Chapter 4:

Applications of the Derivative I: Graphing

Chapter 4 Overview: Derivatives and Graphs

There are two main contexts for derivatives: graphing and motion. In this chapter, we will consider the graphical applications of the derivative. Much of this is a review of material covered last year. Key topics include:

- Finding extremes
- The First and Second Derivative Tests
- Finding intervals of increasing and decreasing
- Finding intervals of concavity
- Optimization
- Graphing using the derivatives
- Making inferences regarding the original graph from the graph of its derivative.

Several multiple-choice questions and at least one full free response question (often parts of others), have to do with this general topic.

4.1: Extrema and the First Derivative Test

One of the most valuable aspects of Calculus is that it allows us to find extreme values of functions. The ability to find maximum or minimum values of functions has wide-ranging applications. Every industry has uses for finding extremes, from optimizing profit and loss, to maximizing output of a chemical reaction, to minimizing surface areas of packages. This one tool of Calculus eventually revolutionized the way the entire world approached every aspect of industry. It allowed people to solve formerly unsolvable problems.

OBJECTIVES

Find critical values and extreme values for functions.
Use the 1st and 2nd derivative tests to identify maxima vs. minima.

It will be helpful to keep in mind a few things from last year for this chapter (and all other chapters following).

REMINDER: *Vocabulary*:

1. *Critical Value*--The *x*-coordinate of the extreme
2. *Maximum Value*--The *y*-coordinate of the high point.
3. *Minimum Value*--The *y*-coordinate of the low point.
4. *Relative Extremes*--the highest or lowest points in any section of the curve.
5. *Absolute Extremes*--the highest or lowest points of the whole curve.
6. *Interval of Increasing*--the interval of *x*-values for which the curve is rising from left to right.
7. *Interval of Decreasing*--the interval of *x*-values for which the curve is dropping from left to right.

Critical Values of a function occur when

i. $\frac{dy}{dx} = 0$

ii. $\frac{dy}{dx} \quad dne$

iii. The *x*-coordinates of any domain restriction.

It is also helpful to remember that a **critical value** is referring specifically to the value of the x, while the **extreme value** refers to the value of the y.
The first derivative is what allows us to algebraically find extremes, and the first derivative test allows us to interpret critical values as maxima or minima. Since the first derivative of a function tells us when that function is increasing or decreasing, we can figure out if a critical value is associated with a maximum or minimum depending on the sign change of the derivative.

EX 1 Find the critical values algebraically of $y = 3x^4 + 2x^3 - 39x^2 + 36x - 4$.

$$\frac{dy}{dx} = 12x^3 + 6x^2 - 78x + 36 = 0$$

$$2x^3 + x^2 - 13x + 6 = 0$$

$$(2x-1)(x-2)(x+3) = 0$$

$$x = \frac{1}{2}, 2, \text{ or} - 3$$

One of the easiest ways to determine if a critical value is at a maximum, a minimum, or neither is through the First Derivative Test:

The 1st Derivative Test

As the sign pattern of the 1st derivative is viewed left to right, the critical value represents a
1) relative maximum if the sign changes from + to -
2) relative minimum if the sign changes from - to +
or 3) neither a max. or min. if the sign does not change.

Of course, we need to remember how to create a sign pattern in order to apply the 1st Derivative Test:

Process to create a sign pattern:

1. Factor the polynomial to find the zeros.
2. Put the zeros on a number line.
   Be sure to label the sign pattern.
3. Determine the sign on the right end of the sign pattern.
4. Moving to the left, alternate the signs as you cross a zero, unless that zero's factor is raised to an even power.

Ex 2 Find the critical values of $y = 5x^4 - 10x^2$ and determine if each is at a maximum, a minimum, or neither.

$$\frac{dy}{dx} = 20x^3 - 20x = 0$$

$$20x\left(x^2 - 1\right) = 0$$

$$x = -1, 0, 1$$

$$\begin{array}{cccccccc} y' & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -1 & & 0 & & 1 & \end{array}$$

So, there are critical values at x values of –1, 0, and 1. When we look at the sign pattern, we can see we have a minimum at –1, a maximum at 0, and another minimum at 1.

The First Derivative Test Corollary for Endpoints

An endpoint is at a maximum if
- i) it is the left end and followed by a –, or
- ii) it is the right end and preceded by a +.

An endpoint is at a minimum if
- i) it is the left end and followed by a +, or
- ii) it is the right end and preceded by a –.

Or, more simply:

	Left End	Right End
$\frac{dy}{dx} > 0$	Minimum	Maximum
$\frac{dy}{dx} < 0$	Maximum	Minimum

EX 3 (Ex 1 again) Find the critical values algebraically of $y = 3x^4 + 2x^3 - 39x^2 + 36x - 4$ on $x \in [-2,6]$.

$$\frac{dy}{dx} = 12x^3 + 6x^2 - 78x + 36 = 0$$

$$2x^3 + x^2 - 13x + 6 = 0$$

$$(2x-1)(x-2)(x+3) = 0$$

$x = \frac{1}{2}, 2, \text{ or} -3$, but –3 is not in the restricted domain.

$x = -2 \text{and } 6$ are the endpoints of the arbitrarily stated domain.

Therefore, $x = 2, \frac{1}{2}, -2,\text{and } 6$ are the critical values.

Note that the domain of $y = 3x^4 + 2x^3 - 39x^2 + 36x - 4$ **is** $x \in (-\infty,\ \infty)$**.**
$x = 2\text{and } 6$ **are the endpoints of the restriction.**

Ex 4 Find the maximum values for the function $g(x) = \sqrt{x^3 - 9x}$

Domain: $x^3 - 9x \geq 0$

$$\begin{array}{cccccccc} y & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 3 & \end{array}$$

$$x^3 - 9x \geq 0 \rightarrow x \in [-3, 0] \cup [3, \infty)$$

$$g'(x) = \frac{3x^2 - 9}{2\sqrt{x^3 - 9x}}$$

$g'(x) = 0$ when $3x^2 - 9 = 0$, so $x = \pm\sqrt{3}$
$g'(x)$ does not exist when $x^3 - 9x = 0$, so $x = \pm 3, 0$

$$\begin{array}{cccccccccc} y' & & dne & + & 0 & - & dne & & dne & + \\ \hline x & & -3 & & -\sqrt{3} & & 0 & \sqrt{3} & 3 & \end{array}$$

Since $x = \sqrt{3}$ is not in the domain of the function, our critical values are at $x = \pm 3, 0, -\sqrt{3}$. From our sign pattern, we can conclude that at $x = -\sqrt{3}$, we have a maximum value, and by substituting this value into the function, we find that the maximum value is at $y = 3.224$.

The first derivative test is not the only way to test whether critical values are associated with maximum or minimum values. If you recall, the second derivative can show us intervals of concavity. This is very useful because if we know that the curve is concave down at a critical value, it must be associated with a maximum value of the function. Similarly, if the curve is concave up, the critical value must be associated with a minimum value.

4.1 Free Response Homework

Find the critical values.

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1. $y = 2x^3 + 9x^2 - 168x$

2. $y = 3x^4 + 2x^3 + 12x^2 + 12x - 42$

3. $y = \dfrac{x^2+1}{x^3-4x}$

4. $y = \dfrac{x-5}{x^2+9}$

5. $y = \sqrt{9x^3 - 4x^2 - 27x + 12}$

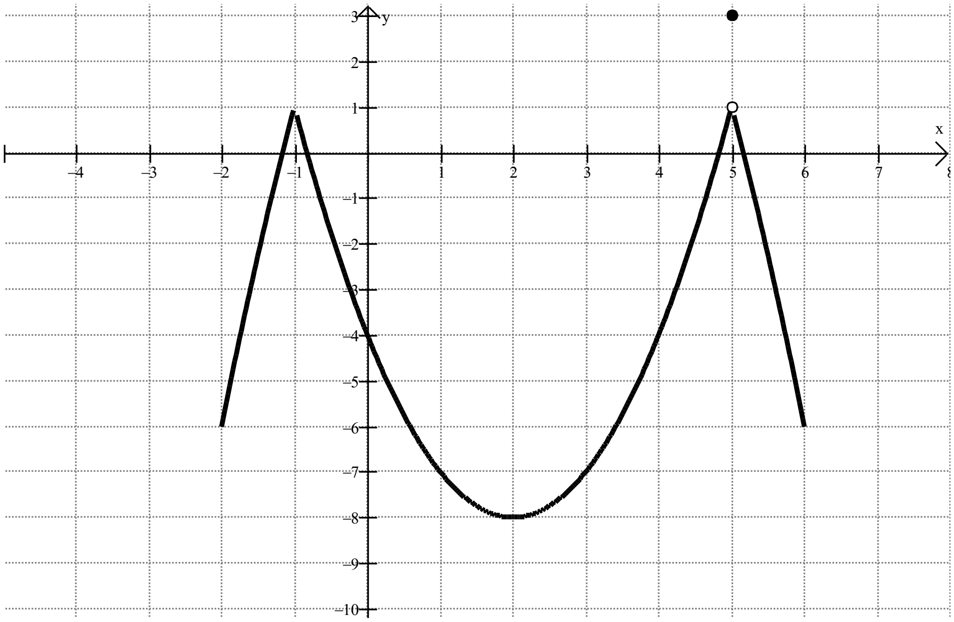
6. $y = \sqrt{\dfrac{3x}{9-x^2}}$

7. $y = \left(x^2\right)\sqrt[3]{9-x^2}$

8. $y = \left(x - x^2\right)3^x$

9. $y = \cos^{-1}x + x\sqrt{1-x^2}$

10.



Find the critical values of each of the following and determine if each is at a maximum point, a minimum point, or neither.

11. $y=x^3-4x^2+x+6$ 12. $y=x^3-x^2-4x+4$

13. $y=x^3-7x^2+11x+3$ 14. $y=3x^3-x^2-40x+48$

15. $y=-3x^4+2x^3$ 16. $y=3x^4+16x^3+24x^2$

17. $y=\frac{1}{4}x^4-\frac{7}{3}x^3+\frac{11}{2}x^2+3x-2$18. $y=\frac{1}{5}x^5+x^4-x^3-7x^2-8x-5$

19. $y=2x^5-35x^3+135x-29$ on $x\in[-2,\ 6]$

20. $y=x^4-6x^2-12$ on $x\in[-1,2]$

21. $y=3x^3+2x^2-27x-18$ on $x\in[-3,3)$

22. $y=x^4-4x^3$on $x\in[-1,2]$

For 23 – 27, find the absolute maximum and minimum values of *f* on the given intervals.

23. $f(x)=\dfrac{x}{x^2+1},[0,2]$

24. $f(t)=\sqrt[3]{t}(8-t),[0,8]$

25. $f(z)=ze^{-z},[0,2]$

26. $f(x)=\dfrac{\ln(x)}{x},[1,3]$

27. $f(x)=e^{-x}-e^{-2x},[0,1]$

4.1 Multiple Choice Homework

1. Give the value of x where the function $f(x) = x^3 - 9x^2 + 24x + 4$ has a relative maximum point.

a) 4 b) –2 c) 2 d) –4 e) 3

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2. Give the value of x where the function $f(x) = x^3 - \frac{33}{2}x^2 + 84x - 2$ has a relative minimum point.

a) –4 b) –7 c) 4 d) 5 e) 7

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3. Give the approximate location of a relative maximum point for the function $f(x) = 3x^3 + 5x^2 - 3x$.

a) $(-1.357, 5.779)$ b) $(0.2457, -0.3908)$c) $(-1.357, 5.713)$

d) $(0.2457, -0.3216)$ e) $(-1.357, -0.3908)$

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4. Given this sign pattern $\begin{array}{c} \frac{dy}{dx} \\ \\ x \end{array} \begin{array}{c} +0 \quad 0 \quad - \quad 0 \quad + \quad 0 \quad 0- \\ \longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\longrightarrow \\ -2-\sqrt{2} \qquad 0 \quad \sqrt{2} \quad 2 \end{array}$, at what value of x does f have a local minimum?

a) -2 b) $-\sqrt{2}$ c) 0 d) $\sqrt{2}$ e) 2

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5. The absolute maximum of $y = -\sqrt{25 - x^2}$ on $x \in [-2, 4]$ is

a) -2 b) 0 c) -5 d) $-\sqrt{21}$ e) -3

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6. Find the absolute maximum value of $y = \sqrt{36 - x^2}$ on the interval $x \in [-2,2]$.

a) 5 b) 6 c) 7 d) 0 e) 1

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7. The graph of the function $f(x) = 2x^{5/3} - 5x^{2/3}$ is increasing on which of the following intervals

I. $1 < x$ II. $0 < x < 1$ III. $x < 0$

a) I only b) II only c) III only

d) I and II only e) II and III only

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8. If $f(x) = x^2 e^{-2x}$, then the graph of *f* is increasing for all *x* such that

a) $0 < x < 1$ b) $0 < x < \frac{1}{2}$ c) $0 < x < 2$

d) $x < 0$ e) $x > 0$

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9. If $f(x) = \dfrac{x}{\ln(7x)}$, then what is the interval of decreasing?

a) $\left(0, \dfrac{1}{7}\right)$ b) $\left(0, \dfrac{1}{7}\right) \cup \left(\dfrac{1}{7}, \dfrac{1}{7}e\right)$ c) $(1, 7e)$

d) $(1, 7)$ e) $\left(1, \dfrac{1}{7}e\right)$

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10. Suppose $f'(x) = \dfrac{(x+1)^3(x-4)^2}{\left(x^4+1\right)}$. Which of the following statements must be true?

I. The slope of the line tangent to $y = f(x)$ at $x = 1$ is 36.
II. $f(x)$ is increasing on $x \in (1, 4)$
III. $f(x)$ has a minimum at $x = -1$

a) II only b) I and III only c) II and III only

d) I and II only e) I, II, and III

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11. Given this sign pattern $\begin{array}{cc} f'(x) & -0+0-0- \\ & \longleftrightarrow \\ x & -4-12 \end{array}$, at what value of *x* does *f* has a relative minimum point?

a) –4 b) –1 c) 2 d) 1 e) no value

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12. Let $f(x)$ and $g(x)$ be twice-differentiable functions with selected values given in the table below.

x	$g(x)$	$g'(x)$	$g''(x)$	$f(x)$	$f'(x)$	$f''(x)$
0	2	3	-2	2	1	-2
2	4	2	0	4	0	3
4	1	-1	3	0	2	-1

At $x=2$, $f(x)$ has a:

a) Local Minimum b) Local Maximum c) Point of Inflection

d) Zero e) None of these

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13. Let $f(x)$ and $g(x)$ be twice-differentiable functions with selected values given in the table in #17. Let $h(x)=g(f(x))$. At $x=2$, $h(x)$ has a:

a) Local Minimum b) Local Maximum c) Point of Inflection

d) Zero e) None of these

4.2 Optimization

Optimization is a practical application of finding maxima and minima for functions. As I mentioned before, this revolutionized thinking and is a critical component of all industries. You might remember this topic from chapter 2 of book 2 last year; the word problems many of you avoided on the test last year. This year, they form a much more fundamental part of what we need to be able to do, so we can no longer simply skip these problems on tests.

OBJECTIVES

Solve optimization problems.

Every optimization problem looks a bit different, but they all follow a similar progression. You must first identify your variables and any formula you need. Use algebra to eliminate variables, and **take the derivative of the function you are trying to optimize**. This is the most common mistake in optimization problems; taking the derivative of the wrong function.

