Chapter 8:

AP Free Response Review

Chapter 8: Derivatives and Integrals on the AP Calculus AB Exam

As with any cumulative exam, the AP Exam mixes all the Derivative and Integral material together. Sometimes, it is hard for a student to separate the questions into the digestible pieces in which the material was learned. To that end, this chapter will investigate the most common topics on the free response questions of AP Calculus AB Exam. To that end, a content analysis was performed on the 2011 through 2023 exams (excluding 2020 due to Covid 19), looking for common topics and subtopics. The following were the broad topics:

- Tabular Data Problems (all 12 exams). Note that sometimes these questions overlap with Accumulation of Rates, Rectilinear Motion, or Area and Volume.
- Differential Equations (all 12 exams)
- Accumulation of Rates (11 exams)
- Graphical Analysis (11 exams)
- Rectilinear Motion (11 exams)
- Area & Volume (9 exams)
- Related Rates (6 exams), later as part of the Volume problem
- Implicit Differentiation (5 exams)
- Miscellaneous
  - Derivatives and Integrals from tables and/or graphs
  - Limits and continuity
  - L'Hopital's Rule

Keep in mind the CollegeBoard's goals for AP Calculus:

- Students should be able to work with functions represented in a variety of ways: Graphical, numerical, verbal, and analytic (algebraic). They should understand the connections between these representations.
- Students should understand the meaning of the derivative in terms of a rate of change and local linear approximation and should be able to use derivatives to solve a variety of problems.
- Students should understand the meaning of the definite integral both as a limit of Riemann sums and as a net accumulation of change and should be able to use integrals to solve a variety of problems.

- Students should understand the relationship between the derivative and the definite integral as expressed in the Fundamental Theorem of Calculus.
- Students should be able to communicate mathematics and explain solutions to problems both verbally and in written sentences.
- Students should be able to model a written description of a physical situation with a function, a differential equation, or an integral.
- Students should be able to use technology to help solve problems, experiment, interpret results, and support conclusions.
- Students should be able to determine the reasonableness of solutions, including sign, size, relative accuracy, and units of measure.

These goals dictate the design of the free response questions in particular.

8.1 Accumulation of Rates

Common Sub-Topics:

- Total change of input or output
- Increasing or decreasing of the change of amount vs change of the rate
- Total amount
- Average value
- Absolute minimum or maximum
- Interpretation of units

Key Ideas:

- Realize that each part of the problem (a through d) is a separate question, and the questions are not necessarily sequential.
    - Reread the base information as you start each new part.
- Integrating a rate results in total change.
- The units/labels tell everything.
- Slow down because critical reading is big part of what is being tested.
    - Pay particular attention to details buried in the text.
- Reread the problem again and answer the question that was asked.
- In an explanation, use ALL the units involved in the problem.
- Know how and when to use the calculator to find the definite integral.

Key Phrases:

Instantaneous rate of change: $\frac{dx}{dt}$

Average rate of change: $\frac{f(b)-f(a)}{b-a}$

Average value: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

Total change: $\int_a^b R(t)dt$ or $\int_a^t [\textit{incoming rate} - \textit{outgoing rate}]\, dx$

Total rate of change: $\textit{incoming rate} - \textit{outgoing rate}$

Total Amount: $Total(t) = \textit{initial value} + \int_a^t [\textit{incoming rate} - \textit{outgoing rate}]\, dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(\textit{Rate of Change})$ is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(\textit{Rate of Change})$ is positive

Always consider the Units!!!

If we consider the units involved in integrating a rate, this becomes more apparent.

$\int_a^b R(t)\,dt = \int_a^b \frac{units}{time}(time) = sum\ of\ units =$ total units.

Amount increasing or decreasing would be $\frac{d}{dt}(units) = \frac{units}{time}$

Rate would be given in $\frac{units}{time}$

$\frac{f(b)-f(a)}{b-a}$ would be in $\frac{units}{time}$

$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$ would be $\frac{1}{time}\int_a^b \frac{units}{time}(time) = \frac{units}{time}$

Amount increasing or decreasing would be $(units)\frac{1}{time} = \frac{units}{time}$

Amount increasing or decreasing at an increasing or decreasing rate would be $(units)\frac{1}{time}\cdot\frac{1}{time} = \frac{units}{time^2}$

Rate increasing or decreasing would be $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{time}\left(\frac{units}{time}\right) = \frac{units}{time^2}$

Ex 1 At time $t=0$, there are 120 gallons of oil in a tank. During the time interval $0 \le t \le 10$ hours, oil flows into the tank at a rate of $h(t)=10-\frac{t\cos(t)}{2}$ and out of the tank at a rate given by $g(t)=6+\frac{e^{0.52t}}{t+1}$. Both *h* and *g* are measured in gallons per hour.

(a) How much oil flows out of the tank during this 10-hour time period?
(b) Find the value of $h(4.3)-g(4.3)$. Using correct units, explain what this value represents in the context of this problem.

(c) Write an expression for $A(t)$, the total amount of oil in the tank at time *t*.
(d) Find the absolute maximum and minimum amount of oil in the tank during $0 \le t \le 10$ hours.

(a) How much oil flows out of the tank during this 10-hour time period?

"How much oil flows out" means the total amount of the change caused by $g(t)$.

How much oil flows out $= \int_0^{10}\left(6+\frac{e^{0.52t}}{t+1}\right)dt = 100.827\, gallons\ of\ oil$

(b) Find the value of $h(4.3)-g(4.3)$. Using correct units, explain what this value represents in the context of this problem.

$$h(4.3)-g(4.3)=\left(10-\frac{4.3\cos 4.3}{2}\right)-\left(6+\frac{e^{0.53(4.3)}}{5.3}\right)=3.096\ {}^{gal}\!/_{hr}.$$

$h(4.3)-g(4.3)$ is the rate, in gallons per hour, of how fast the amount of oil in the tank is increasing when $t=4.3$ hours.

(c) Write an expression for $A(t)$, the total amount of oil in the tank at time *t*.

$$A(t) = 120 + \int_0^t \left[h(x) - g(x)\right]dx$$

(d) Find the absolute maximum and minimum amount of oil in the tank during $0 \le t \le 10$ hours.

Critical Values: $A'(t) = \frac{d}{dt}\left[120 + \int_0^t \left[h(x) - g(x)\right]dx\right] = h(t) - g(t) = 0$

$t = 5.310$

Endpoints are also critical values: $t = 0 \text{ } or \text{ } 10$

cv	$A(t) = 120 + \int_0^t \left[h(x) - g(x)\right]dx$
$t = 0$	$A(t) = 120 + \int_0^0 \left[h(x) - g(x)\right]dx = 120$
$t = 5.310$	$A(t) = 120 + \int_0^{5.310} \left[h(x) - g(x)\right]dx = 103.176$
$t = 10$	$A(t) = 120 + \int_0^{10} \left[h(x) - g(x)\right]dx = 117.188$

absolute maximum = 120 absolute minimum = 103.176

8.1 Free Response Homework

AP Handout: 2011AB #1, 2012AB #1, 2014AB #1, 2015AB #1, 2018AB #1, 2019AB #1

8.2: Reasoning from Tabular Data (Recap of 3.2, 3.4, 3.5 and 3.6)

Reasoning from Tabular Data (aka Table Problems) has been one of the most commonly recurring topics on the AP Exam. It has been the first question on the Free Response part of the Exam for the past six years (2014 - 2019).

There are three kinds of table problems:

1. The most common is a word problem involving an unknown function but with specific data points.
   - These are very much in line with the Accumulation of Rates, Rectilinear Motion and/or Volume problems.

2. Multiple choice questions often have tables of values to plug into the Chain, Product, or Quotient Rule.
   - The trick here is that much of the data are distractors.

3. Occasionally there have been graphing problems where a table is used to present data instead of sign patterns.

Common Sub-Topics:

- Riemann and Trapezoidal Sums
- Overestimate vs underestimate
- Tangent approximations using secant lines
- Graphing problems with the First Derivative Test.
- Interpretation of units
- MVT and IVT

Ex 1 The C&A Smelter opened in Douglas, Arizona in 1904 to process copper ore from the Copper Queen Mine in Bisbee and the Nacozari Mine in Sonora. By 1907, the Smelter was processing 10,000 tons of ore per month. During a 10-hour shift on a given day, copper ore arrived at Douglas at a rate modeled by

$$E(t) = 18 + 71\cos^2 \frac{2}{3}t \text{ tons of ore per hour.}$$

Workers arrive at 4am and begin to process the ore. At that time, there were 17 tons of unprocessed ore left from the last shift.

