(x) = \dfrac{5}{\left(x^2+5\right)^{3/2}}$

18. $F''(x) = \dfrac{2}{\left(3x^2-2x+1\right)^{3/2}}$

19. $\dfrac{d^2y}{dx^2} = \dfrac{14\left(3x^2+10\right)}{\left(x^2-10\right)^3}$

20. $\dfrac{d^2y}{dx^2} = \dfrac{6\left(7x^2-7x+2\right)}{\left(x^2-x+1\right)^3}$

21. $6x+2$

22. $36x^2-120x^2+84$

23. $y'' = \dfrac{-8x^3-32x}{\left(x^2+4\right)^3}$

24. $y'' = \dfrac{-18x^2+24}{\left(x^2-4\right)^3}$

25. $y'' = \dfrac{-2x^4+4x^3+8x^2-32x}{\left(8-x^2\right)^{3/2}}$

26. $\dfrac{d^2y}{dx^2} = -\cos x$

27. $e^{-x}(x-2)$

28. $e^{-x^2}\left(4x^2-2\right)$

29. $y'' = \dfrac{2x^3 + 54x}{\left(x^2 - 9\right)^3}$

30. $\dfrac{2}{9}x^{-4/3}$

1.5 Multiple Choice Answers

1. E 2. C 3. B 4. E 5. B

1.6 Free Response Homework

1. Imp: $\dfrac{dy}{dx} = \dfrac{-6x - y - 2}{x}$; Exp: $\dfrac{dy}{dx} = \dfrac{-4 - 3x^2}{x^2}$

2. Imp: $\dfrac{dy}{dx} = \dfrac{-y^2}{x^2}$; Exp: $\dfrac{dy}{dx} = \dfrac{-1}{\left(1 - x\right)^2}$

3. Imp: $\dfrac{dy}{dx} = \dfrac{-y^{1/2}}{x^{1/2}}$; Exp: $\dfrac{dy}{dx} = \dfrac{-4 + \sqrt{x}}{\sqrt{x}}$

4. $\dfrac{dy}{dx} = \dfrac{-x}{y}$

5. $\dfrac{dy}{dx} = \dfrac{-3x^2 - 20xy}{10x^2 + 14}$

6. $\dfrac{dy}{dx} = \dfrac{-\sin y - 2xy^2}{2x^2 y + x\cos y}$

7. $\dfrac{dy}{dx} = \tan\left(x\right)\tan\left(y\right)$

8. $\dfrac{dy}{dx} = \dfrac{1 - 2xye^{x^2 y}}{x^2 e^{x^2 y} - 1}$

9. $\dfrac{dy}{dx} = \dfrac{\sec\left(x - y\right)\left(1 + x^2\right)^2 + 2xy}{1 + x^2 + \sec\left(x - y\right)\left(1 + x^2\right)^2}$

10. $y = \dfrac{\sqrt{3}}{3}\left(x - \sqrt{3}\right)$

11. $y + 2 = \dfrac{3}{2}x$

12. $y+3=\frac{-6}{5}(x+1)$

13. $Tangent:\ y-0=-\frac{4}{\pi}\left(x-\frac{\pi}{2}\right)$

$Normal:\ y-0=\frac{\pi}{4}\left(x-\frac{\pi}{2}\right)$

14. $y-2=-\frac{10}{17}(x-1)$

15. $y-2=-\frac{8}{7}(x-1)$

16. $\frac{2y(x+y)}{(x+2y)^3}$

17. $\frac{-36y^2-16x^2}{81y^2}$

1.6 Multiple Choice Answers

1.	D	2.	A	3.	A	4.	D	5.	C	6.	E
7.	C	8.	B	9.	A	10.	B	11.	E	12.	A
13.	E										

1.7 Free Response Homework

1. $\frac{dy}{dx}=(2x+1)^4(x^3-3)^5\left(\frac{8}{2x+1}+\frac{15x^2}{x^3-3}\right)$

2. $\frac{dz}{dy}=(y^3-3)e^{(2y+1)}\left[\frac{3y^2}{y^3-3}+2\right]$

3. $\frac{dy}{dx}=\frac{\sin^2 x\ \tan^4 x}{(x^2+5)^2}\left(2\cot x+\frac{4}{\sin x\cos x}+\frac{4x}{x^2+5}\right)$

4. $g'(t)=1+\ln(t)$

5. $y=\ \ln^x(x)\left[\frac{1}{\ln x}+\ln(\ln x)\right]$

6. $\frac{dp}{dv}=v^{e^v}\left[\frac{e^v}{v}+e^v\ln v\right]$

7. $\frac{dt}{du}=\frac{u^{t-1}Lnu}{t^{u-1}}$

8. $y-e=2(x-e)$ *and* $f(2.8)\approx 2.682$

9. $y-e=2(x-e)$

$f(2.7)\approx e+2(2.7-e)=2.682$ $\qquad$ $f(2.7)=(2.7)^{\ln(2.7)}=2.682$

% Difference = 0

<u>Practice Chapter Test Key</u>

1. D 2. A 3. D 4. B

5. D 6. D 7. D 8. B

9. $7x^6-\frac{7}{2}x^{-1/8}+7^x\ln 7+\frac{4}{7}x^{-11/7}-\frac{1}{5}x^{-2}$

10. $2e^{3x^2}(3\cos 4x-2\sin 4x)$

11. 16

12a. $f(1)=6,\ f'(1)=k-4,\ f''(1)=k^2+12$

12b. $f'''(1)=k^3x^{k(x-1)}+8g'''(2x)\rightarrow f^{IV}(1)=k^4x^{k(x-1)}+16g^{IV}(2x)$