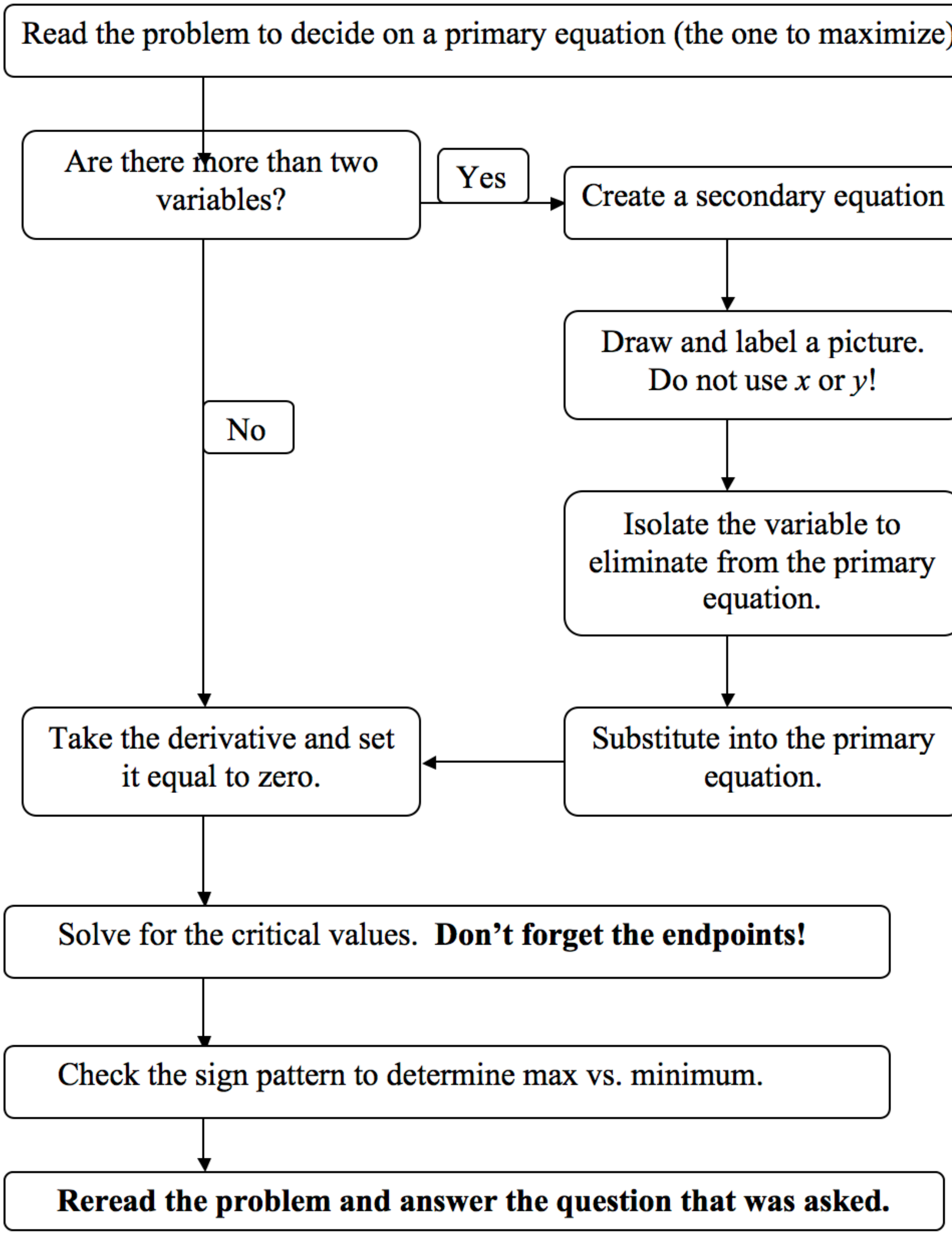
Key Ideas:

I. Find the variable to be maximized or minimized. That is the equation/variable to differentiate.

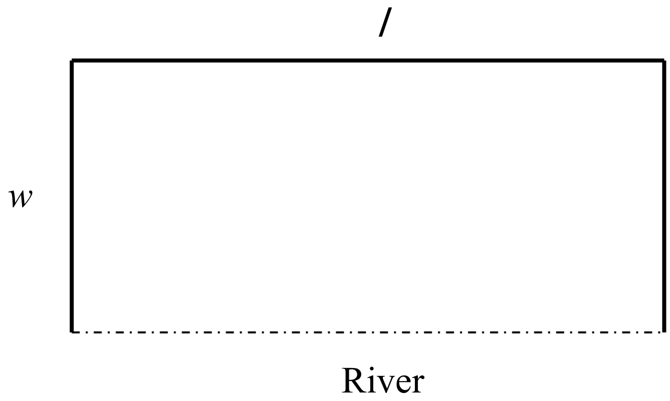
II. The equation which comes from the sentence with the superlative usually has too many variables. Other sentences help to substitute out the extra variables.

The following flowchart might help:

Strategy for Approaching Optimization Problems with Calculus



Ex 1 The owner of the Rancho Grande has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is the area?



$A = lw$ The problem with maximizing this area formula lies in the fact that we have two independent variables (*l* and *w*). We need the fact about perimeter to complete the problem.

$$A = lw$$

$$P = l + 2w$$

$$3000 = l + 2w$$

$$3000 - 2w = l$$

$$A = (3000 - 2w)w$$

$$A = 3000w - 2w^2$$

Now, since we have an equation with one independent variable, we can take the derivative easily.

$$\frac{dA}{dw} = 3000 - 4w$$

$$\frac{dA}{dw} = 3000 - 4w = 0$$

$$w = 750$$

$$l = 1500$$

So we would want a width of 750 yards and a length of 1500 yards. This would give us an area of 1,125,000 yards^2.

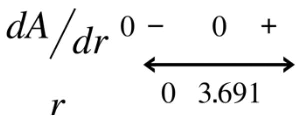
Ex 2 A cylindrical cola can has a volume $32\pi \text{in}^3$. What is the minimum surface area?

$$S = 2\pi r^2 + 2\pi rh$$

$$V = \pi r^2 h$$
$$32\pi = \pi r^2 h$$
$$\frac{32}{r^2} = h$$

$$S = 2\pi r^2 + \pi r\left(\frac{64}{r^2}\right)$$
$$S = 2\pi r^2 + 64\pi r^{-1}$$
$$\frac{dS}{dr} = 4\pi r - 64\pi r^{-2} = 0$$
$$4\pi r\left(1 - \frac{16\pi}{r^3}\right) = 0$$
$$r = 0\text{or } 3.691$$

There is an implied domain here. You cannot have a radius of 0 inches, so 3.691 inches is the radius for the minimum area. The sign pattern verifies this:



So the minimum surface area would be

$$S = 2\pi(3.691)^2 + 2\pi(3.691)\left(\frac{32}{(3.691)^2}\right) = 140.058\text{in}^2$$

Ex 3 Find the point on the curve $y=\dfrac{e^{-x^2}}{2}$ that is closest to the origin.

We want to minimize the distance to the origin, so we will be using the Pythagorean theorem to find the distance.

$$D=\sqrt{x^2+y^2}$$

$$y=\frac{e^{-x^2}}{2}$$

$$D=\sqrt{x^2+\left(\frac{e^{-x^2}}{2}\right)^2}$$

$$\frac{dD}{dx}=\frac{1}{2}\left(x^2+\frac{e^{-2x^2}}{4}\right)^{-1/2}\left(2x-xe^{-2x^2}\right)=0$$

$$x=0,\ \pm.841$$

x=	D=
-.841	.876
0	.5
.841	.876

So the minimum distance from the origin is at the point (0, .5).

Given the diverse nature of optimization problems, it is helpful to remember all the formulas from geometry.

4.2 Free Response Homework

1. A cylindrical cola can has volume $16\pi \text{in}^3$. (a) What radius would minimize the surface area? (b) If the top and bottom of the can cost 6 cents/square inch and the sides cost 3 cents/square inch, what radius would minimize the cost?

2. A cylindrical cola can has volume $32\pi \text{in}^3$. What is the minimum surface area?

3. Find two positive numbers whose product is 110 and whose sum is a minimum.

4. Find a positive number such that the sum of the number and its reciprocal is a minimum.

5. Find the maximum area of a rectangle inscribed between the *x*-axis and the parabola $y = 16 - x^2$.

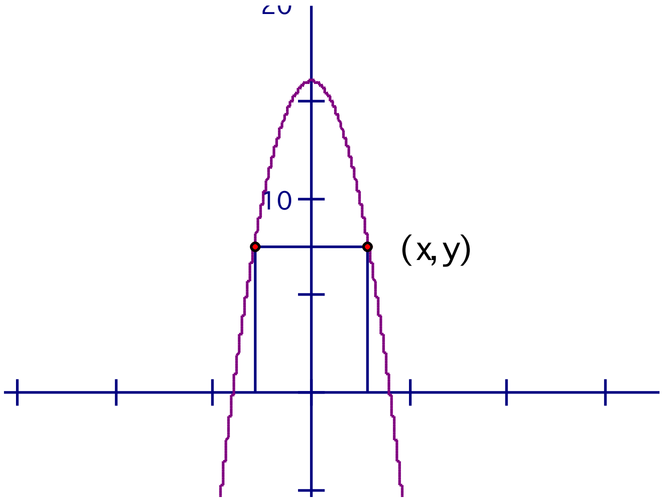


Figure for problem 13

6. A person's sensitivity to medicine is sometimes described by $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$, where *R* is the temperature reaction, *M* is the amount of medicine and *C* is a positive constant. How much medicine *M* would cause the highest reaction *R*?

7. What is volume of the largest cone with surface area of $120\pi\ in^2$? [note: $V = \frac{\pi}{3} rh$ and $A = \pi r^2 + \pi rl$]

8. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?

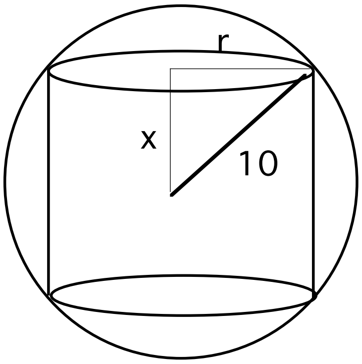
9. If 1200 cm^2 of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.

10. Find the point on the line $y = 4x + 7$ that is closest to the origin.

11. Find the points on the curve $y = \frac{1}{x^2 + 1}$ that are closest to the origin.

12. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

13. Find the maximum volume of a cylinder inscribed in a sphere with radius 10.



14. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. Let *x* be the length of a side of the triangle. (a) Find where the wire should be cut to maximize the area enclosed, then (b) find where the wire should be cut to minimize the area enclosed.

4.2 Multiple Choice Homework

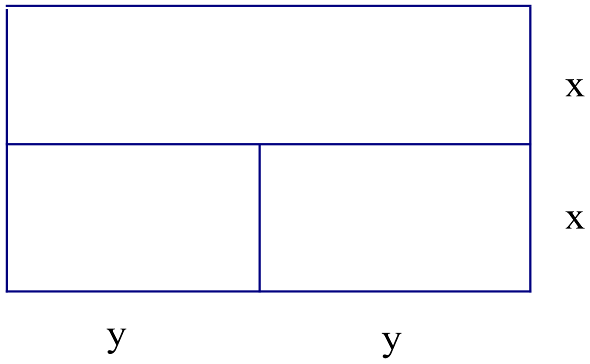
1. The sum of two positive integers *x* and *y* is 150. Find the value of *x* that minimizes $P = x^3 - 150xy$

a) $x = 25$
b) $x = 75$
c) $x = 50$
d) $x = 125$
e) $x = 100$

2. The sum of two positive integers *x* and *y* is 120. Find the value of *x* that minimizes $P = x^3 - 120xy$

a) $x = 20$
b) $x = 40$
c) $x = 60$
d) $x = 80$
e) $x = 100$

3. A farmer has 100 yards of fence to enclose a field, subdivide it into two equal pens, and further subdivide one of those pens into two equal fields as shown below.



What value of y produces the maximum total area?

a) 12.5 b) 10 c) $\frac{100}{11}$ d) $\frac{25}{3}$ e) None of these

4. A farmer with 890 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?

a) 19,825.5 ft^2 b) 19,802.5 ft^2 c) 19,801.5 ft^2

d) 19,902.5 ft^2 e) 19,791.5 ft^2

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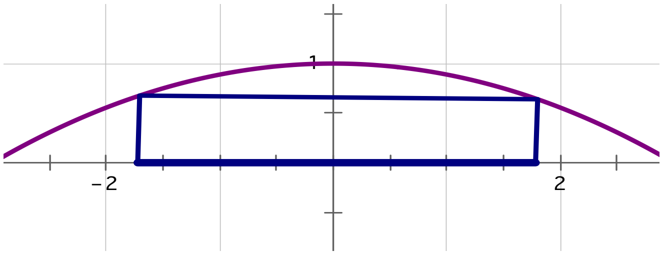
5. A rectangle with one side on the x-axis has its upper vertices on the graph of $y = 1 - \frac{x^2}{9}$, as shown in the figure below. What is the maximum area of the rectangle?



a) $\sqrt{3}$ b) $\frac{\sqrt{3}}{2}$ c) $\frac{2}{3}$ d) $\frac{4\sqrt{3}}{3}$ e) None of these

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6. A piece of wire 10 meters long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut for the square so that the total area enclosed is a minimum? (Area of an equilateral triangle: $A = \frac{\sqrt{3}}{4}s^2$, where s is the length of a side of the triangle)

a) 5.35 meters b) 4.40 meters c) 4.35 meters

d) 0 meters e) 3.25 meters

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7. In a certain community, an epidemic spreads in such a way that the percentage P of the population that is infected after t months is modeled by

$$P(t) = \frac{kt^2}{(C+t^2)^2},$$

where C and k are constants. Find t, such that P is most.

a) 0 b) $\sqrt{C}$ c) $\sqrt{k}$ d) $\sqrt{\frac{C}{3}}$ e) None of these

4.3 The Mean Value, Extreme Value, and Rolle's Theorems

The Mean Value Theorem is an interesting piece of the history of Calculus that was used to prove a lot of what we take for granted. The Mean Value Theorem was used to prove that a derivative being positive or negative told you that the function was increasing or decreasing, respectively. Of course, this led directly to the first derivative test and the intervals of concavity.

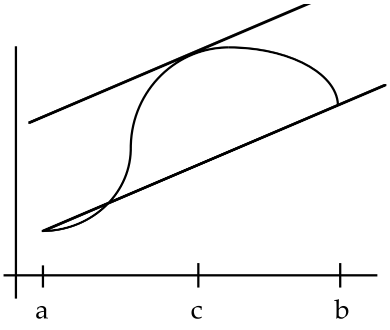
Mean Value Theorem

If f is a function that satisfies these two hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)

Then there is a number c in the interval (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Again, translating from math to English, this just says that, if you have a smooth, continuous curve, the slope of the line connecting the endpoints has to equal the slope of a tangent somewhere in that interval. Alternatively, it says that the secant line through the endpoints has the same slope as a tangent line.



MEAN VALUE THEOREM

Ex 1 Show that the function $f(x) = x^3 - 4x^2 + 1, [-1,\ 3]$ satisfies all the conditions of the Mean Value Theorem and find *c*.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c) = 3c^2 - 8c \qquad \frac{f(3) - f(-1)}{3 - (-1)} = \frac{-8 - (-4)}{4} = -1$$

$$3c^2 - 8c = -1$$

$$3c^2 - 8c + 1 = 0$$

$$c = \frac{8 \pm \sqrt{8^2 - 4(3)(1)}}{2(3)}$$

$$c = 2.535 \quad or \quad 0.131$$

Rolle's Theorem is a specific a case of the Mean Value theorem, though Joseph-Louis Lagrange used it to prove the Mean Value Theorem. Therefore, Rolle's Theorem was used to prove all of the rules we have used to interpret derivatives for the last couple of years. It was a very useful theorem, but it is now something of a historical curiosity.

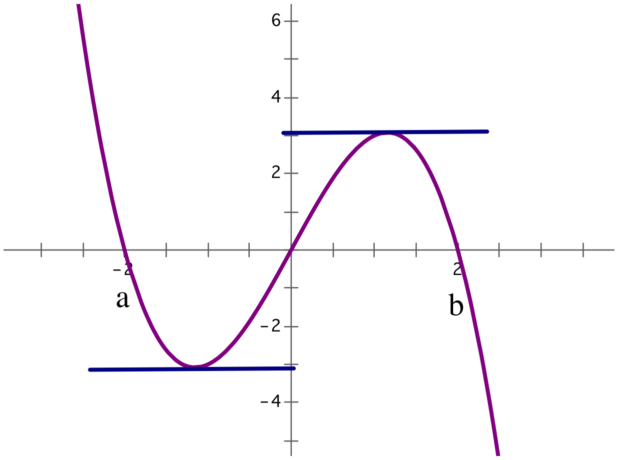
Rolle's Theorem

If f is a function that satisfies these three hypotheses

1. f is continuous on the closed interval $[a,b]$
2. f is differentiable on the closed interval (a,b)
3. $f(a) = f(b)$

Then there is a number c in the interval (a,b) such that $f'(c) = 0$.

Written in this typically mathematical way, it is a bit confusing, but it basically says that if you have a continuous, smooth curve with the initial point and the ending point at the same height, there is some point in the curve that has a derivative of zero. If you look at this from a graphical perspective, it should be pretty obvious.



Rolle's Theorem is sometimes confused with the Extreme Value Theorem.

The Extreme Value Theorem: If *f* is continuous on the closed interval [*a*, *b*], then $f(x)$ attains a maximum and a minimum at least once each.

Note that the Extreme Value Theorem basically says all continuous functions have an extreme value on a closed interval. **This is because the extreme could be at an endpoint.** Rolle's Theorem implies there is an extreme at an interior point, though Rolle's never mentions extremes.

Ex 2 Show that the function $f(x)=x^2-4x+1$ on $x\in[0,4]$ satisfies all the conditions of the Mean Value Theorem and find *c*.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c)=\frac{f(b)-f(a)}{b-a}\rightarrow f'(c)=\frac{f(4)-f(0)}{4-1}=0$$

$$f'(c)=2c-4=0$$

$$c=2$$

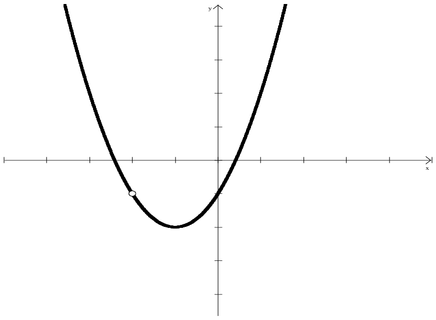
We need to consider when the MVT might not apply to a problem—that is, when is a function not continuous or not differentiable. We saw the first in PreCalculus, but did not always name it "discontinuity." Continuity basically means a function's graph has no breaks in it. The formal definition involved limits and we will explore that in the Limit Chapter.