The shift supervisor measures their rate of output every few hours and records the findings in the chart below.

t = time after midnight in hours	4	6	8	11	14
$P(t)$= Rate of ore processed in tons/hour	0	82.6	72.7	46.7	34.9

The supervisor notices that the workers' rate of processing decreased throughout the day.

(a) How many tons of copper ore arrive at the C&A Smelter between 4am and 2pm?
(b) Use a Right Riemann sum with subintervals indicated by the table to approximate $\int_4^{14} P(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.
(c) Is your approximation in part (b) an under- or over-approximation? Explain.
(d) The workers end their shift at 2pm. At that time, is there still unprocessed ore at the Smelter? Explain your reasoning.

(a) How many tons of copper ore arrive at the C&A Smelter between 4am and 2pm?

$$\int_4^{14} E(t)\,dt = 551.813 \ tons$$

(b) Use a Right Riemann sum with subintervals indicated by the table to approximate $\int_4^{14} P(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

$$\int_4^{14} P(t)\,dt \approx 2(82.6)+2(72.7)+3(46.7)+3(34.9)=562.6\ tons$$

Approximately 562.6 tons of copper were processed between 4 a.m. and 2 p.m.

(c) Is your approximation in part (b) an under- or over-approximation? Explain.

The approximation is an under-estimate because the data on the table show a decreasing function and right-hand Riemann rectangles under-estimate a decreasing function.

(d) The workers end their shift at 2pm. At that time, is there still ore in the smelter left to process? Explain your reasoning.

Yes, there was still ore left at the end of the day because, while 562.2 tons of ore were processed, there were 568.813 tons on site—17 tons at the beginning of the day and 551.813 tons which were delivered.

or

$$\begin{aligned} Total(t) &= initial\ value + \int_a^t [incoming\ rate]\,dx - \int_a^t [outgoing\ rate]\,dx \\ &= 17 + \int_4^{14} E(x)\,dx - \int_4^{14} P(x)\,dx \\ &= 17 + 551.813 - 562.2 \\ &= 6.613\ tons \end{aligned}$$

Ex 2 On May 15, the weather in the town of Apcalc changes at a rate of $W(t)$ degrees Fahrenheit per hour. $W(t)$is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight, t=0, the weather in Apcalc is 40 degrees Fahrenheit.

t (in hours since midnight)	0	1	3	6	8
$W(t)$ (in degrees Fahrenheit per hour)	-2.4	-2.1	-1.2	1.8	4.5

a) At approximately what rate is the rate of change of the temperature changing at 2am (t=2)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Is there a time when the rate of change of the temperature equals 7? Justify your answer.

d) Is there a time in $0 \le t \le 8$ when $W(t) = 0$? Justify your answer.

a) $W'(2) = \dfrac{-1.2-(-2.1)}{3-1} = 0.45$ °F/hr^2

b) $\int_0^8 W(t)\,dt \approx 1(-2.1)+2(-1.2)+3(1.8)+2(4.5) = 9.9.$

The total temperature change between midnight and 8am is 9.9°F.

c) $W(t)$ is twice differentiable, so the MVT applies. $\dfrac{1.8-(-2.4)}{6-0} = \dfrac{42}{6} = 7,$ therefore there must be a c where $W'(c) = 7$.

d) $W(t)$ is twice differentiable, so it is continuous and the IVT applies. $W(0)=-2.4<0$ and $W(8)=4.5>0$, so there must be a c in $0\le t\le 8$ when $W(c)=0$

x	$-3<x<-1$	$x=-1$	$-1<x<1$	$x=1$	$1<x<3$
$f'(x)$	Positive	DNE	Negative	0	Negative
$f''(x)$	Positive	DNE	Positive	0	Negative

Ex 3: A function f is continuous on the interval $x\in[-3, 3]$ such that $f(-3)=4$ and $f(3)=1$. The functions f' and f'' have the properties given above.

(a) Find all the values of x for which has a maximum or a minimum on $x\in[-3, 3]$. Justify your answer.

(b) Find all the values of x for which f has a point of inflection on $x\in[-3, 3]$. Justify your answer.

(c) Sketch a graph of $f(x)$ on $x\in[-3, 3]$.

(a) Find all the values of x for which has a maximum or a minimum on $x\in[-3, 3]$. Justify your answer.

$x=-1$ is at a maximum because $f'(-1)$ does not exist and the signs of f' change from positive to negative.

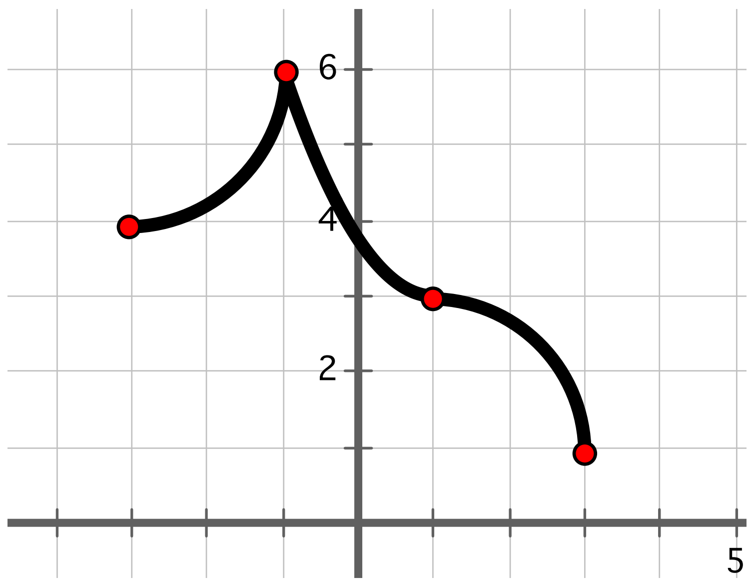
$x=-3$ is at a minimum because it is the left endpoint and f' is positive after $x=-3$.

$x=3$ is at a minimum because it is the right endpoint and f' is negative before $x=3$.

(b) Find all the values of x for which f has a point of inflection on $x \in [-3, 3]$. Justify your answer.

$x = 1$ is a point of inflection because the signs of f'' change there.

(c) Sketch a graph of $f(x)$ on $x \in [-3, 3]$.



8.2 Free Response Homework

AP Handout: 2013AB #3, 2016AB #1, 2017AB #1, 2018AB #4, 2019AB #2

8.3 Graphical Analysis

OBJECTIVES

Interpret information in the graph of a derivative in terms of the graph of the "original" function.

Key Ideas:

1. Within the graph of $f'(x)$ are hidden the sign patterns of f' and f''.
   - The f' signs come from where the graph of $f'(x)$ is above or below the x-axis.
   - The f'' signs come from where the graph of $f'(x)$ is increasing or decreasing.
2. Endpoints cannot be Points of Inflection.
3. Always make justification based on what is given.
4. If $g(x)$ is defined as the $\int f(x)dx$, be sure to state that $f(x)=g'(x)$.
5. If $g(x)=\int_a^x f(t)dt$, you must state that $g'(x)=f(t)$.
6. The y-values for $g(x)$ come from the area between the curve and the x-axis.
   - Pay attention to the lower bound on the integral. That will determine positive and negative values for $g(x)$.

Common Sub-Topics:

- Relative maximums and minimums on $g(x)$.
- POIs on $g(x)$.
- Intervals of increasing, decreasing, upward concavity, or downward concavity on $g(x)$.

- Sketching $g(x)$.
- Finding values of $g(x)$, $g'(x)$, and $g''(x)$
- Absolute extrema.