Since all the families of functions investigated in PreCalculus were continuous in their domain, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

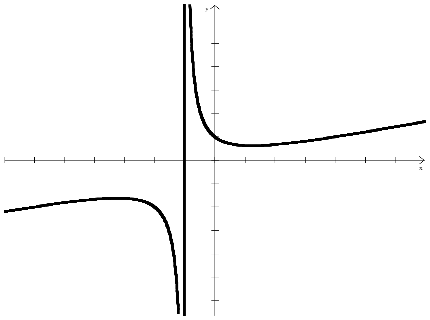
Removable Discontinuity
($\underset{x \to a}{Lim} f(x)$ does exist)

Essential Discontinuity
($\underset{x \to a}{Lim} f(x)$ does not exist)

f(a) does not exist

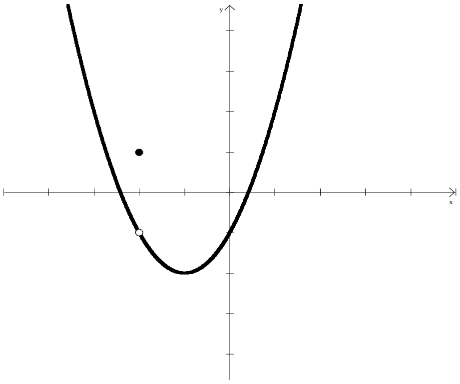


Point of Exclusion (POE)

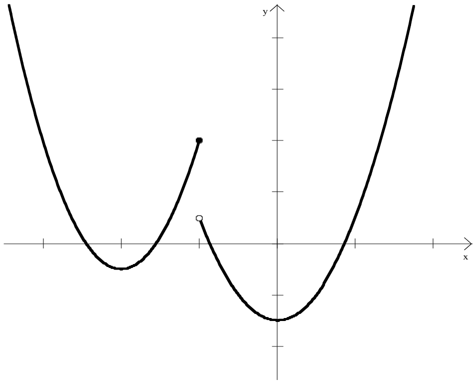


Vertical asymptotes

f(a) exists



Point of Displacement (POD)

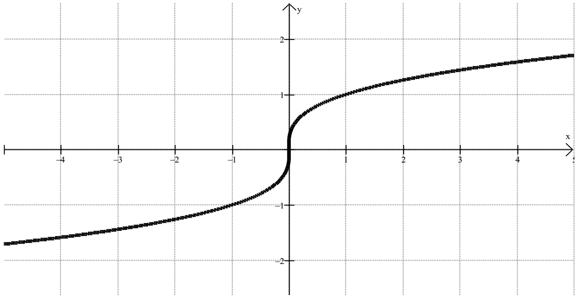


Jump Discontinuity

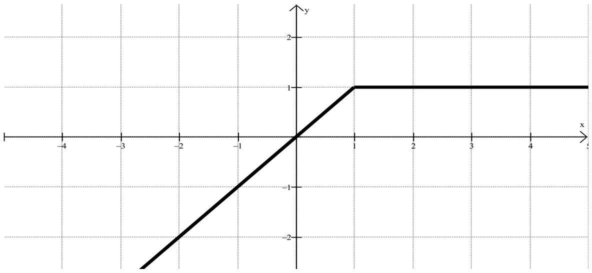
"Not differentiable" simply means the derivative does not exist. This can happen one of three ways.

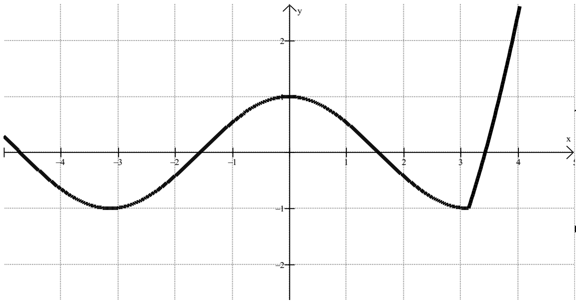
1. A function is not differentiable if it is not continuous.
   For pictures, see above.

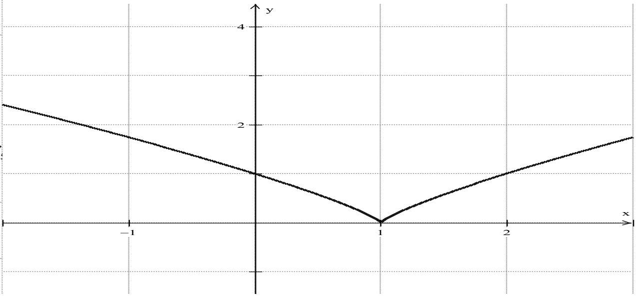
2. A function is not differentiable if the tangent line is vertical.



3. A function is not differentiable if it is not "smooth."







NB. All the families of functions studied in PreCaculus are continuous and differentiable in their domains.

Ex 3 To which of these functions does the Mean Value Theorem apply on the interval $[-1,3]$?

The MVT often arises in AP Table Problems like this one:

Ex 4 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(d) Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$

(d) Explain why there are at least two times between $t=0$ and $t=20$ when $V'(t)=0$.

Because $V(t)$ is a continuous and differentiable function, the Mean Value Theorem applies. Since $V(0)=83=V(6)$, there must be a *c*-value between $t=0$ and $t=6$ where $V'(t)=0$. Similarly, since $V(13)=38=V(26)$, there must be a c-value between $t=13$ and $t=20$ where $V'(t)=0$. There might be more *c*-values where $V'(t)=0$, but the MVT guarantees at least two.

Other Theorems that can be confused with The Mean Value Theorem

The Average Value Theorem: The average value of a function *f* on a closed interval [*a*, *b*] is defined as $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$.

Average Rate of Change: The average rate of change of a function *f* on a closed interval [*a*, *b*] is defined as $= \frac{f(b)-f(a)}{b-a}$.

The Intermediate Value Theorem: If *f* is continuous on the closed interval [*a*, *b*], then $f(x)$ attains every height between $f(a)$ and $f(b)$.

One consequence of the Intermediate Value Theorem is that if $f(a)$ and $f(b)$ are opposite signs, there is a zero in the closed interval.

This next example shows how the Intermediate Value Theorem might come into play.

Ex 5 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function. Explain why there are at least two times when $V(t) = 74$.

Because $V(t)$ is continuous and $68 < 74 < 83$, according to the IMT, there must be at least one time between $t = 0$ and $t = 4$ and at least one time between $t = 4$ and $t = 6$ when $V(t) = 74$.

4.3 Free Response Homework

Verify that the following functions fit all the conditions of the Mean Value Theorem, and then find all values of *c* that satisfy the conclusion of the Mean Value Theorem.

1. $f(x)=x^3-3x^2+2x+5,[0,\ 2]$

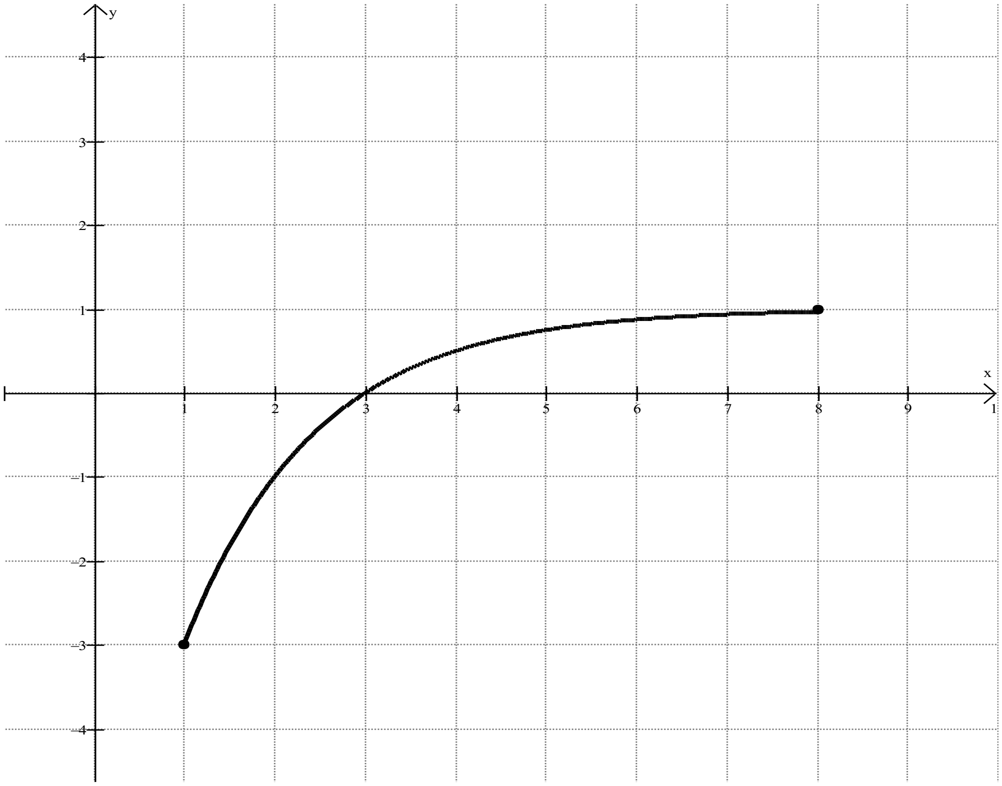
2. $f(x)=\sin(2\pi x),[-1,\ 1]$

3. $g(t)=t\sqrt{t+6},[-4,\ 0]$

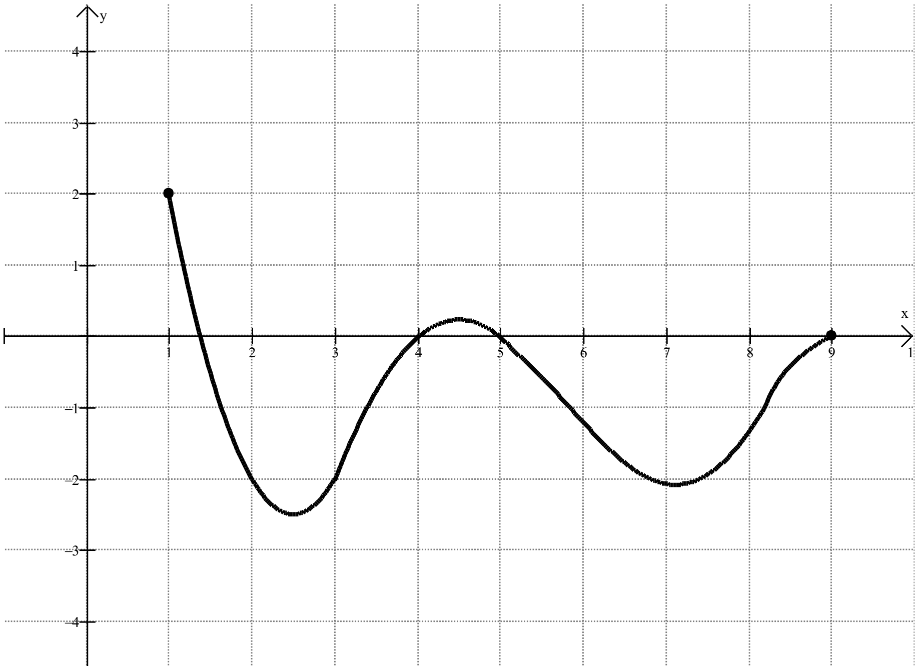
4. $H(t)=\dfrac{t}{(t-6)^2},[2,\ 8]$

Given the graph of the function below, estimate all values of c that satisfy the conclusion of the Mean Value Theorem for the interval [1,8].

5.



6. Given the graph of the function below, estimate all values of c that satisfy the conclusion of the Mean Value Theorem for the interval [1,9].



Determine if the following functions fit the conditions of the Intermediate Value Theorem, and determine the range of the results.

7. $f(x) = x^3 - 3x^2 + 2x + 5, [0,\ 4]$

8. $f(x) = \dfrac{\sin(2\pi x)}{x}, [-1,\ 1]$

9. $g(t) = t\sqrt{t+6}, [-4,\ 0]$

10. $H(t) = \dfrac{t}{\sqrt{t+6}}, [2,\ 8]$

11. On May 15, the weather in the town of Apcalc changes at a rate of $W(t)$ degrees Fahrenheit per hour. $W(t)$is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight, t=0, the weather in Apcalc is 40 degrees Fahrenheit.

t (in hours since midnight)	0	1	3	6	8
$W(t)$ (in degrees Fahrenheit per hour)	-2.4	-2.1	-1.2	1.8	4.5

a) At approximately what rate is the rate of change of the temperature changing at 2am (t=2)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Is there a time when the rate of change of the temperature equals 0.7? Justify your answer.

d) Is there a time in $0 \le t \le 8$ when $W(t) = 0$? Justify your answer.

12. Star Formation Rate (*SFR*) observations of red-shift allow scientists to track the gains and approximate future gains. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

Assume the *SFR* data represents a continuous and differentiable function. *SFR* is measured in solar masses per giga-years (millions of years) per cubic parsecs and time *t* is measured in giga-years.

a) Use midpoint rectangles to approximate the total star formation. Using the correct units, explain the meaning of your result.

b) Is there a time when *SFR* = 0? Justify your answer.

c) Is there a time when *SFR* = 0.0056? Justify your answer.

13. Dr. Quattrin's paternal grandmother's family originated in the Alpine town of Sauris, Italy, where the temperature in January changes at a rate of $W(t)$degrees Celsius per hour. $W(t)$is a twice-differentiable, increasing and concave up

function with selected values in the table below. At midnight ($t = 0$), the temperature in Sauris is $-8°C$.

t (in hours after midnight)	0	1	3	6	8
$W(t)$ (in degrees Celsius per hour)	-2.6	-3.1	-1.2	1.9	2.5

Is there a time when the rate of change of the temperature equals 0.3? Justify your answer.

14. AP Handout: AP Calc AB/BC 2004B #3, AP Calc AB 2002 #6

4.3 Multiple Choice Homework

1. Let f be a polynomial function with degree greater than 2. If $a \neq b$ and $f(a) = f(b) = 1$, which of the following must be true of at least one value of x between a and b?

I. $f(x) = 0$ II. $f'(x) = 0$ III. $f''(x) = 0$

a) I only b) II only c) III only

d) II and III only e) I, II, and III

---

2. Which of the following functions **fail** to meet the conditions of the Mean Value Theorem?

I. $3x^{2/3} - 1$ on $[-1,\ 2]$ II. $|3x - 2|$ on $[1,\ 2]$

III. $4x^3 - 2x + 3$ on $[-1,\ 2]$

a) I only b) II only c) III only d) I and II only e) II and III only

---

3. $y = -2x^2 + x - 2$ is defined on $x \in [1,\ 3]$. Find the c-value determined by the Mean Value Theorem.

a) $x = \frac{1}{2}$

b) $x = \frac{5}{4}$

c) $x = \frac{3}{2}$

d) $x = 2$

e) $x = \frac{9}{4}$

---

4. Let $g(x)$ be a continuous function on the interval $x \in [0,\ 1]$ and let $g(1) = 0$ and $g(0) = 1$. Which of the following statements is not necessarily true?

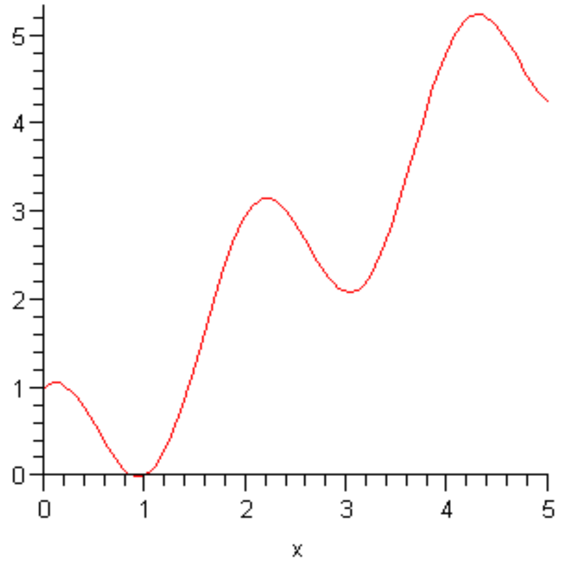
a) The exist a number c on $[0,\ 1]$ such that $g(c) \geq g(x)$ for all $x \in [0,\ 1]$

b) For all a and b in $[0,\ 1]$, if $a = b$, then $g(a) = g(b)$.

c) The exist a number c on $[0,\ 1]$ such that $g(c) = \frac{1}{2}$

d) The exist a number c on $[0,\ 1]$ such that $g(c) = \frac{3}{2}$

e) For all c in $(0,\ 1)$, $\lim_{x \to c} g(x) = g(c)$



5. The graph of a differentiable function *f* is shown above on the closed interval [0, 5]. How many values of *x* in the open interval (0, 5) satisfy the conclusion of the Mean Value Theorem for *f* on [0, 5] ?

a) Two b) Three c) Four d) Five

---

6. To which of the following phrases does the equation $Y = \dfrac{f(b) - f(a)}{b - a}$ pertain?

a) The Mean Value Theorem

b) The Average Value Theorem

c) The Average Rate of Change

d) The Intermediate Value Theorem

e) Rolle’s Theorem

---

t hours	0	2	4	6	8	10	12	14
$E(t)$	10	4	10	5	7	3	7	5

7. The table above shows particular values of a differentiable function $E(t)$. At how many times on the interval $t \in [0,\ 12]$ does the Mean Value Theorem guarantee that $E'(t) = 0$

a) None b) One c) Two d) Three

4.4 Intro to AP: Accumulation of Rates Revisited

Key Phrases for decoding Accumulation of Rates questions:

Total change: $\int_a^b R(t)dt$ or $\int_a^t \left[\textit{incoming rate} - \textit{outgoing rate}\right] dx$

Total rate of change: $\textit{incoming rate} - \textit{outgoing rate}$

Total Amount: $\textit{Total}(t) = \textit{initial value} + \int_a^t \left[\textit{incoming rate} - \textit{outgoing rate}\right] dx$

Instantaneous rate of change: $\frac{dx}{dt}$ or $R(t)$

Average rate of change: $\frac{f(b)-f(a)}{b-a}$

Average value of $f(x)$: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(\textit{Rate of Change})$is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(\textit{Rate of Change})$is positive

*Note that it can be a function that is increasing or decreasing, or it could be a derivative that is increasing or decreasing, etc.

In Ex 1 & 2 in Sections 3.4, we skipped the last question until later. Now it is later.

Objective

Find extremes Accumulation of Rates Problems.

Ex 1 The rate at which people enter a park is given by the function $E(t) = \dfrac{15600}{t^2 - 24t + 160}$, and the rate at which they are leaving is given by $L(t) = \dfrac{9890}{t^2 - 38t + 370} - 76$. Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8 \le t \le 24$. At $t = 8$ there are no people in the park.

(a) How many people have entered the park at 4 pm ($t = 16$)? Round your answer to the nearest whole number.
(b) The price of admission is \$36 until 4 pm ($t = 16$). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.
(c) Let $H(t) = \int_8^t E(x) - L(x)dx$ for $8 \le t \le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.
(d) At what time t, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

(d) At what time t, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

To find the maximum number of people, we have to set the derivative $H'(t)$

equal to zero. **We must also check the endpoints**, which are also critical values (we could just do a sign pattern instead to verify that the zero is the maximum).

$$H'(t) = E(t) - L(t) = 0$$

Using a graphing calculator, we find the zero is at t = 16.046. Or, we can find the point of intersection of $E(t)$ and $L(t)$. The critical values are t = 0, 16.046, and 24.

At $t = 0$, H = 0 (this was given at the beginning of the problem)

At $\mathrm{t} = 16.046$, $H(16.046) = \int_{8}^{16.046} E(t) - L(t)dt = 5023$

Att $= 24$, $H(16.046) = \int_{8}^{24} E(t) - L(t)dt = 1453.$

Because 5023 is the highest of these three H values, the maximum number of people in the park occurs at t =16.046.

Ex 2 A certain industrial chemical reaction produces synthetic oil at a rate of $S(t) = \dfrac{15t}{1+3t}$. At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of $R(t) = 2 + 5\sin\left(\dfrac{4\pi}{25}t\right)$. Both functions have units of gallons per hour, and the reaction runs from $t = 0$ to $t = 6$. At time of $t = 0$, the reaction vessel contains 2500 gallons of oil.

(a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measure.
(b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time *t*.
(c) Find the rate at which the total amount of oil is changing at $t = 4$.

(d) For the interval indicated above, at what time *t* is the amount of oil in the reaction vessel at a minimum? What is the minimum value? Justify your answers.

(d) For the interval indicated above, at what time *t* is the amount of oil in the reaction vessel at a minimum? What is the minimum value? Justify your answers.

Critical values occur when $S(t) - R(t) = 0$and at the endpoints. Our graphing calculator shows the only time when $S(t) - R(t) = 0$ is $t = 5.117$.

t	$P(t)$
0	2500
5.117	2492.367
6	2493.277

The minimum is 2492.367 gallons and occurs at $t = 5.117$.

Ex 3: <u>The Donner Summit Snowfall Problem I</u>

The snowfall in Donner Summit is tracked by the US Weather Service. For the month of March, 2022, $S(t)$ represents the rate of snowfall in inches per day and its data is presented in the table below. $M(t)=0.65-0.35\cos\left(\frac{5x^{0.95}}{6}\right)$ represents the rate at which the snow melts in inches per day, where t is measured in days.

t in days	1	3	4	7	11	15	21
$S(t)$ in inches per day	1.4	4.9	4.4	0.1	4.6	0.2	2.7

(a) Find $\int_1^{21} M(t)\,dt$. Using the correct units, explain the meaning of $\frac{1}{21-1}\int_1^{21} M(t)\,dt$.

(b) Using a Midpoint Reimann Sum, find $\int_1^{21} S(t)\,dt$. State the correct units.

(c) Approximate $S'(5)$. Using the correct units, explain $S'(5)$ in context of the problem.

(d) Assume $S(t)=2.5-2.5\cos\left(\frac{10x^{.9}}{9}\right)$ would model the snow fall. If there were 9 inches of snow on the ground at the beginning of Day 1, find the minimum amount of snow on the ground between $t=1$ and $t=7$.

(a) $\int_1^{21} M(t)\,dt = 13.023$ inches. $\frac{1}{21-1}\int_1^{21} M(t)\,dt$ is the approximate average snow melt, in inches per day, between $t=1$ and $t=21$ days.

(b) $\int_1^{21} S(t)\,dt \approx 3(7)+7(2)+10(2)=55 \text{ } inches \text{ } of \text{ } snow$.

(c) $S'(5) \approx \frac{6-2}{4-7} = -\frac{4}{3}\ {}^{in}\!/_{day^2}$. The rate of snowfall, in inches per day, is decreasing by 4/3 inches per day per day on day 5.

(d) Assume $S(t) = 2.5 - 2.5\cos\left(\frac{10x^{.9}}{9}\right)$ would model the snow fall. If there were 9 inches of snow on the ground at the beginning of Day 1, find the minimum amount of snow on the ground between $t = 1$ and $t = 7$.

$$A(t) = 9 + \int_1^t [S(x) - M(x)]dx \rightarrow A'(t) = S(t) - M(t) = 0$$

$$t = 0.541,\ 5.751$$

t	$A(t)$
0	9
0.541	8.781
5.751	22.955
7	22.096

The absolute minimum amount of snow on the ground is 9.781 inches.

NB. Be sure to answer the question asked. In this case, they asked for the time of the relative minimum.

Summary of Extrema Process

1. Determine the amount equation.

2. Find critical values:

i) $\frac{dy}{dx} = 0$

ii) $\frac{dy}{dx} = dne$

iii) Endpoints of the domain restriction.

3. Find the amount values for the critical values and table the values.

4. Answer the question that was asked

4.4 Free Response Homework

1. The number of parts per million (ppm), $C(t)$, of chlorine in a pool changes at the rate of $C'(t) = 1 - 3e^{-0.2\sqrt{t}}$ ppm per day, where t is measured in days. There are 50 ppm of chlorine in the pool at time $t = 0$. Chlorine should be added to the pool if the level drops below 40 ppm.

(a) Is the amount of chorine increasing or decreasing at $t = 9$? Why or why not?

(b) For what value of t is the amount of chlorine at a minimum? Justify your answer.

---

2. In 1920, Dr. Quattrin's grandfather Andrea returned to America from Italy after fighting in World War I. He arrived in New York Harbor on the *SS Pannonia* and, despite having established residency in 1913, had to be processed through the Immigration Center at Ellis Island. There were 1123 non-citizen, third-class passengers on the *Pannonia* that had to go through processing. (First- and second-class passengers passed through without processing.) Immigrants entered the processing line at a rate modeled by the function $E(t) = 8843\left(\frac{t}{5}\right)^4\left(1 - \frac{t}{10}\right)^5$, where t is measured in hours after the ship began opffloading immigrants and $0 \le t \le 10$. The new arrivals were processed out at a rate of 250 people per hour. The *Pannonia* was the third ship in port, so there were already 2500 people in line when the *Pannonia* passengers got into line.

Assuming no new ships entered the harbor, what was the absolute maximum number of people in the processing line? Justify your answer.

---

3. Letters arrive at a post office at a rate of $P(t) = 8 + t\sin\frac{t^3}{80}$ hundred letters per hour over the course of a workday. The day begins at 9am ($t = 0$) and ends at 5pm ($t = 8$). There are 3 hundred letters in the office at 9am. Workers send letters out of the office at a constant rate of 5 hundred letters per hour.

(a) Write an expression for $L(t)$, the total number of letters in the post office at time *t*.
(b) What is the maximum number of letters in the office over the course of the workday ($0 \le t \le 8$)?

---

4. A ski resort uses a snow machine to control the snow level on a ski slope. Over a 24-hour period the volume of snow added to the slope per hour is modeled by the equation $S(t) = 24 - t\sin^2\left(\frac{t}{14}\right)$. The rate that the snow melts is modeled by $M(t) = 10 + 8\cos\left(\frac{t}{3}\right)$. Both $M(t)$ and $S(t)$ are measured in $\frac{yd^3}{h}$ and t is measured in hours for $0 \le t \le 24$. At time t = 0 the slope holds $50yd^3$ of snow.

(a) Write an expression for $A(t)$, the total amount of snow on the mountain at time *t*.
(b) Find the absolute maximum and minimum amount of snow on the mountain during $0 \le t \le 24$ hours. Include the units.

5. More than 30% of observed star systems have multiple stars, and 70% of those have more than two stars. When stars are close together, they exchange mass in a process known as accretion. Consider a trinary system where S_1 is larger than S_2, and S_2 is larger than S_3. S_3 will lose mass to S_2, and S_2 will lose mass to S_1 While scientific readings are not available because of the time scale, let us suppose that S_2 loses mass to the larger S_1 at a rate of $L(t) = 1 + (.01t)^2 + .23\sin\left(\frac{\pi}{25}t\right)$ and gains mass from the smaller S_3 at a rate of $G(t) = 0.2 + 0.15\sqrt{t}$ where $0 \le t \le 100$ years. $L(t)$ and $G(t)$ are measured in yottatons per year $\left(\mathrm{Y}/yr\right)$. (A yottaton is 10^{26} tons, or 10^{-7} solar masses.)

(a) Write an expression for $A(t)$, the total mass of S_2 at time *t*.

(b) If the mass of S_2 is 1,000,000 yottatons at $t = 0$, find the minimum mass of S_2 on the time interval $0 \le t \le 100$.

---

6. The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(t/3\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

(a) Write an expression for $A(t)$, the total gallons of water in the basement at time *t*.

(b) Find the minimum amount of water in the basement on the time interval $0 \le t \le 18$.

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7. The King Philip Problem

On Ocean Beach at the foot of Noriega is a shipwreck buried in the sand. In 1858, the clipper King Philip ran aground. It was stripped of salvageable material, but the 45% of its hull is still buried on the beach, and it appears every few years as the tide washes the sand in and out.

As the sand washes in or is added by the National Park Service, the height changes at a rate of

$$A(t) = \frac{6t}{1+2t},$$

and, as the tide washes sand out, the height changes at a rate of

$$B(t) = 2 + 7\cos\left(\frac{\pi}{15}t\right)\sin\left(\frac{3\pi}{16}t\right)$$

Both $A(t)$ and $B(t)$ are both measured in inches above the wreck per year for $0 \le t \le 10$. At $t = 0$, the height is 10 inches above the wreck.

(a) Find the total inches of sand above the wreck which are washed out in the first five years. Indicate the correct units.
(b) Is the rate of change of the height increasing or decreasing at $t = 6$ years? Justify your answer.
(c) Write an expression for $H(t)$, the total number of inches above the wreck at time *t*.
(d) What is the absolute minimum number of inches above the wreck over the course of the ten years described?

---

8. Cat Population Problem

The Peninsula Humane Society (PHS) is dedicated to the care and adoption of as many animals who they receive as possible. Since cats breed seasonally, the number of cats and kittens they receive into their facility in a given year varies roughly sinusoidally with time. The data available from 2019 shows the rate $R(\mathrm{t})$, measured in healthy cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$R(\mathrm{t}) = 120 - 88\cos\left[\frac{\pi}{6}(t-2)\right].$$

The rate $A(\mathrm{t})$ at which adoption occur, measured in cats per month, varies with time, measured in months after New Year's Day, according to the equation

$$A(\mathrm{t}) = 125 - 85\cos\left[\frac{\pi}{6}(t-3)\right].$$

On New Years' Day ($\mathrm{t}=0$), there were 131 cats in the PHS Nursery waiting to be adopted.

(a) How many cats and kittens were received at PHS in 2019?
(b) Find $A'(10.3)$. Using the correct units, explain the meaning of $A'(10.3)$ in context of the problem.
(c) Find the number of healthy cats and kittens predicted by the models to be in the PHS facility at the end of 2019.
(d) Find the time when the number of healthy cats and kittens in the PHS facility during 2019 was at an absolute maximum. Include the units.

---

9. The Oak Island Spoils Problem

The search for treasure on Oak Island has been ongoing since 1795. In the last ten years, modern engineering techniques have been brought to bear by the Lagina brothers. They hired ROC Drilling to sink 8 foot-diameter metal tubes (called caissons) down to 200 feet and then use a hammer grab to excavate the "spoils" (what comes out of the caissons). The hammer grab brings up the spoils at a rate of

$$H(t) = 9 + 4\cos\left(\frac{t^2}{14}\right).$$

The spoils are then taken to a wash plant where they are rinsed and sorted into piles by size so they can be visually inspected to find artifacts. A shift supervisor measures their rate of output every few hours and records the findings in the chart below.

t = time in hours	0	1	3	5	8
$P(t)$= Rate of spoils washed in yds^3/hour	0	8.1	14.5	12.3	3.2

At the beginning of the day, there are 2 yds^3 at the wash plant from the day before.

(a) Find the total amount of spoils pulled from the caisson by the hammer grab during the eight hours described on the table. Indicate the correct units.

(b) Use a Right Riemann sum with subintervals indicated by the table to approximate $\int_0^8 \mathrm{P(t)dt}$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Assume the equation $W(t) = 0.1x(10-x)^2$ models the $P(t)$ data on the table. Write an expression for $U(t)$, the amount of unwashed spoils at time t.

(d) What is the absolute maximum amount of unwashed spoils waiting to be cleaned?

10. The Callaghan Ranch Harvest Problem

In 1867, Dr. Q's great-great grandfather Michael Callaghan bought 160 acres of public land in Solano County to raise wheat and sheep. By 1890, the ranch had grown to 1210 acres. Prior to the advent of the combine harvester, reaping and threshing were done by different machines. The table below shows the rate at which wheat on the ranch was gathered by a McCormick Reaper on a given 12-hour workday.

t	0	2	5	9	12
$R(t)$	3.3	2.5	1.6	2.7	1.6

Time t is measured in hours after 6am $(t = 0)$ and $R(t)$ is measured in acres per hour. The wheat was then delivered to the barn and put through the threshing machine. At the start of the day, there are 2.4 acres of wheat in the barn.

(a) Is the rate at which the wheat is being harvested at 9:30 $(t = 3.5)$ increasng or decreasing? Explain the result in context of the problem, using the appropriate units.
(b) Use a right-hand sum to approximation to find the total amount of wheat harvested on this particular day.

(c) The thresher processed the wheat into grain and hay at a rate of $P(t) = \dfrac{2.1x}{\sqrt{x^2+1}}$ acres per hour. How much wheat has been thresher by noon $(t = 6)$ when the farm hands break for lunch?

(d) Write an equation for $0 \le t \le 12$ which would determine the amount of wheat in the barn at any time *t*. Using your estimate of the amount of wheat reaped in b) above, find the amount of unthreshed wheat is in the barn at the end of the day.

(e) Assume $R(t)$ can be modeled by the equation $Q(t) = 3.3 - \frac{3}{20}t - .9\sin\left(\frac{\pi}{7}t\right)$. What is the minimum amount of unthreshed wheat in the barn?

---

11. The Yuma Desalting Problem Ib

The desalting plant at Yuma, AZ, removes alkaline (salt) products from the Colorado River the make the water better for irrigation downstream in Mexico. Data from a Pilot Run of the plant shows that water enters the plant at a rate $W(t)$ as shown on the table below:

t in Month	0	1	2	3	4	5	6	7	8	9	10
$W(t)$ in foot-acres per month	0	2375	3189	3411	3207	2169	2269	2151	2167	3022	2293

The rate $P(t)$ of outflow of processed water, in foot-acre per month is modeled by

$$P(t) = -0.55t^4 + 15t^3 - 158t^2 + 722t + 1032$$

For $0 \le t \le 10$. Based on supplies available, not all the water gets processed before returning to the Colorado River.

a) Using a Midpoint Reimann Sum, approximate the volume of water that enters the plant during these ten months.

b) Set up an equation for $U(t)$ which would define the amount of unprocessed water that exits the plant. Using your answer in part a), approximate $U(10)$. Indicate units.

c) Approximate $W'(6)$. Using the correct units, explain the meaning of your answer.

d) Assuming $W(t)$ can be modeled by $E(t) = 2800 + 750\sin\left(\frac{2\pi}{11}t\right)$, find the time at which there is an absolute maximum amount of unprocessed water flowing through the plant for $0 \le t \le 10$. Justify your answer.

---

12. The 49er Sack Leader Problem II

NB's Games completed	0	19	21	41	57
$B(g)$ in sacks per game	0	0.68	0.62	0.98	1.16

The table above shows Nick Bosa's sack rate, in sacks per game, over his first four seasons.

(a) Using a Right-hand Reimann Sum, determine the approximate number of sacks Nick Bosa had during these four years. Round to the neares whole number.

(b) Using the data on the table, estimate $B'(32)$. Based on this estimate and the data on the table, was Bosa's sack total increasing at an increasing or decreasing rate? Using the correct units, explain your answer.

(c) Disregarding his injury-shortened 2020 season, $N(g) = 0.178\sqrt{g}$ models Bosa's sack rate per game during the first four years of his career. Find

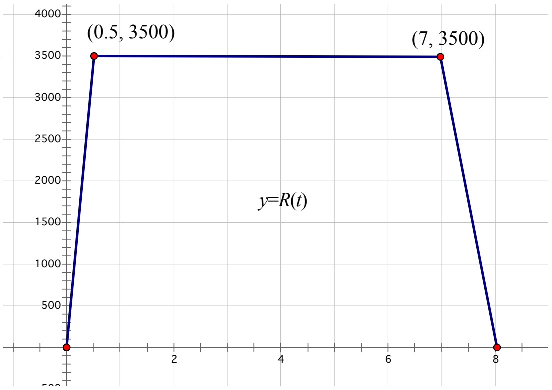
$\frac{1}{57}\int_0^{57} N(g)\,dg$. Using the correct units, explain the the result in context of the problem.

(d) Assume that $A(g) = -.001g^2 + 0.073g - 0.004$ models the rate of Aldon Smith's sacks over his first 50 games from 2011 to 2014. During what game is the difference between Bosa's and Smith's sack totals at a maximum? Show your calculations.

---

13. The Groundwater Replenishment System Problem I

Disparagingly referred to as Toilet to Tap, Orange County's GWRS (Groundwater Replenishment System) has converted 400 billion gallons of raw sewage into drinkable water over the past fifteen years. That is about 25,000 gallons every eight hours. The process occurs in three phases: microfiltration, reverse osmosis, and ultraviolet disinfection. Let us assume that the rate $R(t)$, in gallons per hour, at which raw sewage enters the process as modeled by the graph below, formed by three linear functions.