$g(x)=\int_0^x f(t)\,dt$	The y-values of $g(x)$	Increasing EXTREME Decreasing	Concave up POI Concave down
$f(x)$	Area under *F(x)*	Positive ZERO Negative	Increasing EXTREME Decreasing
$f'(x)$			Positive ZERO Negative

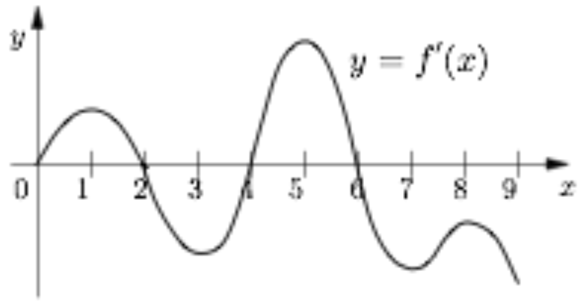
Summary of Key Phases for Justification

1. "$x=a$ is at a maximum on $f(x)$ because $f'(x)$ switches from positive to negative."

2. "$x=a$ is at a minimum on $f(x)$ because $f'(x)$ switches from negative to positive."

3. "$x=a$ is at a point of inflection on $f(x)$ because $f'(x)$ switches from increasing to decreasing (or decreasing to increasing)."

Ex 1 The domain of a function f is the interval $x \in [0, 9]$. The graph of the function f', the derivative of f, is given below.



a) Find all the values of x for which f has a relative maximum or a relative minimum on $x \in [0, 9]$. Justify your answer.
b) For what values of $x \in [0, 9]$ is f concave up? Justify your answer.
c) Using the information found in a) and b) and the fact that $f(0) = 0$, sketch a graph of f on $x \in [0, 5]$.

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a) Find all the values of x for which f has a relative maximum or a relative minimum on $x \in [0, 9]$. Justify your answer.

The zeros and endpoints of $f'(x)$ are the extremes of $f(x)$. Therefore, which f has a relative maximum or a relative minimum

$$x = 0, 2, 4, 6 \text{ and } 9$$

Maximums occur where $f'(x)$ switches from positive to negative. So, the maximums on $f(x)$ occur at $x = 2$ *and* 6.
Minimums occur where $f'(x)$ switches from negative to positive. So, the minimums on $f(x)$ occur at $x = 4$.
There is minimum at $x = 0$ because $x = 0$is a left endpoint and f is increasing after that endpoint because $f'(x)$ was positive.
There is minimum at $x = 9$ because $x = 9$is a right endpoint and f was decreasing before that endpoint because $f'(x)$ was negative.

b) For what values of $x \in [0, 9]$ is f concave up? Justify your answer.

$f(x)$ is concave up when $f'(x)$ is increasing. Therefore, f is concave up on

$$x \in (0,1), (3,5), \textit{and } (7,8)$$

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c) Using the information found in a) and b) and the fact that $f(0)=0$, sketch a graph of f on $x \in [0, 5]$.

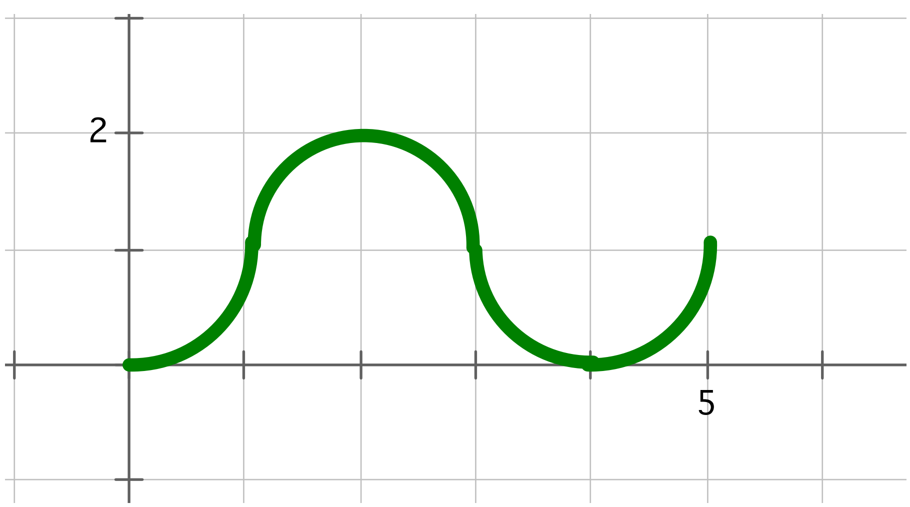
On $x \in (0,1)$, f is increasing and concave up.

On $x \in (1,2)$, f is increasing and concave down.

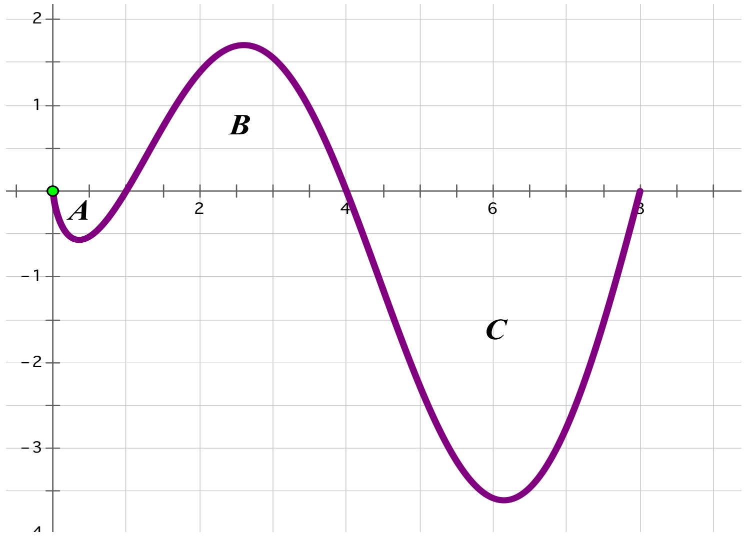
On $x \in (2,3)$, f is decreasing and concave down.

On $x \in (3,4)$, f is decreasing and concave up.

On $x \in (4,5)$, f is increasing and concave up.



Ex 2 Let $g(x)=\int_0^x f(t)\,dt$, where the graph of $f(x)$ is shown below. $f(x)$ has horizontal tangent lines at $x=\frac{1}{2}, \frac{5}{2}$, and 6.



Let $A=1$, $B=6$, and $C=14$. A, B, and C represent the areas between $f(x)$ and the x-axis.

(a) Find $g(1)$ and $g'(1)$

(b) On what intervals, if any, is $g(x)$ increasing and concave down? Explain your reasoning.

(c) For $x\in(0,\,8]$, find the value of x at which $g(x)$ has an absolute minimum. Justify your answer.

(a) Find $g(1)$ and $g'(1)$

$$g(1)=\int_0^1 f(t)\,dt=-(A)=-1$$
$$g'(1)=f(1)=0$$

(b) On what intervals, if any, is $g(x)$ increasing and concave down? Explain your reasoning.

$g(x)$ increasing and concave down when $g'(x)$ is positive and decreasing.

$g' = f$, so $g(x)$ increasing and concave down on $x \in \left(\frac{5}{2},\ 4\right)$

(c) For $x \in (0,\ 8]$, find the value of x at which $g(x)$ has an absolute minimum. Justify your answer.

Critical values on $g(x)$ are where $g'(x) = f(x) = 0$ and at the endpoints.

That is, $x = \{0, 1, 4, 8\}$

c.v.	$g(x)$
$x = 0$	1
$x = 1$	0
$x = 4$	5
$x = 8$	-9

The absolute minimum is -9 and occurs at $x = 8$.

8.3 Free Response Homework

AP Handout: 2015AB #5, 2016AB #3, 2017AB #3, 2018AB #3, 2019AB #3

8.4 Rectilinear Motion

Key Ideas:

- Know the relations among position, velocity, and acceleration.
- Velocity has direction (positive or negative) but speed does not.

Common Sub-Topics:

- Given velocity, find acceleration at a given time.
- Given velocity, find position at a given time.
- Finding time when particle switches direction
- Total distance versus displacement
- Speeding up or slowing down

Formulas:

Position = $x(t)$ *or* $y(t)$

Velocity = $x'(t)$ *or* $y'(t)$

Acceleration = $x''(t)$ *or* $y''(t)$

Therefore:

Position = $\int x'(t)dt$ *or* $\int y'(t)dt$

Velocity = $\int x''(t)dt$ *or* $\int y''(t)dt$

$$\text{Displacement} = \int_a^b v\,dt \qquad \text{Total Distance} = \int_a^b |v|\,dt$$

$$\text{Position at } x = a \quad = x(a) + \int_a^b v\,dt$$

Also remember from PreCalculus

1. The sign of the velocity determines the direction of the movement:

   Velocity > 0 means the movement is to the right (or up)
   Velocity < 0 means the movement is to the left (or down)
   Velocity = 0 means the movement is stopped.