Further, let us assume the rate at which the raw sewage is converted to treated sewage, in gallons per hour, in the microfiltration process is modeled by $T(t) = 83e^{0.5t}\sqrt{8t - t^2}$ for $t \in [0,\ 8]$ hours.

(a) How many gallons of raw sewage have entered the process in these eight hours?

(b) Find $T'(6.2)$. Using the correct units, explain the meaning of the result in terms of the situation.

(c) Assuming there is no raw sewage left from the previous process, write an expression for the $A(t)$, the total amount of raw sewage present within the process at any time *t*.

(d) Find the maximum amount of raw sewage in process during the micro filtration phase.

(e) Was all the raw sewage converted? Explain your reasoning.

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14. Groundwater Replenishment System Problem II

Disparagingly referred to as Toilet to Tap, Orange County's GWRS (Groundwater Replenishment System) has converted 400 billion gallons of raw sewage into drinkable water over the past fifteen years. That is about 25,000 gallons every eight hours. The process occurs in three phases: microfiltration, reverse osmosis, and ultraviolet disinfection. Let us assume that the rate $T(t)$, in gallons per hour, at which treated sewage enters the reverse osmosis process as modeled by

$T(t) = 3800e^{-0.4t}\sqrt{8t - t^2}$ gallons for $0 \le t \le 8$ hours.

Further, let us assume the rate $N(t)$ at which the treated sewage is converted to non-potable (non-drinkable) water, in gallons per hour, in the reverse osmosis process is modeled by the table below.

T in hours	0	1.8	4.4	6.3	8
$N(t)$ in gallons per hour	0	2301	3107	4320	5204

(a) How many gallons of treated sewage have entered the process in these eight hours? Round to the nearest gallon.

(b) Find $N'(5.2)$. Using the correct units, explain the meaning of the result in terms of the situation.

(c) Use a trapezoidal sum to approximate the amount of non-potable water that exits the system in these eight hours. Round to the nearest gallon.

(d) Assuming there is 1200 gallons of treated sewage in the system at the left from the previous process, use the answers to (a) and (c) above to approximate, to the nearest gallon, the amount of treated sewage still in the system at the end of these eight hours.

(e) Assume $M(t) = -1 + 1650\sqrt{t} + 352\cos\left(\frac{\pi}{4}x\right)$ is a model of the $N(t)$ data given above. What is the maximum amount, to the nearest gallon, of water in the system for $0 \le t \le 8$ hours? Justify your answer.

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15. Groundwater Replenishment System Problem III

Disparagingly referred to as Toilet to Tap, Orange County's GWRS (Groundwater Replenishment System) has converted 400 billion gallons of raw sewage into drinkable water over the past fifteen years. That is about 25,000 gallons every eight hours. The process occurs in three phases: microfiltration, reverse osmosis, and ultraviolet disinfection. Let us assume that $N(t)$ and $P(t)$, in gallons per hour, are the rates at which non-potable water enters and the potable (drinkable) water exits the ultraviolet disinfection process. Further, let us assume that $N(t)$ and $P(t)$ are modeled by the equations

$$N(t) = 3750 - 3000\cos\left[\frac{\pi}{8}x^{1.5}\right]$$

and

$$P(t) = 1125\sqrt{8x - x^2}$$

for $t \in [0, 8]$ hours.

(a) How many gallons of non-potable water have entered the process in these eight hours?

(b) Find $P'(5.3)$. Using the correct units, explain the meaning of the result in terms of the situation.

(c) Assuming there are 4500 gallons of non-potable water left from the previous process, write an expression for the $A(t)$, the total amount of non-potable water present within the process at any time *t*.

(d) Find the time at which the minimum amount of non-potable water in process during the micro filtration phase.

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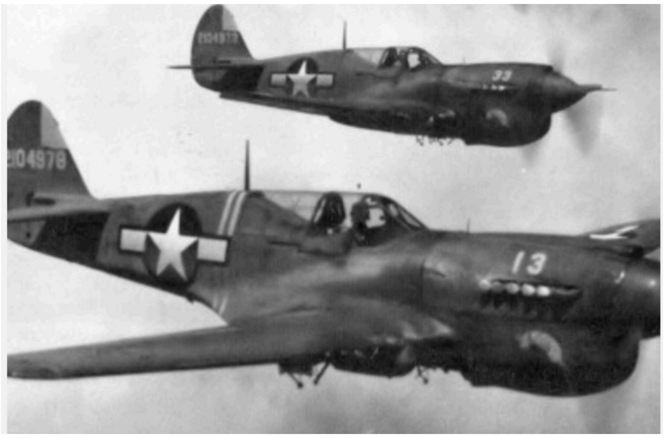
16. WWII Aircraft Problem I

t in Months	6	18	30	42	54	66
$A(t)$ in thousand of Axis-built planes per month	1.06	1.3	1.4	2.1	3.5	5.7
$U(t)$ in thousand of US-built planes per month	0.18	0.5	1.6	4.0	7.2	8.1



The table above shows the rates, in thousand of military planes per month, at which aircraft were built during WWII, where t is measured in months after the beginning of 1939. $A(t)$ represents the rate at which Germany and Japan were building aircraft and $U(t)$ is the rate at which the United States was building aircraft. During the first six months of the War, the Axis powers had produced 4.9 thousand planes and the US had produced 1.2 thousand planes.

(a) Using a Right-Hand Reimann Sum, approximate the number of aircraft build by the Axis powers during these 66 months.

(b) Approximate $U'(20)$. Using the correct units, explain the meaning of this result.

(c) Assume $X(t) = 0.757e^{0.028t}$ models the data for $A(t)$ and $S(t) = 0.164e^{0.067t}$ models the data for $U(t)$. Set up and equation that would model the difference between the Axis and the US aircraft production.

(d) Find the maximum difference between the Axis and the US aircraft production.

4.5: The Second Derivative, Points of Inflection, and Concavity

In terms of graphing, the main use of the Second Derivative is to determine concavity and Points of Infection.

Vocabulary

Concavity--Means: The way a curve bends

Concave Up--Means: The curve bends counter-clockwise.

Concave Down--Means: The curve bends clockwise.

Inflection Value--Defn: The x-value at which the concavity of the curve switches. This can be a vertical asymptote.

Point of Inflection--Defn: The point at which the concavity of the curve switches from up to down or down to up.

A Point of Inflection (POI) occurs if and only if

the sign of $f''(x)$ changes.

N*B*. *An endpoint cannot be a point of inflection.*

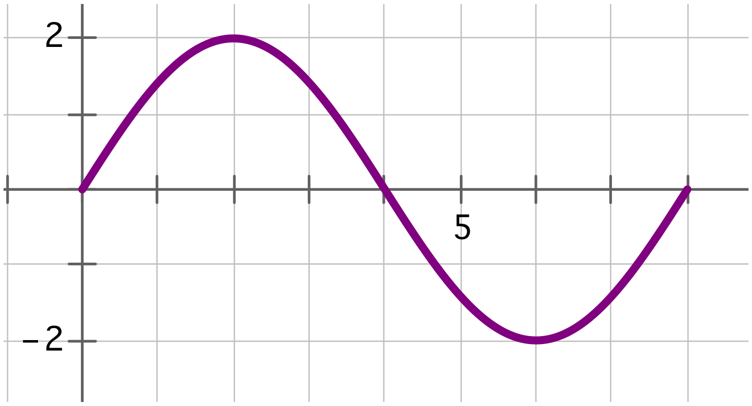
The Second Derivative and Intervals of Concave Up or Down are comparable to the First Derivative and Intervals of Increasing or Decreasing, respectively. In other words,

Derivative	Positive	Negative
$f'(x)$	Increasing	Decreasing
$f''(x)$	Concave up	Concave down

OBJECTIVE

Find Points of Inflection and Intervals of Concavity.
Determine intervals or concavity and increasing/decreasing.
Apply the Second Derivative Test.

Take this graph, for instance:



The curve appears to be increasing on $x \in [0, 2] \cup [6, 8]$ and decreasing on $x \in [2, 6]$. The curve also appears to be concave down $x \in (0, 4)$ and concave up on $x \in (4, 8)$. So, $x = 4$ would be at a point of inflection.

Note on intervals: *For increasing and decreasing, AP always uses brackets and for intervals of concavity AP always uses parentheses. There is an explanation based on formal definitions of increasing, decreasing and concavity that, at this level, appear trivial. We will be consistent with the AP system.*

EX 1 Find the Intervals of Concavity for $y = xe^{2x}$

$$\frac{dy}{dx} = x\left(e^{2x}(2)\right) + e^{2x}(1) = e^{2x}(2x+1)$$

$$\frac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$$

$$= e^{2x}[(2) + (2x+1)(2)]$$

$$= e^{2x}(4x+4) = 0$$

The critical value for the POI is $x = -1$ and there are no Vertical Asymptotes.

So $\begin{array}{cc} y'' & - \quad 0 \quad + \\ & \longleftrightarrow \\ x & -1 \end{array}$ and

y is concave down on $x \in (-\infty, -1)$ and concave down on $x \in (-1, \infty)$.
$x = 1$ would be at a point of inflection because the concavity changes.

EX 2 Find the Points of Inflection of $y = x^3 + 4x^2 - 3x + 1$

$$y' = 3x^2 + 8x - 3$$

$$y'' = 6x + 8 = 0$$

$$x = -\frac{4}{3}$$

$$\begin{array}{cc} y'' & - \quad 0 \quad + \\ & \longleftrightarrow \\ x & -\frac{4}{3} \end{array}$$

$$x = -\frac{4}{3} \rightarrow y = 5.889$$

$\left(-\frac{4}{3}, 5.889\right)$ is the POI.

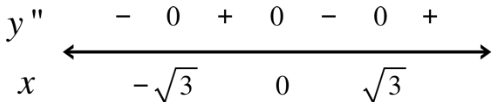
EX 3 Find the Points of Inflection of $y = \dfrac{4x}{x^2+1}$

$$y' = \frac{\left(x^2+1\right)(4) - (4x)(2x)}{\left(x^2+1\right)^2} = \frac{-4x^2+4}{\left(x^2+1\right)^2}$$

$$\begin{aligned} y'' &= \frac{\left(x^2+1\right)^2(-8x) - \left(-4x^2+4\right)2\left(x^2+1\right)(2x)}{\left(x^2+1\right)^4} \\ &= \frac{(8x)\left(x^2+1\right)\left[-\left(x^2+1\right) - \left(-2x^2+2\right)\right]}{\left(x^2+1\right)^4} \\ &= \frac{(8x)\left(x^2-3\right)}{\left(x^2+1\right)^3} = 0 \end{aligned}$$

$$x = 0,\ \pm\sqrt{3}$$

Check the sign pattern to make sure we have POIs and not something else.

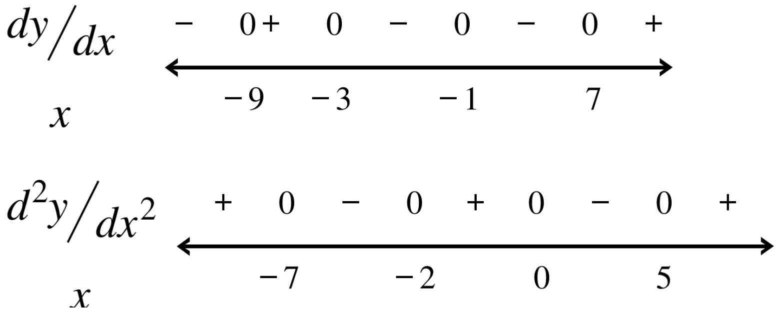


So, all three *x*-values represent POIs, and those POIs are

$$\left(-\sqrt{3},\ -\sqrt{3}\right), (0, 0),\ and\ \left(\sqrt{3},\ \sqrt{3}\right)$$

AP Calculus does not approach these problems (or any problems) only algebraically. They also look at the topic numerically (with sign patterns) and graphically.

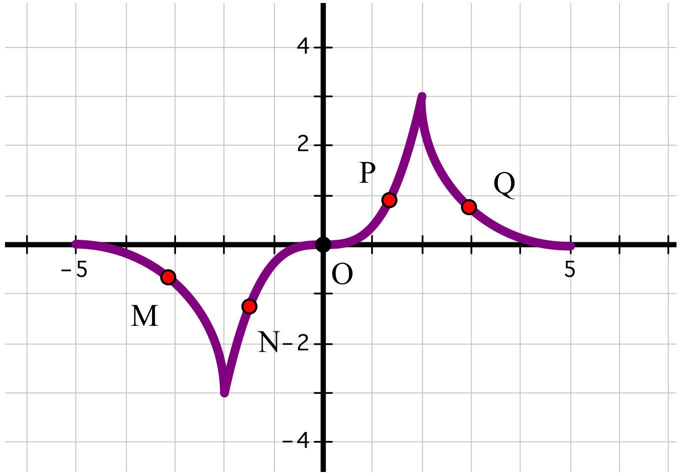
EX 4 Given these two sign patterns:



a) On what interval(s) is *y* decreasing?
b) On what interval(s) is *y* concave up?
c) On what interval(s) is *y* both increasing and concave down?

a) $(-\infty, -9)\cup(-3, -1)\cup(-1, 7)$
b) $(-\infty, -7)\cup(-2, 0)\cup(5, \infty)$?
c) $(-7, -3)$

Ex 5 At what point on the curve below is the interval decreasing and concave up?



The correct answer is point Q.

The 2nd Derivative Test

In Section 4.1, we were reminded that the sign pattern of the First Derivative will reveal whether a critical value is at a maximum, a minimum, or neither.

The 2nd Derivative Test

For a function *f*,

1) If $f'(c)=0$ and $f''(c)>0$, then f has a relative minimum at c.
2) If $f'(c)=0$ and $f''(c)<0$, then f has a relative maximum at c.

Ex 5 Determine if the critical values of $y=(3-x^2)e^x$ are at a maximum or a minimum value.

First, find the critical values:

$$\frac{dy}{dx}=(3-x^2)e^x+e^x(-2x)$$

$$=-e^x(x^2+2x-3)=0$$

The derivative is never DNE.

Critical values: $x=-3, 1$

Find the 2nd derivative and substitute the critical values:

$$\frac{dy}{dx}=-e^x(x^2+2x-3)$$

$$\frac{d^2y}{dx^2}=-e^x(2x+2)+(x^2+2x-3)(-e^x)$$

$$=-e^x(x^2+4x-1)$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=1}=-4e \qquad \left.\frac{d^2y}{dx^2}\right|_{x=-3}=4e^{-3}$$

Therefore, $x=1$ is at a maximum value and $x=-3$ is at a minimum value.

Ex 6 Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that they have values given on the table below.

x	$f'(x)$	$f''(x)$	$g'(x)$	$g''(x)$
2	0	-2	8	0
4	8	0	0	3
8	0	0	0	-4

Then at $x = 8$, $g(x)$ has a:

a) Relative Maximum

b) Relative Minimum

c) Point of Inflection

d) Zero

The correct answer is A because, at $x = 8$, the $g'(8)=0$ and $g''(8)$ is negative.

In this problem, the equation for $g(x)$ is not give, so it is not possible to create a sign pattern for $g'(x)$ and the First Derivative Test cannot be applied.

4.5 Free Response Homework

1. Given this sign pattern for the derivative of $F(x)$, on what interval(s) is $F(x)$ concave down?

$F''(x)$	$-$	0	$+$	0	$-$	0	$+$
x		-3		0		3	

2. The sign pattern for the derivative of $H(x)$ is given. (a) Is $x = -4$ a point of inflection? (b) Is $x = -1$ a point of inflection?

d^2H/dx^2	$+$	0	$-$	0	$-$	0	$+$
x		-4		-1		2	

Find Points of Inflection and Intervals of Concavity.

3. $y = x^3 + x^2 - 7x - 15$

4. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

5. $y = \dfrac{-4x}{x^2 + 4}$

6. $y = \dfrac{x^2 - 1}{x^2 - 4}$

7. $y = \ln\left(4x - x^3\right)$

8. $y = x\sqrt{8 - x^2}$

9. $y = \dfrac{1}{2}x + \sin xonx \in (0, 2\pi)$

10. $y = xe^{-x}$

11. $y = e^{-x^2}$

12. $y = \dfrac{x}{x^2 - 9}$

13. $y = 2x - x^{2/3}$

For each of the following functions, apply the 2nd Derivative Test to determine if each critical value is at a maximum, a minimum, or neither.

14. $y = x^3 - 7x^2 + 11x + 3$

15. $y = 3x^4 + 16x^3 + 24x^2$

16. $f(x) = \dfrac{3x}{x^2 + 4}$

17. $y = x^4 - 4x^3$

18. $f(x) = e^{-x} - e^{-2x}$

19. $f(x) = x^2 \ln x$

20. $h(t) = t^3 - 12t + 21$

21. $f(x) = xe^{-x^2}$

4.5 Multiple Choice Homework

1. $f(x) = x^3 - 6x^2$ is concave up when

a) $x > 2$ b) $x < 2$ c) $0 < x < 4$

d) $x < 0$ or $x > 4$ e) $x > 6$

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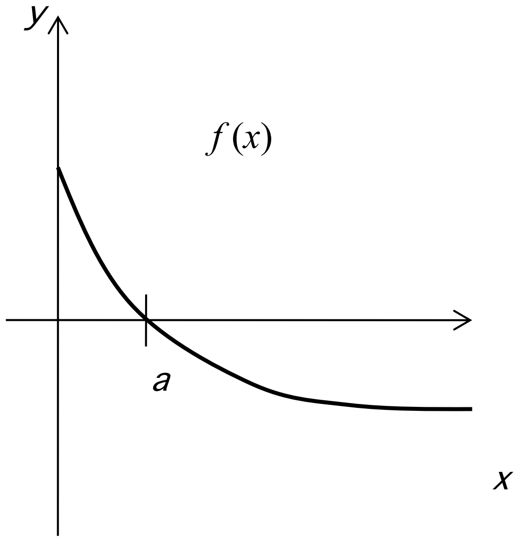
2. Given the twice differentiable function f, which of the following statements is true?



a) $f'(a) < f''(a) < f(a)$ b) $f'(a) < f(a) < f''(a)$

c) $f''(a) < f(a) < f'(a)$ d) $f''(a) < f'(a) < f(a)$

e) $f(a) < f'(a) < f''(a)$

---

3. For $x > 0$, let $f'(x) = \dfrac{\ln x}{x}$ and $f''(x) = \dfrac{1 - \ln x}{x^2}$. Which of the following is true?

a) f is decreasing for $x > 1$ and the graph of f is concave down for $x > e$

b) f is decreasing for $x > 1$ and the graph of f is concave up for $x > e$

c) f is increasing for $x > 1$ and the graph of f is concave down for $x > e$

d) f is increasing for $x > 1$ and the graph of f is concave up for $x > e$

e) f is increasing for $0 < x < e$ and is concave down for $0 < x < e^{3/2}$

---

4. Consider the function $f(x) = \left(x^2 - 5\right)^3$ for all real numbers x. The number of inflection points for the graph of f is

a) 1 b) 2 c) 3 d) 4 e) 5

---

5. If $\dfrac{d}{dx}\left[f(x)\right] = g(x)$ and $\dfrac{d}{dx}\left[g(x)\right] = f(3x)$, then $\dfrac{d^2}{dx^2}\left[f\left(x^2\right)\right]$ is

a) $4x^2 f\left(3x^2\right) + 2g\left(x^2\right)$ b) $f\left(3x^2\right)$ c) $f\left(x^4\right)$

d) $2xf\left(3x^2\right) + 2g\left(x^2\right)$ e) $2xf\left(3x^2\right)$

---

6. If $y = e^{kx}$, then $\dfrac{d^5y}{dx^5} =$

a) k^5e^x b) k^5e^{kx} c) $5!e^{kx}$ d) $5!e^x$ e) $5e^{kx}$

---

7. The graph of $y = 3x^5 - 10x^4$ has an inflection point at

a) $(0,0)$ and $(2,-64)$ b) $(0,0)$ and $(3,-81)$

c) $(0,0)$only d) $(3,-81)$only e) $(2,-64)$only

---

8. An equation of the line tangent to the graph of $y = x^3 + 3x^2 + 2$ at its point of inflection is

a) $y = -3x + 1$ b) $y = -3x + 7$ c) $y = x + 5$

d) $y = 3x + 1$ e) $y = 3x + 7$

---

9. The number of inflection points for the graph of $y = 2x + \cos\left(x^2\right)$ in the interval $0 \le x \le 5$ is

a) 6 b) 7 c) 8 d) 9 e) 10

---

10. On which of the following intervals is the graph of the curve $y = x^5 - 5x^4 + 10x + 15$ concave up?

I. $x < 0$ II. $0 < x < 3$ III. $x > 3$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

11. Let *f* be a function with a second derivative given $f''(x)=x^2(x-3)(x-6)$. What are the *x*-coordinates of the points of inflection of the graph of *f*?

a) 0 only b) 3 only c) 0 and 6 only

d) 3 and 6 only e) 0, 3, and 6

---

12. The derivative of the function is given by $f'(x)=x^2\cos(x^2)$. How many points of inflection does the graph of f have on the open interval $(-2,2)$?

a) One b) Two c) Three d) Four e) Five

---

13. How many points of inflection does the graph of $y=\cos x+\frac{1}{3}\cos 3x-\frac{1}{5}\cos 5x$ have on the interval $0\le x\le \pi$?

a) 1 b) 2 c) 3 d) 4 e) 5

---

14. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that they have values given on the table below.

x	$f'(x)$	$f''(x)$	$g'(x)$	$g''(x)$
2	0	2	-8	0
4	8	0	0	3
8	0	-12	0	4

Then at $x = 4$, $g(x)$ has a:

a) Relative Maximum

b) Relative Minimum

c) Point of Inflection

d) Zero

---

15. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that they have values given on the table below.

x	$f'(x)$	$f''(x)$	$g'(x)$	$g''(x)$
2	0	2	-8	0
4	8	0	0	3
8	0	-12	0	4

Then at $x = 4$, $f(x)$ most likely has a:

a) Relative Maximum

b) Relative Minimum

c) Point of Inflection

d) Zero

4.6: Graphing with Derivatives

All last year, we concerned ourselves with sketching graphs based on traits of a function. We tended to look at the one key aspect of the derivative – that is finding extremes – as it applied to **a function expressed by an equation**.

REMEMBER TRAITS:

1. Domain
2. Range
3. y-intercept
4. Zeros
5. Vertical Asymptotes
6. Points of Exclusion
7. End Behavior
8. Extreme Points

Toward the end of the year in PreCalculus H, we began to look at the first and second derivatives as the more important traits of the function and used sign patterns to determine how the points were connected. We were still viewing the function as expressed by an equation, but **the emphasis became how the points were connected,** instead of the points themselves. And the sign patterns of the derivatives told us that.

Interpretation of Sign Patterns:

Sign	+	0 or dne	–
$f(x)$	Curve above x-axis	Zero / VA	Curve below x-axis
$f'(x)$	Increasing	Critical Value	Decreasing
$f''(x)$	Concave Up	POI	Concave Down

Ex 1 Make a table of key traits and a complete sketch of $y = x^4 - 2x^3$.

Zeros: $x^4 - 2x^3 = 0 \to x = 0\text{or}2 \to (0,0),(2,0)$

$$\begin{array}{c|ccccc} y & + & 0 & - & 0 & + \\ \hline x & & 0 & & 2 & \end{array}$$

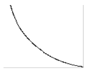
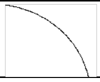
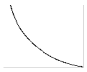
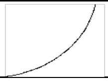
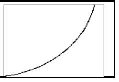
Extreme Points: $\frac{dy}{dx} = 4x^3 - 6x^2 = 0 \to x = 0\text{or}\frac{3}{2} \to (0,0),\left(\frac{3}{2},-\frac{27}{16}\right)$

$$\begin{array}{c|ccccc} y' & - & 0 & - & 0 & + \\ \hline x & & 0 & & \frac{3}{2} & \end{array}$$

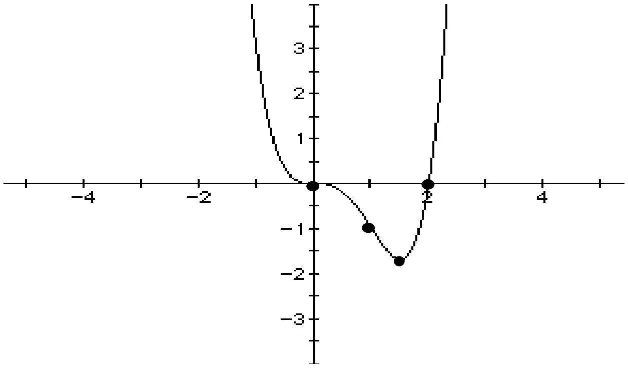
POI: $\frac{d^2y}{dx^2} = 12x^2 - 12x = 0 \to x = 0\text{or}1 \to (0,0),(1,-1)$

$$\begin{array}{c|ccccc} y'' & + & 0 & - & 0 & + \\ \hline x & & 0 & & 1 & \end{array}$$

The three sign patterns can be put into a table so how they interact can be seen:

	$x<0$	$x=0$	$0<x<1$	$x=1$	$1<x<\frac{3}{2}$	$x=\frac{3}{2}$	$\frac{3}{2}<x<2$	$x=2$	$x>2$
y	+	0	–	-1	–	$-\frac{27}{16}$	–	0	+
y'	–	0	–	-	–	0	+	+	+
y''	+	0	–	0	+	+	+	+	+
									

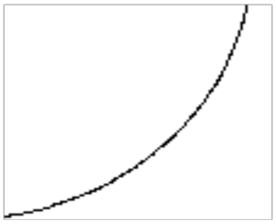
Connect the dots with the appropriately shaped curve and the sketch is:



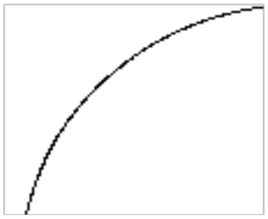
$y = x^4 - 2x^3$

To emphasize the concept that the meanings of the signs for the first and second derivative have a direct reflection in the shape of the graph, the questions are often asked with no equation involved.

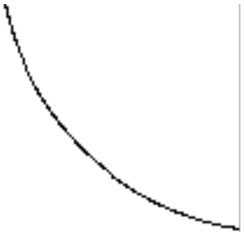
The emphasis in Calculus is on what is occurring between the points and the functions are not necessarily based on an equation.



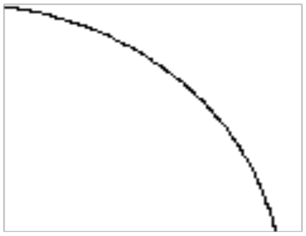
Increasing and Concave Up



Increasing and Concave Down



Decreasing and Concave Up



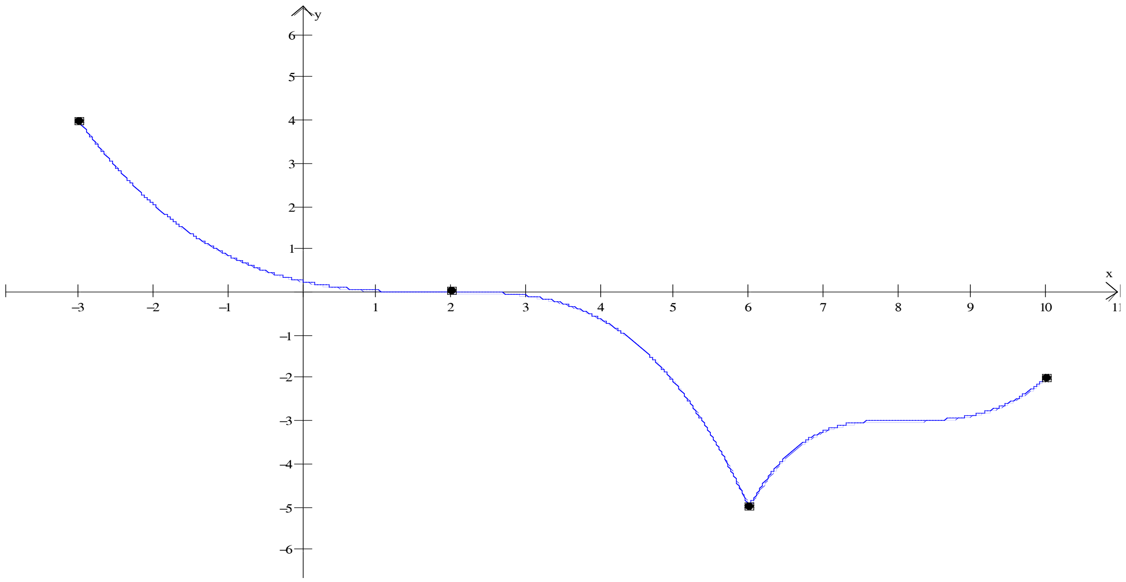
Decreasing and Concave Down

OBJECTIVE

Sketch the graph of a function using information from its first and/or second derivatives.

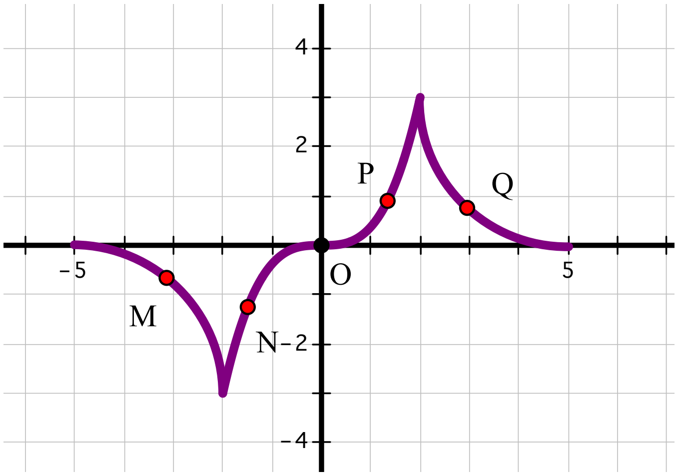
EX 2 Sketch the graph of the function whose traits are given below.

x	–3		2		6		8		10
$f(x)$	4	+	0	–	–5	–	–3	–	–2
$f'(x)$	DNE	–	0	–	DNE	+	+	+	DNE
$f''(x)$	DNE	+	0	–	DNE	–	0	+	DNE



Note that the information given in the table became a graph which is not smooth. Most of the graphs analyzed in PreCalculus were smooth (that is, continuous and differentiable), but there are not there are many more functions and non-functions which are not smooth.

Ex 3   At what point on the curve below is $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$



a) M b) N c) O d) P e) Q

This is the same question as Ex 5 in the last section, only they words "increasing" and "concave up" have been replaced with their mathematical equivalents $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$, respectively.

The correct answer is point Q.

4.6 Free Response Homework

1. The table of values and signs of $f(x)$ is given below. Sketch a graph of $f(x)$ on the domain represented on the table.

x	–3		2		6		8		10
$f(x)$	4	+	0	–	–5	–	–3	–	–2
$f'(x)$	DNE	–	0	–	DNE	+	+	+	DNE
$f''(x)$	DNE	+	0	–	DNE	–	0	+	DNE

2. The table of values and signs of $f(x)$ is given below. Sketch a graph of $f(x)$ on the domain represented on the table.

x	–3		0		3		6		8		10
$f(x)$	0	+	3	+	0	+	3	+	0	+	3
$f'(x)$	DNE	+	0	–	DNE	+	DNE	–	0	+	DNE
$f''(x)$	DNE	–	–	–	DNE	0	DNE	+	+	+	DNE

3. The table of values and signs of $f(x)$ is given below. Sketch a graph of $f(x)$ on the domain represented on the table.

x	–3		0		3		6		8		10
$f(x)$	0	+	3	+	0	–	–3	–	0	+	3
$f'(x)$	DNE	+	DNE	–	–	–	DNE	+	0	+	DNE
$f''(x)$	DNE	+	DNE	–	DNE	+	DNE	–	0	+	DNE

4. The table of values and signs of $f(x)$ is given below. Sketch a graph of $f(x)$ on the domain represented on the table. [Be careful at $x = -2$.]

x	–6		-4		-2		0		2		4
$f(x)$	2	+	0	+	2	+	0	–	–2	–	0
$f'(x)$	DNE	–	0	+	DNE	–	–	–	DNE	+	0
$f''(x)$	DNE	+	+	+	DNE	–	0	+	DNE	–	–

5. A function f is continuous on the interval $x \in [-3,\ 3]$ such that $f(-3) = 4$ and $f(3) = 1$. The functions f' and f'' have the properties given below.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Positive	DNE	Negative	0	Negative
$f''(x)$	Positive	DNE	Positive	0	Negative

6. A function f is continuous on the interval $x \in [-3,\ 3]$ such that $f(-3) = 6$ and $f(3) = 1$. The functions f' and f'' have the properties given below. Sketch a graph of $f(x)$. Sketch a graph of $f(x)$ on $x \in [-3,\ 3]$.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Negative	0	Negative	DNE	Positive
$f''(x)$	Positive	0	Negative	DNE	Negative

7. A function f is continuous on the interval $x \in [-3,\ 3]$ such that $f(-3) = 6$ and $f(3) = 1$. The functions f' and f'' have the properties given below. Sketch a graph of $f(x)$ on $x \in [-3,\ 3]$.

x	$-3 < x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$1 < x < 3$
$f'(x)$	Negative	dne	Negative	0	Positive
$f''(x)$	Negative	dne	Positive	3	Positive

8. A function f is continuous on the interval $x \in [-3,\ 10]$ such that $f(-3) = -1$ and $f(3) = -2$. The functions f' and f'' have the properties given below. Sketch a graph of $f(x)$ on $x \in [-3,\ 10]$.

x	–3		–1		1		3		5		8		10
$f'(x)$	+	+	DNE	–	0	+	DNE	0	0	–	DNE	–	0
$f''(x)$	DNE	–	DNE	+	+	+	DNE	0	0	+	DNE	+	DNE

Sketch the possible graph of a function that satisfies the conditions indicated below

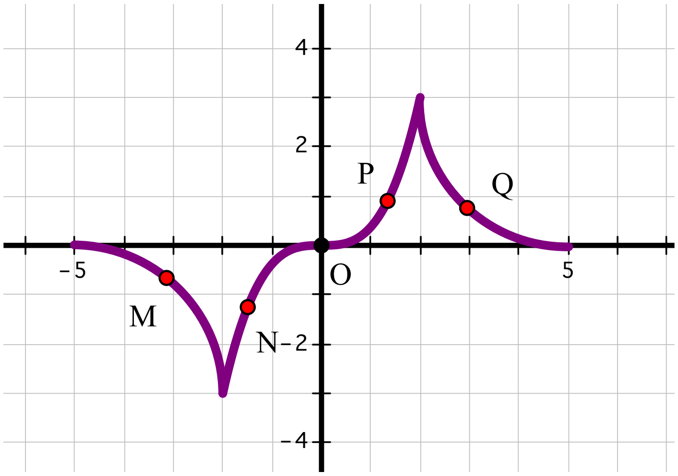
9. Increasing from $(-\infty,5) \cup (7,10)$, decreasing from $(5,7) \cup (10,\infty)$.