2. Speeding up and slowing down is not determined by the sign of the acceleration.

An object is speeding up when $v(t)$ and $a(t)$ have the same sign.
An object is slowing when $v(t)$ and $a(t)$ have opposite signs.

Summary of Key Phases

When = solve for *t*
Where = solve for position
Which direction = is the velocity positive or is the velocity negative
Speeding up or slowing down = are the velocity and acceleration in the same direction or opposite (do they have the same sign or not)

Ex 1: Consider the velocity equation $v(t) = t^2 \sin t^3$ on $x \in [0, 3]$.

a) For what values of *t* is the particle moving up?
b) What is the acceleration at $t = 2$? Show the derivative work.
c) Find the particular position equation if $y(0) = 3$.

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a) For what values of *t* is the particle moving up?

$$v(t) = t^2 \sin t^3 = 0 \rightarrow t = 0,\ or\ \ t^3 = 0 \pm \pi n = \sqrt[3]{\pi}, \sqrt[3]{2\pi}, \sqrt[3]{3\pi}, \ldots$$
$$t = 0,\ 1.465,\ or\ 1.845$$

$v(t)$	0	+	0	−	0	+	0
t	0		$\sqrt[3]{\pi}$		$\sqrt[3]{2\pi}$		3

The particle is moving up when the velocity is positive. So,

$$t \in \left[0, \sqrt[3]{\pi}\right] \cup \left[\sqrt[3]{2\pi}, 3\right]$$

b) What is the acceleration at $t=2$? Show the derivative work.

$$a(t)=\frac{d}{dt}\left[t^2\sin t^3\right]=t^2\left(\cos t^3\right)\left(3t^2\right)+\left(\sin t^3\right)(2t)$$
$$a(2)=16\cos 8+4\sin 8=1.629$$

c) Find the particular position equation if $y(0)=3$.

$$y(t)=\int\left(t^2\sin t^3\right)dt=\frac{1}{3}\int\sin t^3\left(3t^2dt\right)=-\frac{1}{3}\cos t^3+c$$

$$y(0)=3\rightarrow 3=-\frac{1}{3}\cos 0+c\rightarrow 3=-\frac{1}{3}+c\rightarrow\frac{10}{3}=c$$

$$y(t)=-\frac{1}{3}\cos t^3+\frac{10}{3}$$

t	0	.3	.7	1.3	1.7	2.2	2.8	3.3	4
$v(t)$	0	14.1	9.5	17.1	13.3	15.6	12.7	13.7	12.0

Ex 2: Pat takes her bike on a 4-hour ride. She records her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 4$ hours, as shown in the table above. For $0 \le t \le 4$, $v(t) > 0$.

(a) Use the data in the table to approximate Pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\dfrac{3\sin(2\pi t) + 8t}{t^2 + 2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time t = 1.7? Give a reason for your answer.

(a) Use the data in the table to approximate pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

$$a(1.5) \approx \frac{v(1.7) - v(1.3)}{1.7 - 1.3} = \frac{13.3 - 17.1}{1.7 - 1.3} = -\frac{3.8}{.4} = -9.5\ {}^{mi}\!/_{hr^2}$$

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\ dt$ in the context of the problem. Approximate $\int_0^4 v(t)\ dt$ using a left-hand Riemann sum using the values from the table.

$\int_0^4 v(t)\ dt$ would be the approximate number of miles Pat traveled during her four-hour ride.

$$\int_0^4 v(t)\ dt \approx .3(0)+.4(14.1)+.5(9.5)+.4(17.1)+.5(13.3)+.6(15.6)+.5(12.7)+.7(13.7)$$
$$= 49.18\ miles$$

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\dfrac{3\sin(2\pi t)+8t}{t^2+2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

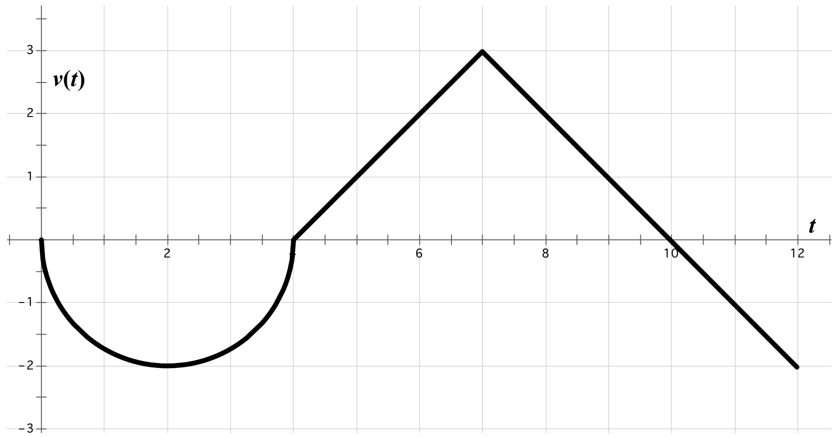
$$AveVelocity = \frac{1}{4-0}\int_0^4 9\sqrt{\frac{3\sin(2\pi t)+8t}{t^2+2}}dt = 13.350\ mph$$

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time $t = 1.7$? Give a reason for your answer.

$a(1.7) = f'(1.7) = -3.288\ {}^{mi}\!/_{hr^2}$

Since it was stated that "For $0 \le t \le 4$, $v(t) > 0$," Pat's speed is decreasing because the velocity and acceleration have opposite signs at $t = 1.7$.

Ex 3 A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t \in [0, 12]$ is shown below.



The figure shown is comprised of a semicircle and two line segments.

(a) At what times, if any, does the particle switch directions?

(b) At what time on $t \in [0, 12]$ is the *speed* the greatest?

(c) What is the total distance traveled by the particle on $t \in [0, 12]$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 10$?

(a) At what times, if any, does the particle switch directions?

$v(t) = 0$ and switches signs at $\text{t} = 4$ and 10.

(b) At what time on $t \in [0, 12]$ is the *speed* the greatest?

The range of $v(t)$ is $v \in [-2, 3]$. Speed is $|v(t)|$ so the greatest speed is 3

(c) What is the total distance traveled by the particle on $t \in [0,\ 12]$

Total distance traveled = $\int_0^{12} |v(t)| dt$.

$$\begin{aligned}\int_0^{12} |v(t)| dt &= -\int_0^4 v(t) dt + \int_4^{10} v(t) dt - \int_{10}^{12} v(t) dt \\ &= -(-2\pi) + 9 - (-2) \\ &= 11 + 2\pi\end{aligned}$$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

$$x(8) = 6 + \int_0^8 v(t) dt = 6 + (-2\pi) + \frac{9}{2} = \frac{21}{2} - 2\pi$$

Ex 4 Two particles move along the x-axis. For $0 \le t \le 8$, the position of particle *P* at time *t* is given by $x_P(t) = \ln(t^2 - 2t + 10)$, while the velocity of particle Q at time *t* is given by $v_Q(t) = t^2 - 8t + 15$. Particle Q is at position $x = 5$ at time $t = 0$.

(a) For $0 \le t \le 8$, when is particle *P* moving to the left?

(b) For $0 \le t \le 8$, find all times *t* during which the two particles travel in the same direction.

(c) Find the acceleration of particle *Q* at time *t* = 2. Is the speed of particle *Q* increasing, decreasing, or neither at time *t* = 2? Explain your reasoning.

(d) Find the position of particle *Q* the first time it changes direction.

(a) For $0 \le t \le 8$, when is particle P moving to the left?

$$x_P(t) = \frac{d}{dt}\ln(t^2 - 2t + 10) = \frac{2t - 2}{t^2 - 2t + 10} = 0 \to t = 1$$

$$\begin{array}{c|ccc} v & - & 0 & + \\ t & & 1 & \end{array} \qquad 0 \le t < 1$$

(b) For $0 \le t \le 8$, find all times *t* during which the two particles travel in the same direction.

$$v_Q(t) = t^2 - 8t + 15 = (t-3)(t-5) = 0 \to t = 3 \; and \; 5$$

$$\begin{array}{c|ccccc} v & + & 0 & - & 0 & + \\ t & & 3 & & 5 & \end{array} \qquad 0 \le t \le 1 \text{ and } 5 \le t \le 8$$

(c) Find the acceleration of particle Q at time $t=2$. Is the speed of particle Q increasing, decreasing, or neither at time $t=2$? Explain your reasoning.

$$a_Q(t)=2t-8 \rightarrow a_Q(2)=-4$$

Since $v_P(2)>0$ and $a_Q(2)=-4$, the speed is decreasing because the velocity and acceleration have opposite signs.

(d) Find the position of particle Q the first time it changes direction.

$$x_Q(t)=5+\int_0^3 \left(t^2-8t+15\right)\,dt=5+\left(\frac{1}{3}t^3-4t^2+15t\right)\Bigg|_0^3=23$$

8.4 Free Response Homework

1. AP Packet: 2013AB #2, 2014AB #4, 2015AB #3, 2016AB #2, 2017 #5, 2018AB #2, 2019AB # 2

8.5: Differential Equations

Key Ideas:

- See **Steps to Solving a Differential Equation** below.

Common Sub-Topics:

- Separation of variables
- Drawing a slope field or sketching one or more solution curve to a slope field.
- Tangent line approximations
  - Over-estimates or underestimates
- Logistic growth
- Related Rates

Summary of Key Phases

- "Use separation of variables"
- "Particular solution to the differential equation"

Steps to Solving a Differential Equation:

1. Separate the variables. Note: Leave constants on the right side of the equation.
2. Integrate both sides of the equation, using u-subs as necessary.
   Note: only write the $+C$ on the right side of the equation.
3. Plug in the initial condition, if you are given one, and solve for C.
   Note: Solve for C immediately if the left integral does not result in a ln. Simplify before solving if there is a ln.
4. Solve for y.
   Note: $e^{\ln|y|} = y$ because e raised to any power is automatically positive, so the absolute values are not necessary.

Ex 1 Consider the differential equation $\frac{dy}{dx} = 3y + 4xy$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = -2$.

(a) Write an equation for the line tangent to the graph of f at $x = 0$ and use it to approximate $f(0.2)$.

(b) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = 3y + 4xy$ with the initial condition $f(0) = -2$.

(a) Write an equation for the line tangent to the graph of f at $x = 0$ and use it to approximate $f(0.2)$.

$$m_{\tan} = \left.\frac{dy}{dx}\right|_{(0,-2)} = 3(-2) + 4(0)(-2) = -6$$

$$y + 2 = -6(x - 0)$$
$$y = -6x - 2$$

$$f(0.2) \approx y(0.2) = -6(0.2) - 2 = -3.2$$

(b) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = 3y + 4xy$ with the initial condition $f(0) = -2$.

$$\frac{dy}{dx} = 3y + 4xy = y(4x + 3)$$
$$\frac{1}{y}dy = (4x + 3)dx$$
$$\int \frac{1}{y}dy = \int (4x + 3)dx$$
$$\ln|y| = 2x^2 + 3x + c$$

$$e^{\ln|y|} = e^{2x^2+3x+c}$$

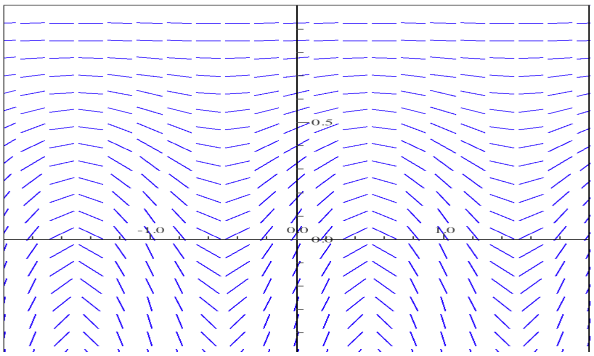
$$y = ke^{2x^2+3x}$$

$f(0) = -2 \rightarrow -2 = ke^0 \rightarrow k = -2$

$$y = -2e^{2x^2+3x}$$

Ex 2: Consider the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$.

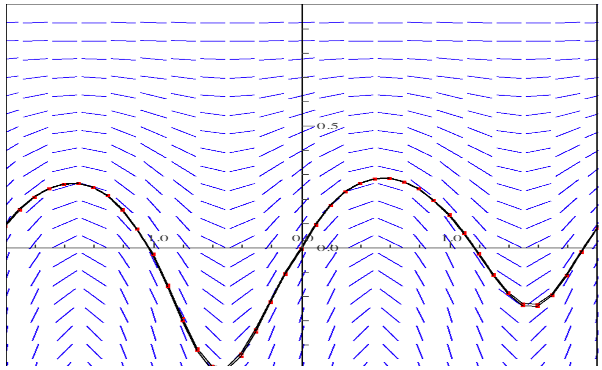
(a) A portion of the slope field of the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$ is shown below. Sketch the solution curves through $(0, 0)$.



(b) Write an equation for the line tangent to solution curve in part a) at the point $(0, 0)$. Use the equation to approximate $f(0.2)$.

(c) Find the particular solution to the differential equation $\frac{dy}{dx} = (y-1)^2 \cos(\pi x)$ at $\left(\frac{1}{2}, 0\right)$.

(a) Sketch the solution curves through $(0,0)$.



(b) Write an equation for the line tangent to solution curve in part a) at the point $(0,-1)$. Use the equation to approximate $f(0.2)$.

At the point $(0,-1)$, $\frac{dy}{dx}=(2)^2\cos(0)=4$.

$$y+1=4(x-0)$$

$$f(0.2)\approx y(0.2)=4(0.2)-1=-0.2$$

(c) Find the particular solution to the differential equation $\frac{dy}{dx}=(y-1)^2\cos(\pi x)$ at $\left(\frac{1}{2},0\right)$.

$$\frac{dy}{dx}=(y-1)^2\cos(\pi x)$$

$$\frac{1}{(y-1)^2}dy=\cos(\pi x)dx$$

$$\int\frac{1}{(y-1)^2}dy=\int\cos(\pi x)dx$$

$$\int(y-1)^{-2}dy=\frac{1}{\pi}\int\cos(\pi x)(\pi dx)$$

$$\frac{(y-1)^{-1}}{-1}=\frac{1}{\pi}(\sin(\pi x))+c$$

$$\left(\frac{1}{2},0\right)\rightarrow\frac{(0-1)^{-1}}{-1}=\frac{1}{\pi}\left(\sin\left(\frac{\pi}{2}\right)\right)+c\rightarrow c=1-\frac{1}{\pi}=\frac{\pi-1}{\pi}$$

$$-\frac{1}{y-1}=\frac{1}{\pi}(\sin(\pi x))+\frac{\pi-1}{\pi}=\frac{\sin(\pi x)+\pi-1}{\pi}$$

$$\frac{1}{y-1}=-\frac{\sin(\pi x)+\pi-1}{\pi}$$

$$y-1=-\frac{\pi}{\sin(\pi x)+\pi-1}$$

$$y=1-\frac{\pi}{\sin(\pi x)+\pi-1}$$

Ex 3: AP 1997 AB #6

Let $v(t)$ be the velocity, in feet per second, of a skydiver at time $t\geq 0$ seconds. After her parachute opens, her velocity satisfies the differential equation $\frac{dv}{dt}=-2v-32$, with the initial condition $v(0)=-50$.

(a) Use separation of variables to find an expression for *v* in terms of *t*, where *t* is measured in seconds.