10. Increasing from$(-\infty,-3)\cup(5,\infty)$, decreasing from$(-3,5)$, concave up from$(-\infty,-2)\cup(2,\infty)$, concave down from $(-2,2)$

11. Decreasing from$(-\infty,-5)\cup(5,\infty)$, increasing from$(-5,5)$, concave down from$(-\infty,-7)\cup(-3,3)\cup(7,\infty)$, concave up from $(-7,-3)\cup(3,7)$

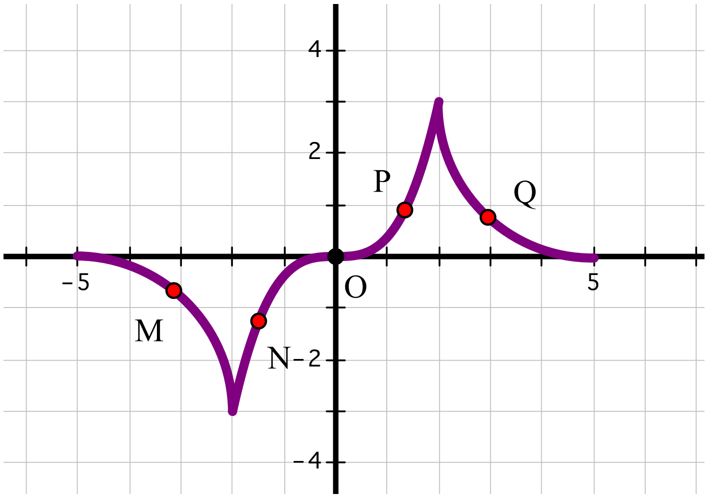
12. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).

4.6 Multiple Choice Homework



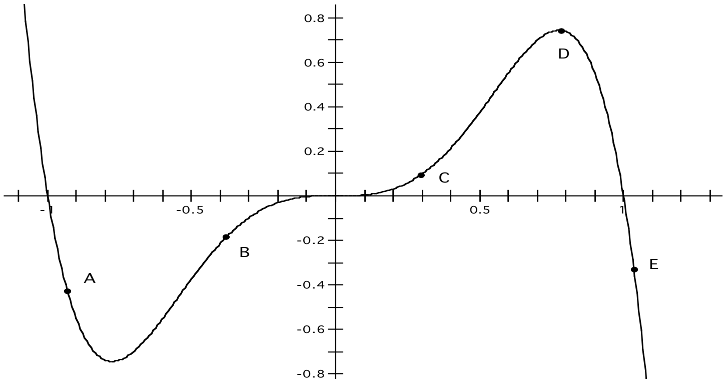
1. At what point on the above curve is $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

a) M b) N c) P d) Q



2. At what point on the above curve is $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$

a) M b) N c) P d) Q



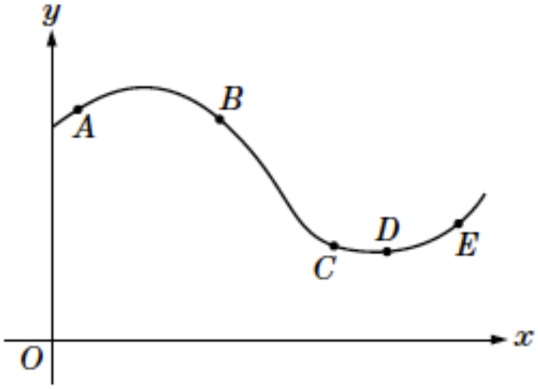
3. The graph of the function $f(x)$ is shown above. At which point on the graph of $f(x)$ is $f'(x) > 0$ and $f''(x) < 0$?

a) A b) B c) C d) D e) E

---

4. At which of the five points on the graph in the figure below are $\frac{dy}{dx}$ and

$\frac{d^2y}{dx^2}$ both negative?



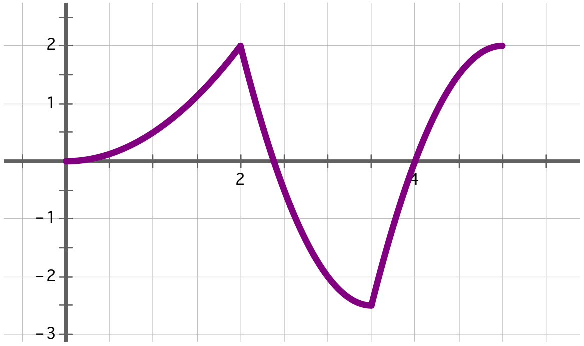
(a) A (b) B (c) C (d) D (e) E

5. Which of the following graphs has these two sign patterns:

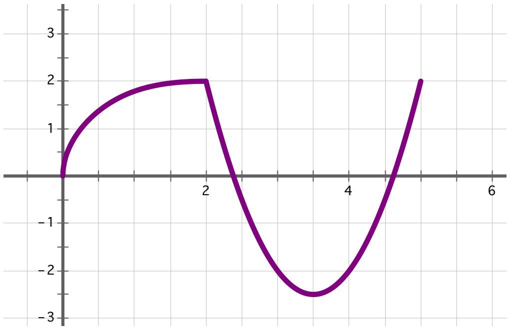
f' + *dne* − dne +
x 2 3.5

f'' + *dne* + dne −
x 2 3.5

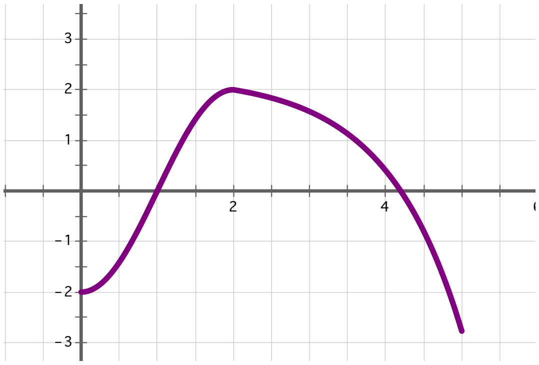
a)



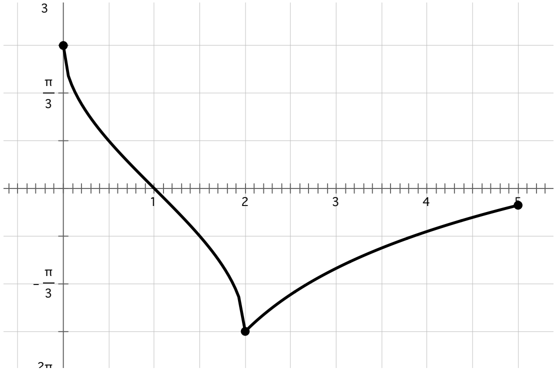
b)



c)



d)

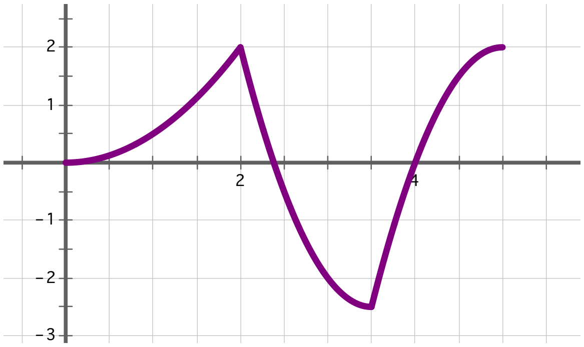


6. Which of the following graphs has these two sign patterns:

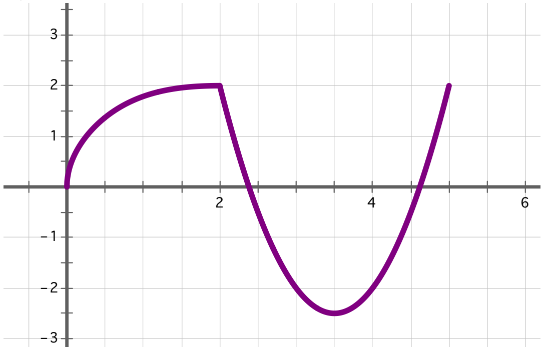
f'	$-$	*dne*	$+$
x		2	

f''	$+$	0	$-$	dne	$-$
x		1		2	

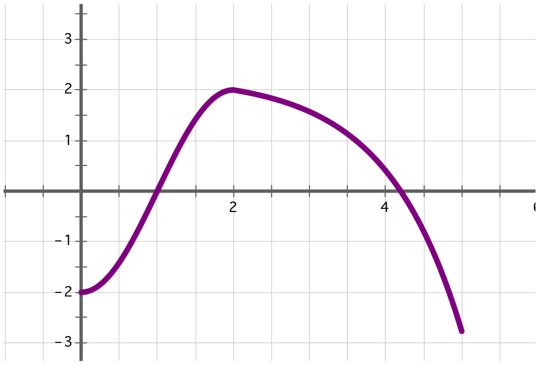
a)



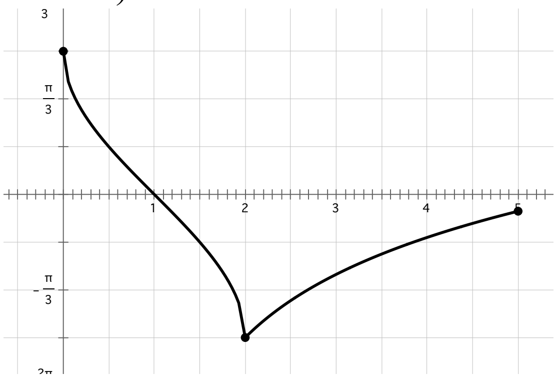
b)



c)



d)

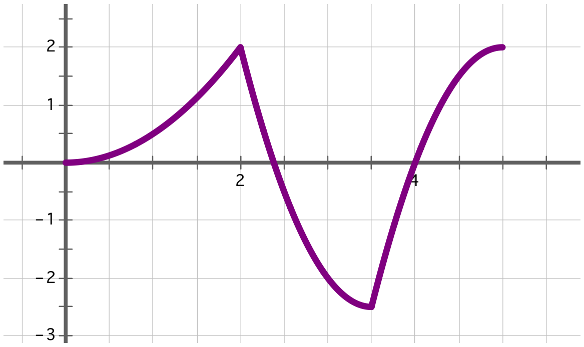


7. Which of the following graphs has these two sign patterns:

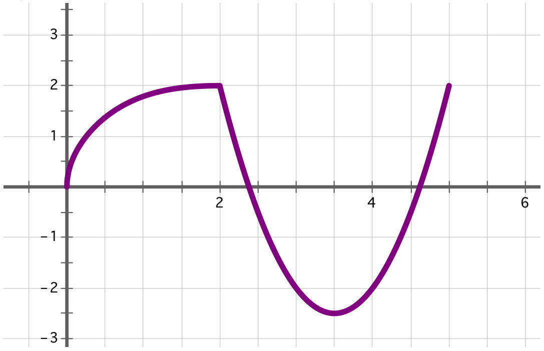
f'	+	*dne*	−
x		2	

f''	+	0	−	dne	−
x		1		2	

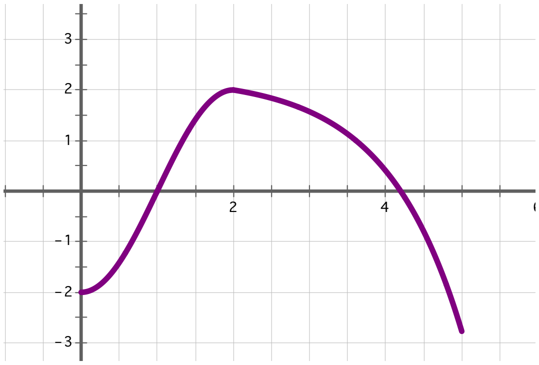
a)



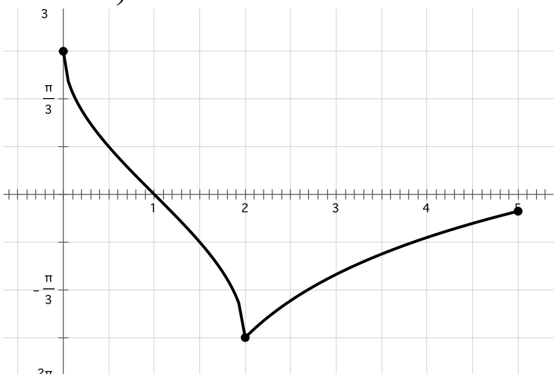
b)



c)



d)

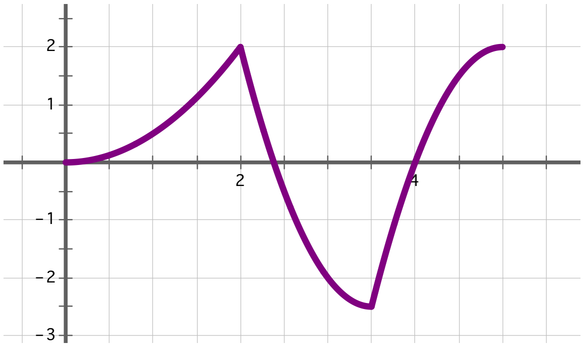


8. Which of the following graphs has these two sign patterns:

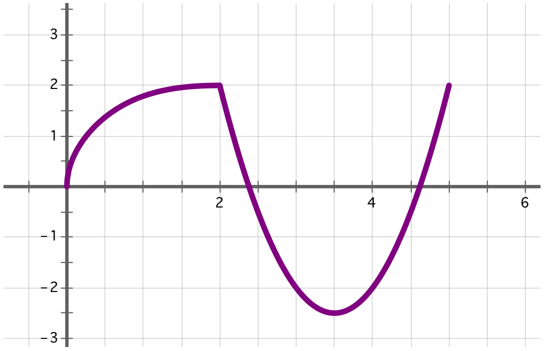
f'	+	*dne*	−	dne	+
x		2		3.5	

f''	−	*dne*	+
x		2	

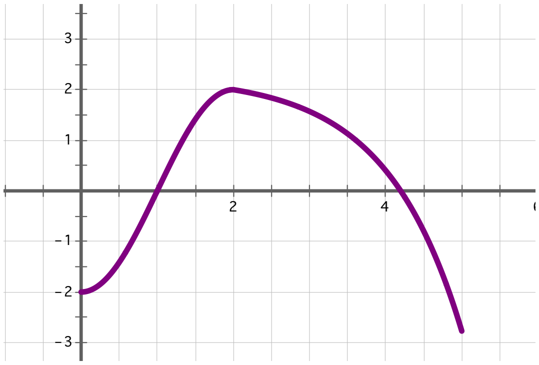
a)



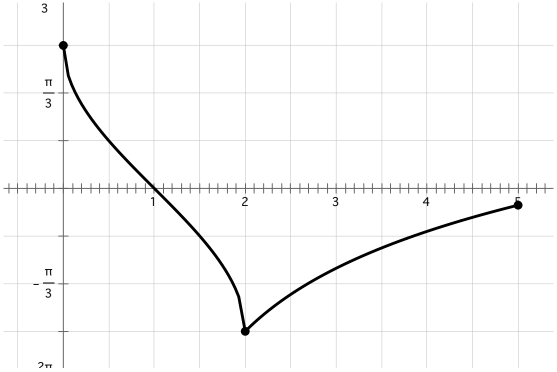
b)



c)



d)



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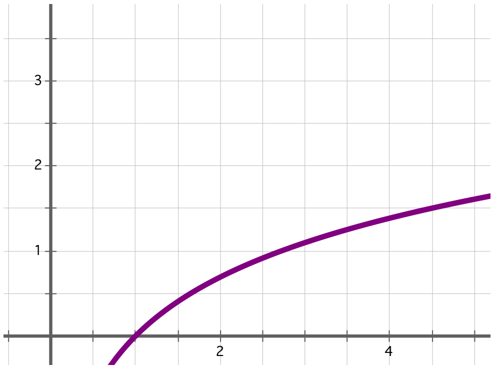
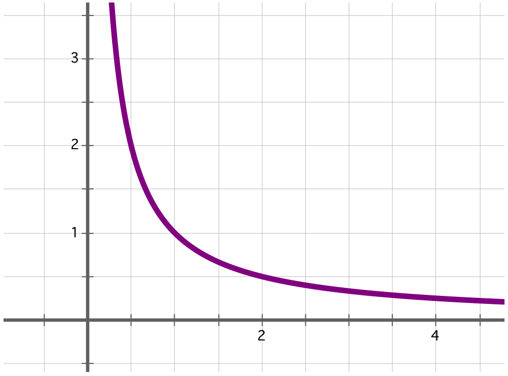
9. Which of the graphs of $y = g(x)$ below has $g'(x) < 0$ and $g''(x) > 0$?

a)



b)



c)

d)

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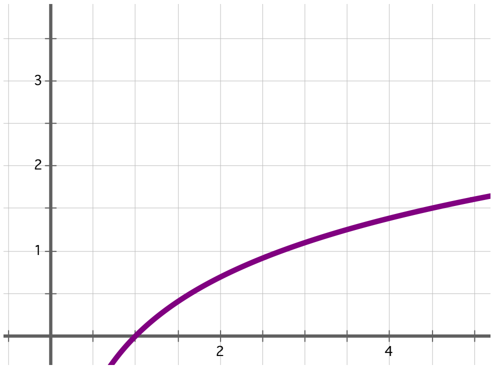
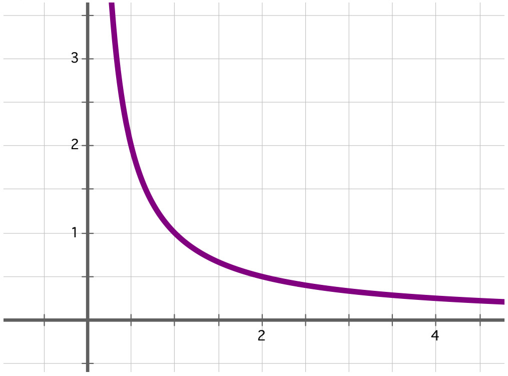
10. Which of the graphs of $y = g(x)$ below has $g'(x) > 0$ and $g''(x) > 0$?