(b) Terminal velocity is defined as $\lim_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

(a) Use separation of variables to find an expression for v in terms of t, where t is measured in seconds.

$$\frac{dv}{dt} = -2v - 32$$

$$\frac{dv}{-2v-32} = dt$$

$$-\frac{1}{2}\int \frac{-2dv}{-2v-32} = \int dt$$

$$-\frac{1}{2}\ln\left|-2v-32\right| = t + c$$

$$\ln\left|-2v-32\right| = -2t + c$$

$$\left|-2v-32\right| = e^{-2t+c}$$

$$-2v - 32 = ke^{-2t}$$

$$v(0) = -50 \to -2(-50) - 32 = ke^{0} \to k = 68$$

$$-2v - 32 = 68e^{-2t}$$

$$-2v = 32 + 68e^{-2t}$$

$$v = -16 - 34e^{-2t}$$

(b) Terminal velocity is defined $\lim\limits_{t\to\infty} v(t)$. Find the terminal velocity of the skydiver to the nearest foot per second.

$$\lim_{t\to\infty}\left(-16 + 34e^{-2t}\right) = -16$$

(c) It is safe to land when her speed is 20 feet per second. At what time t does she reach this speed?

$$-16-34e^{-2t}=-20$$
$$34e^{-2t}=4$$
$$e^{-2t}=\frac{2}{17}$$
$$t=-\frac{1}{2}\ln\frac{2}{17}=1.070$$

<u>8.5 Free Response Homework</u>

AP Handout: 2015AB #6, 2016AB #4, 2017AB #6, 2018AB #6, 2019AB #4

8.6: Area & Volume

Key Ideas:

- Know the formulas and when each one applies.
- The direction of the rectangles determines everything in terms of set-up and which formula to use.
    - If the the rectangles are vertical, x is the variable in the problem.
    - If the the rectangles are horizontal, y is the variable in the problem.
- You can switch the rectangles' direction by rearranging the equations.

Common Sub-Topics:

- Area
- Rotation about an axis
- Rotation about a line
- Cross sections
- Related Rates

Formulas:

Area: $A=\int_a^b [top-bottom]dx \quad or \quad A=\int_c^d [right-left]dy$

Volume by Disk Method: $V=\pi\int_a^b [R(x)]^2 dx \quad or \quad V=\pi\int_c^d [R(y)]^2 dy$

Volume by Washer Method

$$V=\pi\int_a^b \left\{[R(x)]^2-[r(x)]^2\right\} dx \quad or \quad V=\pi\int_c^d \left\{[R(y)]^2-[r(y)]^2\right\} dy$$

$R(x)$ or $r(x)$ are the distances from the further curve and nearer curve, respectively, to the line around which the region is rotating.

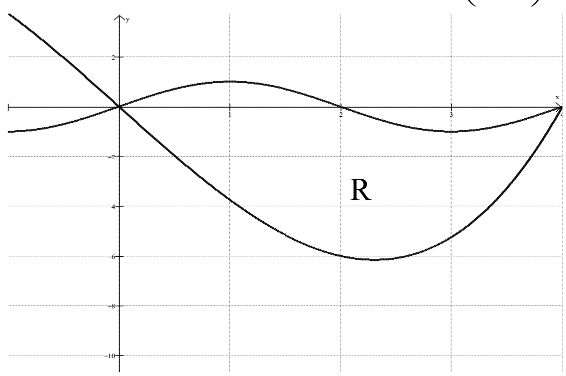
Volume by Cross-section Method

$$V=\int_a^b A(x)dx \qquad or \qquad V=\int_c^d A(y)dy$$

Arc Length $L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ or $L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$

NB. This topic is no longer on the AP Calculus AB Exam.

Ex 1 Let R be the region bounded by the graphs $y=\sin\left(\frac{\pi}{2}x\right)$ and $y=\frac{1}{4}\left(x^3-16x\right)$



a) Find the area of the region R.
b) Find the volume of the solid when the region R is rotated around the line $y=-10$.
c) Find the volume of the solid where the region R is the base and the cross-sections perpendicular to the x-axis are rectangles which are four times as tall as they are wide.

---

a) Find the area of the region R.

$$Area_R=\int_0^4\left[\left(\sin\frac{\pi}{2}x\right)-\left(\frac{1}{4}\left(x^3-16x\right)\right)\right]dx=16$$

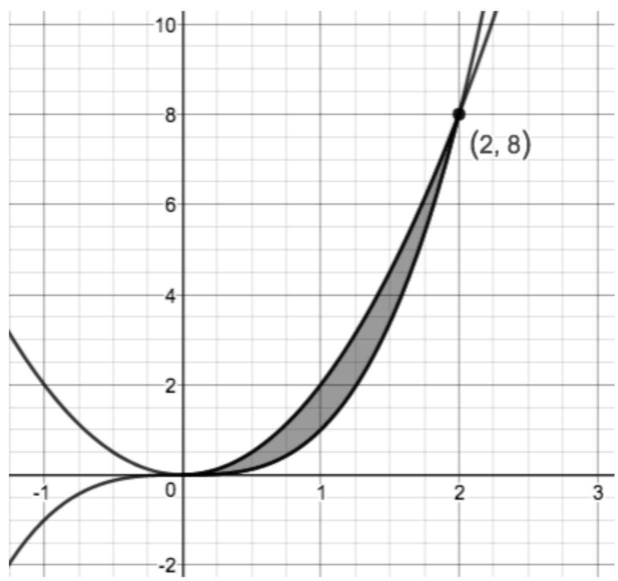
b) Find the volume of the solid when the region R is rotated around the line $y=-10$.

$$V=\pi\int_0^4\left(\sin\frac{\pi}{2}x-(-10)\right)^2-\left(\frac{1}{4}\left(x^3-16x\right)-(-10)\right)^2dx=243.981$$

c) Find the volume of the solid where the region R is the base and the cross-sections perpendicular to the x-axis are rectangles which are four times as tall as they are wide.

$$V=\int_0^4\left(\sin\frac{\pi}{2}x-\frac{1}{4}\left(x^3-16x\right)\right)\cdot 4\left(\sin\frac{\pi}{2}x-\frac{1}{4}\left(x^3-16x\right)\right)dx=307.691$$

Ex 2 Let R be the region bounded by $y=2x^2$ and $y=x^3$, shaded in the picture below. The curves intersect at the origin and at the point $\left(2,8\right)$.



(a) Find the area of R.
(b) Find the volume of the solid formed by revolving R around the y-axis.
(c) Set up, but do not evaluate, an expression involving one or more integrals that would find the volume of the solid whose base is R and whose cross sections parallel to the x-axis are semicircles.

(a) Find the area of R.

$$A = \int top\ curve - bottom\ curve = \int_0^2 \left(2x^2 - x^3\right) dx$$

$$= \left[\frac{2}{3}x^3 - \frac{1}{4}x^4\right]_0^2$$

$$= \frac{16}{3} - \frac{16}{4}$$

$$= \frac{16}{12} = \frac{4}{3}$$

(b) Find the volume of the solid formed by revolving R around the y-axis.

In order to rotate around the y-axis, the equations must have x isolated.

$$y = 2x^2 \rightarrow x = \left(\frac{y}{2}\right)^{1/2} \qquad\qquad y = x^3 \rightarrow x = y^{1/3}$$

$$V = \pi \int_c^d \left(\left[R(y)\right]^2 - \left[r(y)\right]^2\right) dy$$

$$= \pi \int_0^8 \left(\left[y^{1/3}\right]^2 - \left[\left(\frac{y}{2}\right)^{1/2}\right]^2\right) dy$$

$$= \pi \int_0^8 \left(y^{2/3} - \frac{1}{2}y\right) dy$$

$$= \pi \left[\frac{3}{5}y^{5/3} - \frac{1}{4}y^2\right]_0^8$$

$$= \pi \left[\frac{96}{5} - 16\right]$$

$$= \frac{16\pi}{5}$$

(c) Set up, but do not evaluate, an expression involving one or more integrals that would find the volume of the solid whose base is R and whose cross sections parallel to the x-axis are semicircles.

$$A_{semicircle} = \frac{\pi}{2} r^2$$

Parallel to the x-axis means the equations must have x isolated. So the diameter of the semicircles would be $y^{1/3} - \left(\frac{y}{2}\right)^{1/2}$ and $r = \frac{1}{2}\left[y^{1/3} - \left(\frac{y}{2}\right)^{1/2} \right]$

$$\begin{aligned} V &= \int_c^d A(y)\,dy \\ &= \int_c^d \frac{\pi}{2} r^2 dy \\ &= \int_0^8 \frac{\pi}{2}\left[\frac{1}{2}\left[y^{1/3} - \left(\frac{y}{2}\right)^{1/2}\right]^2\right] dy \\ &= \frac{\pi}{4}\int_0^8 \left[y^{1/3} - \left(\frac{y}{2}\right)^{1/2}\right]^2 dy \end{aligned}$$

8.6 Free Response Homework

AP Handout: 2015AB #2, 2016AB #5, 2017AB #1, 2018AB #, 2019AB #5

8.7 Related Rates

Key Ideas (Related Rates):

1. Related Rates problems are implicit differentiation in terms of time.
2. The relationship between variables is usually geometric.
3. The units tell everything in terms of substituting of values.

Process for Related Rates Problems:

1. Determine what is being asked.
    a. Look at the units to determine what is given and what is asked.
2. Determine the equation that relates the variable to each other variable. (NB. This will be the one to be differentiated)
3. Determine what is given.
    a. Look at the units to determine what is given and what is asked.
4. If there is a product of two variables, eliminate the product by either multiplying or substituting a secondary equation.
    a. Note: This is only because we have not learned the Product Rule yet.
5. Differentiate in terms of time.
    a. Do not forget the implicit fractions.
6. Substitute and solve for the missing variable.

Common Sub-Topics (Related Rates):

- Rate of change
- Volume
- Average value

Ex 1 Gravel is being dumped from a conveyor belt at a rate of 35π ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. Remember that $V_{cone} = \frac{\pi}{3} r^2 h$

(a) What is the volume of the gravel pile when the height is 14 ft?
(b) How fast is the height of the pile increasing when the pile is 14 ft high?
(c) How fast is the base area of the pile changing when the pile is 14 ft high?

(a) What is the volume of the gravel pile when the height is 14 ft?

$$diameter = h \rightarrow r = \frac{1}{2} h$$
$$V_{cone} = \frac{\pi}{3}\left(\frac{1}{2}h\right)^2 h = \frac{\pi}{12} h^3 \rightarrow V(14) = \frac{686\pi}{3}\,ft^3$$

(b) How fast is the height of the pile increasing when the pile is 14 ft high?

$$\frac{d}{dt}\left[V = \frac{\pi}{12}h^3\right] \rightarrow \frac{dV}{dt} = \frac{\pi}{4}h^2 \frac{dh}{dt}$$
$$35\pi = \frac{\pi}{4}(14)^2 \frac{dh}{dt}$$
$$\frac{dh}{dt} = \frac{5}{7}\, {}^{ft}\!/_{min}$$

(c) How fast is the base area of the pile changing when the pile is 14 ft high?

$V_{cone} = \frac{\pi}{3}r^2h = \frac{1}{3}Bh$, where $B = \pi r^2$ is the base area.

$$\frac{d}{dt}\left[V = \frac{1}{3}Bh\right]$$
$$\frac{dV}{dt} = \frac{1}{3}B\frac{dh}{dt} + \frac{1}{3}h\frac{dB}{dt}$$

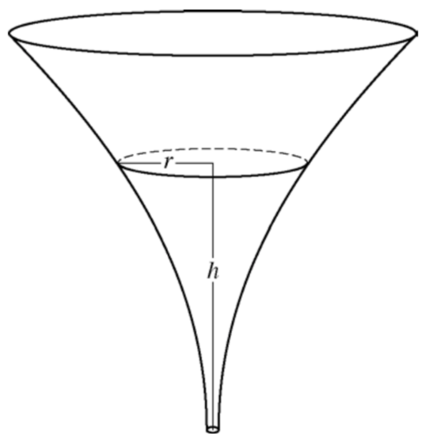
$h = 14 \rightarrow r = 7 \rightarrow B = 49\pi$

$$35\pi = \frac{1}{3}(49\pi)\left(\frac{5}{7}\right) + \frac{1}{3}(14)\frac{dB}{dt}$$
$$35\pi - \frac{35\pi}{3} = \frac{14}{3}\frac{dB}{dt}$$
$$\frac{70\pi}{3} = \frac{14}{3}\frac{dB}{dt}$$
$$\frac{dB}{dt} = 5\pi \ {ft^2}/{min}$$

<u>Related Rates can also be part of a different problem.</u>

Ex 2: AB 2016 #5

Question 5



The inside of a funnel of height 10 inches has circular cross sections, as shown in the figure above. At height h, the radius of the funnel is given by $r = \frac{1}{20}\left(3 + h^2\right)$, where $0 \le h \le 10$. The units of r and h are inches.

(a) Find the average value of the radius of the funnel.

(b) Find the volume of the funnel.

(c) The funnel contains liquid that is draining from the bottom. At the instant when the height of the liquid is $h = 3$ inches, the radius of the surface of the liquid is decreasing at a rate of $\frac{1}{5}$ inch per second. At this instant, what is the rate of change of the height of the liquid with respect to time?

(c) $\frac{dr}{dt} = \frac{1}{20}(2h)\frac{dh}{dt}$ $-\frac{1}{5} = \frac{3}{10}\frac{dh}{dt}$ $\frac{dh}{dt} = -\frac{1}{5}\cdot\frac{10}{3} = -\frac{2}{3}$ in/sec	$3 : \begin{cases} 2 : \text{chain rule} \\ 1 : \text{answer} \end{cases}$

<u>8.7 Free Response Homework</u>

2002AB #6, 2002AB Form B #6, 2003AB #5, 2014AB #4, 2016AB #5, 2017AB #1, 2019AB #4

8.8 Implicit Differentiation

Common Sub-Topics:

Horizontal and vertical tangent lines

Points on a curve with a particular slope

Second Derivative Test

Equation of a tangent line

Key Ideas:

1. Any time the variable being differentiated is different from the variable the derivative is in terms of, an implicit fraction arises.
2. Do not simplify the algebra if you are going to plug numbers in for x and y.

Ex 1 Consider the curve given by $x^2+4xy+y^2=-12$.

(a) Show that $\frac{dy}{dx}=-\frac{x+2y}{2x+y}$.

(b) Find the equations of all the tangent lines which are horizontal.

(c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

(a) Show that $\frac{dy}{dx}=-\frac{x+2y}{2x+y}$.

$$\frac{d}{dx}\left[x^2+4xy+y^2=-12\right]$$

$$2x+4x\frac{dy}{dx}+4y(1)+2y\frac{dy}{dx}=0$$

$$4x\frac{dy}{dx}+2y\frac{dy}{dx}=-2x-4y$$

$$(4x+2y)\frac{dy}{dx}=-2x-4y$$

$$\frac{dy}{dx}=\frac{-2x-4y}{4x+2y}=-\frac{x+2y}{2x+y}$$

Note the use of the Product Rule for $\frac{d}{dx}\left[4xy\right]$.

(b) Find the equations of all the tangent lines which are horizontal.

Horizontal lines have a slope = 0, so $\frac{dy}{dx}=-\frac{x+2y}{2x+y}=0\rightarrow x+2y=0\rightarrow x=-2y$

To be on the curve, $x=-2y$ must make the original equation true:

$$(-2y)^2 + 4(-2y)y + y^2 = -12$$
$$4y^2 - 8y^2 + y^2 = -12$$
$$-3y^2 = -12$$
$$y^2 = 4$$
$$y = \pm 2$$

$$x = -2y \rightarrow (-4, 2) \text{ and } (4, -2)$$

(c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx} = -\frac{x+2y}{2x+y}\right] = -\frac{(2x+y)\left(1+2\frac{dy}{dx}\right) - (x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-4,2)} = -\frac{(2(-4)+2)(1+2(0)) - (-4+4)(2+(0))}{(2(-4)+2)^2} = \frac{6}{(-6)^2} > 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,-2)} = -\frac{(2(4)-2)(1+2(0)) - (4-2(2))(2+(0))}{(2(4)-2)^2} = \frac{-6}{(6)^2} < 0$$

$(-4, 2)$ will be at a minimum because the second derivative is positive.
$(4, -2)$ will be at a maximum because the second derivative is negative

8.8 Free Response Homework

AP Handout: 2012AB #5, 2015AB #6, 2021AB #5, 2023AB #6

8.9 Miscellaneous Topics

Common Sub-Topics:

Derivatives and Integrals from tables and/or graphs

Limits and continuity

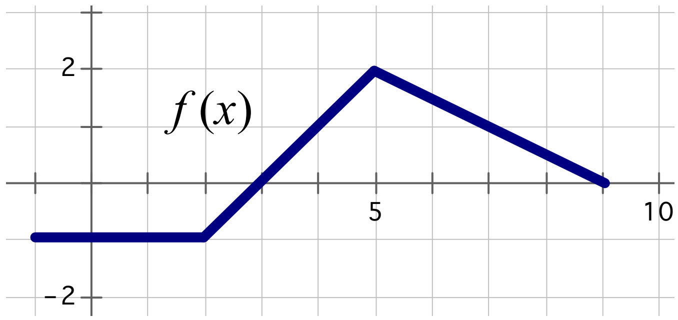
L'Hopital's Rule

Key Ideas (Connecting Representations):

1. Information may be given in algebraic, tabular, or graphical form. Change in delivery does not change what the information means.

Key Ideas for Piece-wise Defined Functions:

1. Know and use the formal (three-part) definition of continuity.
2. Differentiable functions are always continuous, but continuous functions are not always differentiable.
3. Use the Limit symbols properly.
4. Each of the families of functions studied in PreCalc are continuous and differentiable in their domain. SO, know the domains!



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

Ex 1 Let $f(x)$ be the function whose graph is given above and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above. Furthermore, let h be the function defined by $h(x)=\ln(x^2+4)$.

(a) Find the equation of the line tangent to $f(x)$ at $x=4$.

(b) Let K be the function defined by $K(x)=h(f(x))$. Find $K'(3)$.

(c) Let M be the function defined by $M(x)=g(x)\cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x)=\dfrac{g(x)}{h\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

(a) $f(4)=1$ and $f'(4)=1$. The tangent line equation is $y-1=1(x-4)$.

(b) $h(x)=\ln(x^2+4)\to h'(x)=\dfrac{2x}{x^2+4}$

$K(x)=h(f(x))\to K'(x)=h'(f(x))\cdot f'(x)$

$K'(3)=h'(f(3))\cdot f'(3)=h'(0)\cdot f'(3)=(0)\cdot(1)=0$

(c) $M(x)=g(x)\cdot f(x)\rightarrow M'(x)=g(x)\cdot f'(x)+g'(x)\cdot f(x)$

$$M'(6)=g(6)\cdot f'(6)+g'(6)\cdot f(6)=(6)\cdot\left(\frac{1}{2}\right)+(12)\cdot\left(\frac{3}{2}\right)=15$$

(d) $J(x)=\dfrac{g(x)}{h\left(\frac{1}{2}x\right)}\rightarrow J'(x)=\dfrac{h\left(\frac{1}{2}x\right)g'(x)-g(x)\cdot h'\left(\frac{1}{2}x\right)\left(\frac{1}{2}\right)}{\left[h\left(\frac{1}{2}x\right)\right]^2}$

$$J'(8)=\frac{h(4)g'(8)-g(8)\cdot h'(4)\left(\frac{1}{2}\right)}{\left[h(4)\right]^2}=\frac{8\ln 8-\frac{4}{5}}{\ln^2 8}$$

Ex 2   Based on 2012 AP Calculus AB FRQ #4

The function *f* is defined by $f(x)=\sqrt{169-x^2}$ for $x\in[-13,13]$.

(a) Find $f'(x)$.

(b) Write an equation of the line tangent to the graph of *f* at $x=5$.

(c) Let $g(x)=\begin{cases} f(x) & \text{if } -13\le x\le 5 \\ -\frac{3}{2}x+\frac{39}{2} & \text{if } 5\le x\le 13 \end{cases}$. Is $g(x)$ continuous at $x=5$?

Use the definition of continuity to explain your answer.

(d) Find the value of $\int_0^{13} x\cdot g(x)\,dx$.

(a) Find $f'(x)$.

$$f(x)=\sqrt{169-x^2}=\left(169-x^2\right)^{1/2}$$

$$f'(x)=\frac{1}{2}\left(169-x^2\right)^{-1/2}(-2x)=\frac{-x}{\left(169-x^2\right)^{1/2}}$$

(b) Write an equation of the line tangent to the graph of *f* at $x=5$.

$$y-y_1=m(x-x_1)$$

$f(x_1)=y_1\rightarrow f(5)=12$

$m=f'(5)=-\frac{5}{12}$

$$y-12=-\frac{5}{12}(x-5)$$

(c) Let $g(x)=\begin{cases} f(x), & \text{if } -13\le x\le 5 \\ -\frac{3}{2}x+\frac{39}{2}, & \text{if } 5<x\le 13 \end{cases}$. Is $g(x)$ continuous at $x=5$? Use the definition of continuity to explain your answer.

i) Does $g(5)$ exist? Yes, according to the domain of the first line, $x=5$ is in the domain of $g(x)$.

ii) Does the $\lim\limits_{x\to -1} g(x)$ exist? Yes.

$\lim\limits_{x\to 5^-} g(x)=\lim\limits_{x\to 5} f(x)=12=\lim\limits_{x\to 5}\left(-\frac{3}{2}x+\frac{39}{2}\right)=\lim\limits_{x\to 5^+} g(x)$Since both one-sided limits equal 4, then

iii) Does $\lim\limits_{x\to -1} g(x)=g(-1)$? yes. $\lim\limits_{x\to 5} g(x)=12=g(5)$

(d) Find the value of $\int_0^{13} x\cdot g(x)\,dx$.

$$\begin{aligned}
\int_0^{13} x\cdot g(x)\,dx &= \int_0^5 x\left(169-x^2\right)^{1/2}\,dx+\int_5^{13} x\left(-\frac{3}{2}x+\frac{39}{2}\right)dx \\
&= -\frac{1}{2}\int_0^5 \left(169-x^2\right)^{1/2}(-2dx)+\int_5^{13}\left(-\frac{3}{2}x^2+\frac{39}{2}x\right)dx \\
&= \left[-\frac{2}{3}\left(169-x^2\right)^{3/2}\right]_0^5+\left[-\frac{1}{2}x^3+\frac{39}{4}x^2\right]_5^{13} \\
&= -\frac{2}{3}\left[(144)^{3/2}-(169)^{3/2}\right]+\left[\left(-\frac{2197}{2}+\frac{6591}{4}\right)-\left(-\frac{125}{2}+\frac{4875}{4}\right)\right] \\
&= 524\frac{1}{3}
\end{aligned}$$

Ex 3 Let $f(x)=\begin{cases} \ln(1-x), & \text{if } x\le 0 \\ \tan x, & \text{if } 0<x \end{cases}$

(a) Is $f(x)$ continuous at $x=0$? Justify your answer.

(b) Find $f'(-1)$ and $f'\left(\frac{\pi}{4}\right)$.

(c) Express $f'(x)$ as a piecewise-defined function. Explain why $f'(0)$ does not exist.

(d) Find $\lim\limits_{x\to\pi}\frac{f'(x)-2}{\pi-x}$. Justify your answer.

(a) Is $f(x)$ continuous at $x=0$? Justify your answer.

i) Does $f(0)$ exists? $x=0\to\ln(1-0)=\ln 1=0$ $\therefore$yes, $f(0)$ exists.

ii. Does $\lim\limits_{x\to 0} f(x)$ exists? $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0}\ln(1-x)=\ln 1=0$

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0}\tan x=\tan 0=0$

$\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)$ $\therefore$yes, $\lim\limits_{x\to 0} f(x)$ exists

iii. Does $\lim\limits_{x\to 0} f(x)=f(0)$? Yes.

All three statements are true, therefore, $f(x)$ is continuous at $x=0$.

(b) Find $f'(-1)$ and $f'\left(\frac{\pi}{4}\right)$.

$$f'(-1)=\frac{-1}{1-0}=-1$$

$$f'\left(\frac{\pi}{4}\right)=\sec^2\frac{\pi}{4}=\left(\sqrt{2}\right)^2=2$$

(c) $f'(x)=\begin{cases} \dfrac{-1}{(1-x)}, & \text{if } x<0 \\ \sec^2 x, & \text{if } 0<x \end{cases}$. [Note the lack of = in the domains.]

$f'(0)$ does not exist, because $\lim_{x\to 0^-} f'(x) = -1 \neq 2 = \lim_{x\to 0^+} f'(x)$

(d) $\lim_{x\to\pi} \frac{f'(x)-2}{\pi - x} = \lim_{x\to\pi} \frac{\sec^2 x - 2}{\pi - x} \overset{L'H}{=} \lim_{x\to\pi} \frac{2\sec x(\sec x \tan x)}{-1} = 0.$

8.9 Free Response Homework

AP Handout: 2014AB #5, 2015AB #4, 2016AB #6, 2017AB #6, 2018AB #5, 2019AB #6, 2023AB #5.