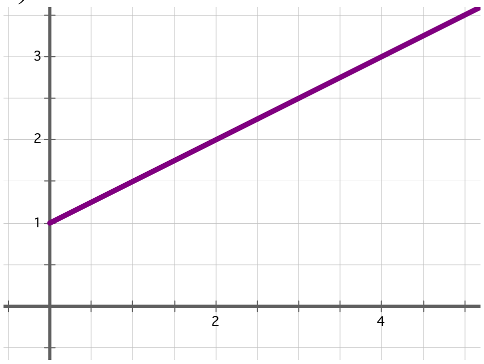
a)



b)



c)



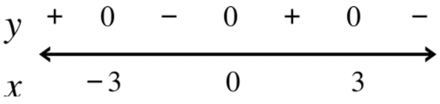
d)

4.7: Graphical Analysis I

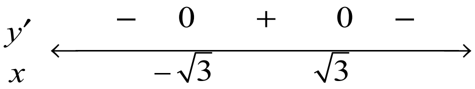
In the last section, we looked at graphing functions and derivatives, but now we will reverse that process. As we noted in the last example of the last section, there is a layering and parallelism between the function, its derivative and its second derivative. The zeros and signs of one tell us about increasing, decreasing, and extremes or the concavity and POIs of another.

EX 1 Find the sign patterns of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$, and compare them.

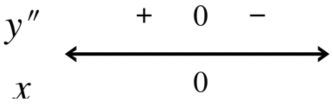
$f(x)=9x-x^3$



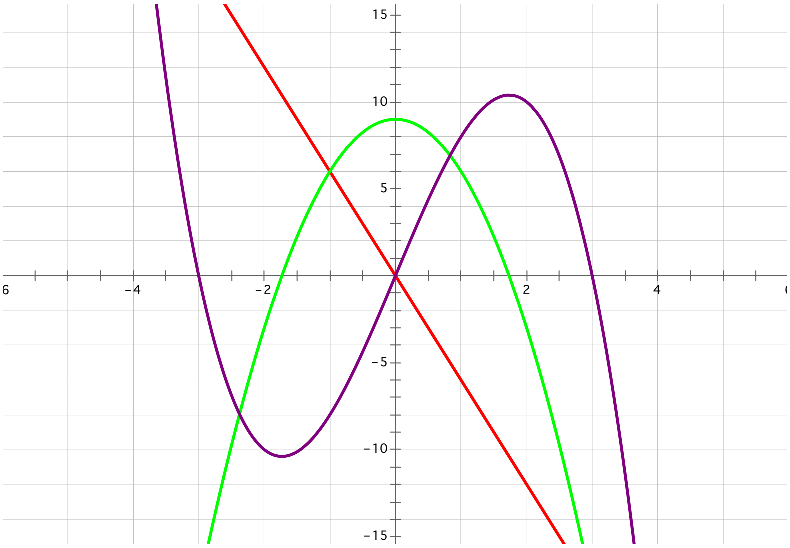
$f'(x)=9-3x^2$



$f''(x)=-6x$



Compare the graphs below of $f(x)=9x-x^3$, $f'(x)=9-3x^2$, and $f''(x)=-6x$.



It can be seen that the extreme points of $f(x)$ line up with the zeros of $f'(x)$, and the extreme point of $f'(x)$ lines up with the zero of $f''(x)$ and the POI of $f(x)$.

Similarly, the positive parts of $f''(x)$ match the increasing part of $f'(x)$ and the concave up part of $f(x)$.

OBJECTIVES

Interpret information in the graph of a derivative in terms of the graph of the "original" function.

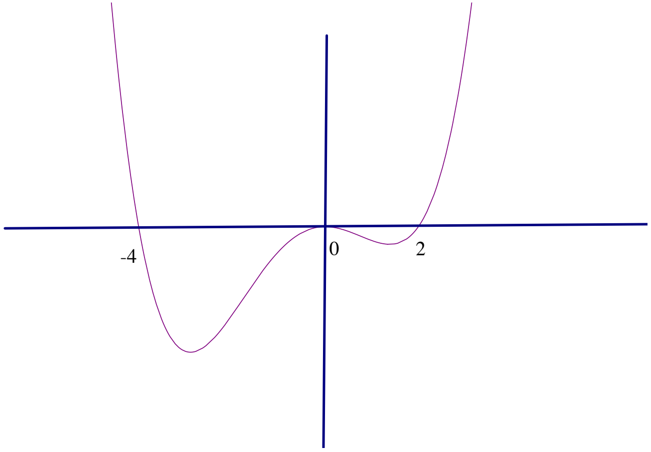
That interconnectedness between a function, its first derivative and its second derivative can be summarized thus:

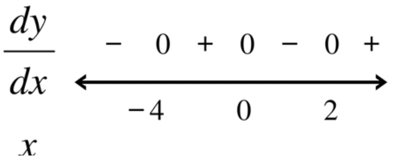
$f(t)$	Positive ZERO of $f(t)$ Negative	Increasing EXTREME of $f(t)$ Decreasing	Concave up POI of $f(t)$ Concave down
$f'(t)$		Positive ZERO of $f'(t)$ Negative	Increasing EXTREME of $f'(t)$ Decreasing
$f''(t)$			Positive ZERO of $f''(t)$ Negative

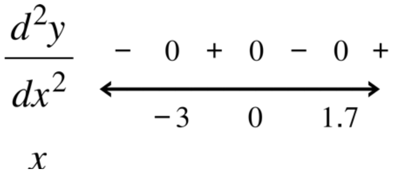
The relationship presented here is upward. The zeros of a function refer to the extremes of the one above it and to the POIs of the one two levels up. If we had a function defined as $\int_0^x F(t)\,dt$ it would be a level above $F(x)$ and it would relate to $F'(x)$ in the same way that $F(x)$ related to $F''(x)$. We will explore this idea further in a later chapter.

Almost any trait can be found from a known first derivative curve, but, unfortunately, the derivative is not enough to tell the exact original curve, because, though the critical values and x-values of the POI are known, their y-values cannot be found. Similarly, the zeros of the original curve cannot be found from its derivative.

Ex 1 The graph below is of $f'(x)$. Find the sign patterns of $f'(x)$ and $f''(x)$.





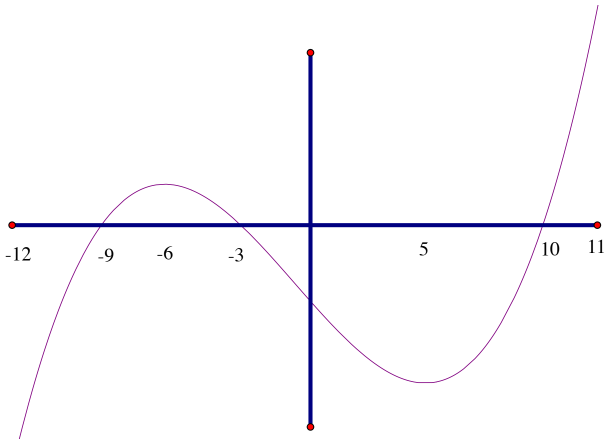


These would then tell us the intervals of increasing and decreasing and of concave up and concave down for $f(x)$. Those relationships can be summarized in the table below.

LEARNING OUTCOME

Determine information about a function from the graph of its derivative.

EX 2 If the curve below is $y = f'(x)$, a) where are the relative maximums and relative minimums of $y = f(x)$ and b) where are the points of inflection of $y = f(x)$? Justify your answer.



What is pictured here is a two dimensional representation of the sign patterns of the first AND second derivatives.

y'	−	0	+	0	−	0	+
x		−9	−	3		10	

y''	+	0	−	0	+
x		−6		5	

a) Where are the relative maximums and relative minimums of $f(x)$? Justify your answer.

Relative maximums are at $x = -12, \; -3, \;$ and $\; 11.$

- At $x = -12$, $f(x)$ starts and is decreasing because $f'(x)$ is negative.
- At $x = -3$, the derivative switches from positive to negative.
- At $x = 11$, $f(x)$ has been increasing before hitting the boundary because the derivative has been positive.

Relative minimums are at $x = -9,$ and 10.

- At $x = -9$ and 10, the derivative switches from negative to positive.

b) Where are the points of inflection of $f(x)$? Justify your answer.

POIs at $x = -6$ and 5

- $f(x)$ has at points of inflection where $f'(x)$ switches from increasing to decreasing or vice versa.

NB. Endpoints cannot be points of inflection.

Key Phrases for Justification

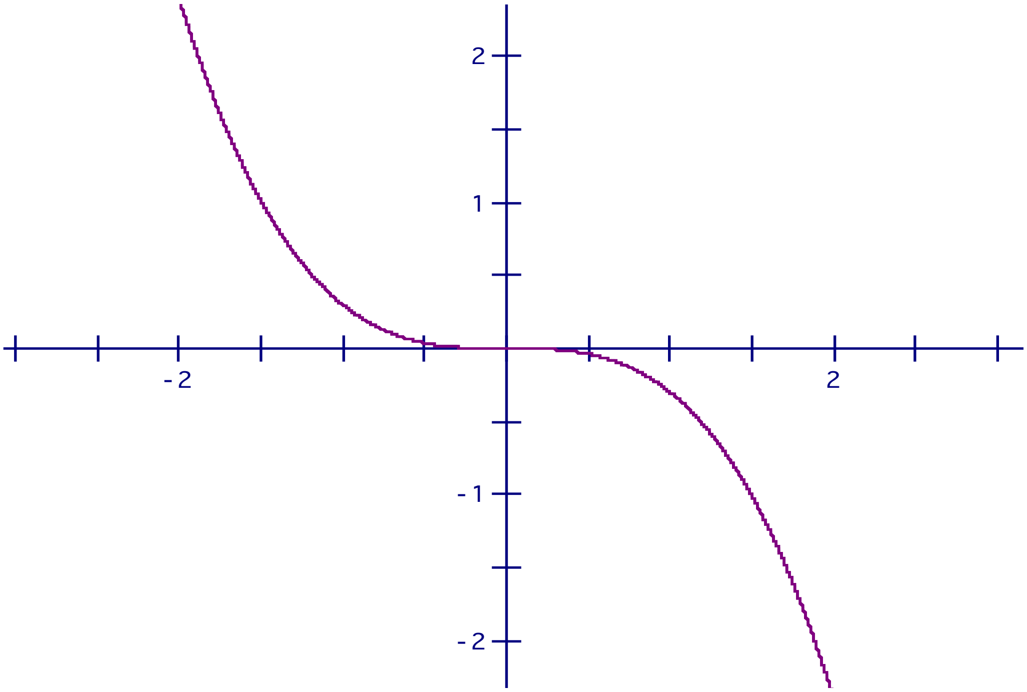
1. “$x = a$ is at a maximum on $f(x)$ because $f'(x)$ switches from positive to negative.”

2. “$x = a$ is at a minimum on $f(x)$ because $f'(x)$ switches from negative to positive.”

3. “$x = a$ is at a point of inflection on $f(x)$ because $f'(x)$ switches from increasing to decreasing (or decreasing to increasing).”

EX 3 Given the same graph of $y = f'(x)$ in EX 2 and $f(0) = 0$, sketch a likely curve for $f(x)$ on $x \in [-2,2]$.

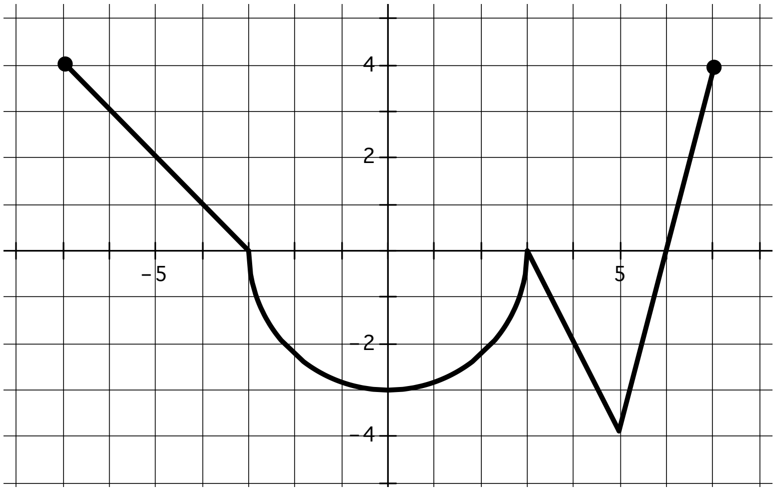
On $x \in [-2,2]$, $f'(x)$ is negative so $f(x)$ in decreasing on that interval. Since $f(x)$ is decreasing and $f(0)$ is the zero, the curve must be above the x-axis on $x \in [-2,0)$ and below the x-axis on $x \in (0,2]$. On $x \in (-2,0)$, the slope of $f'(x)$ (which is $f''(x)$) is positive, so $f(x)$ is concave up. Similarly, on $x \in (0, 2)$, $f''(x)$ is negative since $f'(x)$ is decreasing, so $f(x)$ is concave down. So $(0,0)$ is not only a zero, it is also a point of inflection. Putting it all together, this is a likely sketch:



Note that there are not markings for scale on the y-axis. This is because the y-values of the endpoints of the given domain cannot be known, from the given information.

The graph of $y = f'(x)$ need not be a member of any family of functions either. The sign patterns of the first and second derivatives can be deduced from any graph.

EX 4 The graph below is of $f'(x)$ on $x \in [-7, 7]$.



a) Where are the relative maximums and relative minimums of $y = f(x)$?
b) Where are the points of inflection of $y = f(x)$? Justify your answer.
c) Where is $y = f(x)$ increasing and concave down?

a) $y = f(x)$ has a relative maximum at $x = -3$ because $f'(x)$ switches from positive to negative, and $y = f(x)$ has a relative maximum at $x = 7$ because $f(x)$ is increasing— $f'(x)$ is positive— and then ends.

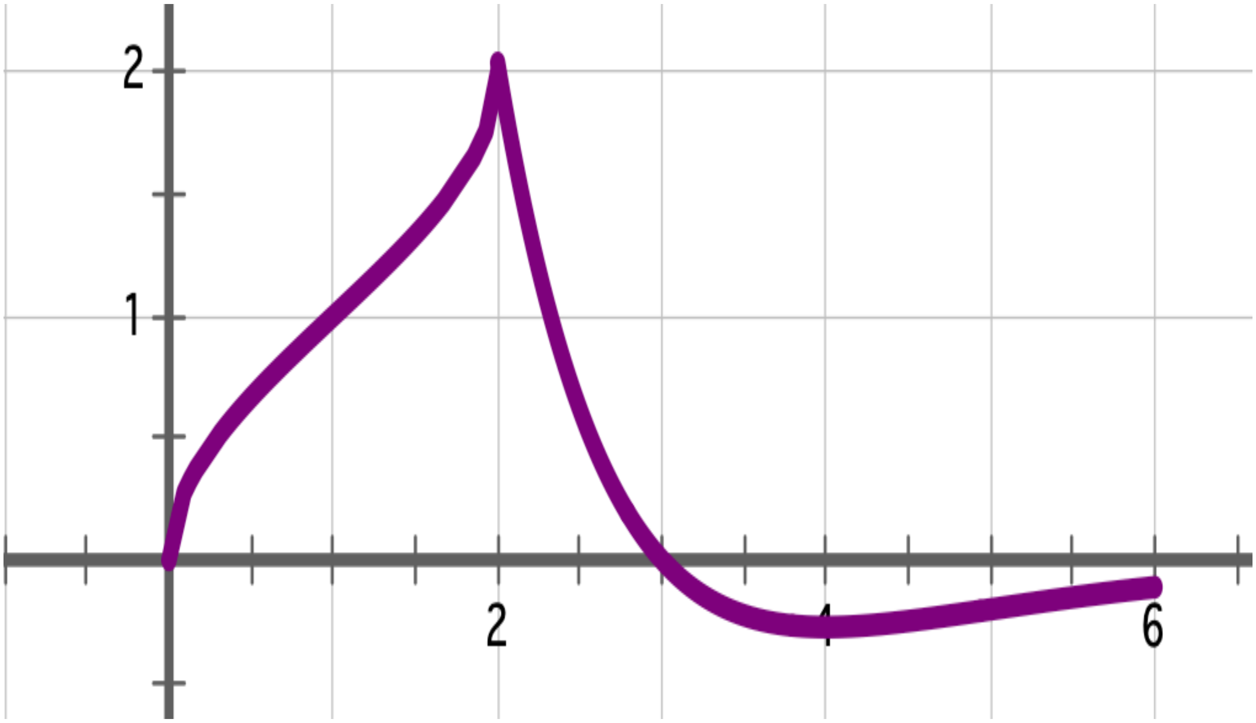
$y = f(x)$ has a relative minimum at $x = 6$ because $f'(x)$ switches from negative to positive, and $y = f(x)$ has a relative minimum at $x = -7$ because $f(x)$ starts and is decreasing— $f'(x)$ is negative.

b) $y = f(x)$ has POIs at $x = 0,\ 3,\ \text{and}\ 5$ because $f'(x)$ switches from increasing to decreasing or vice versa.

c) $y = f(x)$ is increasing and concave down when $f'(x)$ is positive and decreasing, so $x \in (-7,\ -3)$.

4.7 Free Response Homework

1. Let $g(x)$ be a continuous function on $x \in [0, \ \ 6]$ where the graph of $g'(x)$ is the function shown below.



a) Identify the x-value(s) of the relative maximums of $y = g(x)$ on the interval $x \in [0, \ \ 6]$. Justify your answer.

b) Identify the x-value(s) of the relative minimums $y = g(x)$ on the interval $x \in [0, \ \ 6]$. Justify your answer.

c) Where are the points of inflection on $y = g(x)$? Justify your answer.

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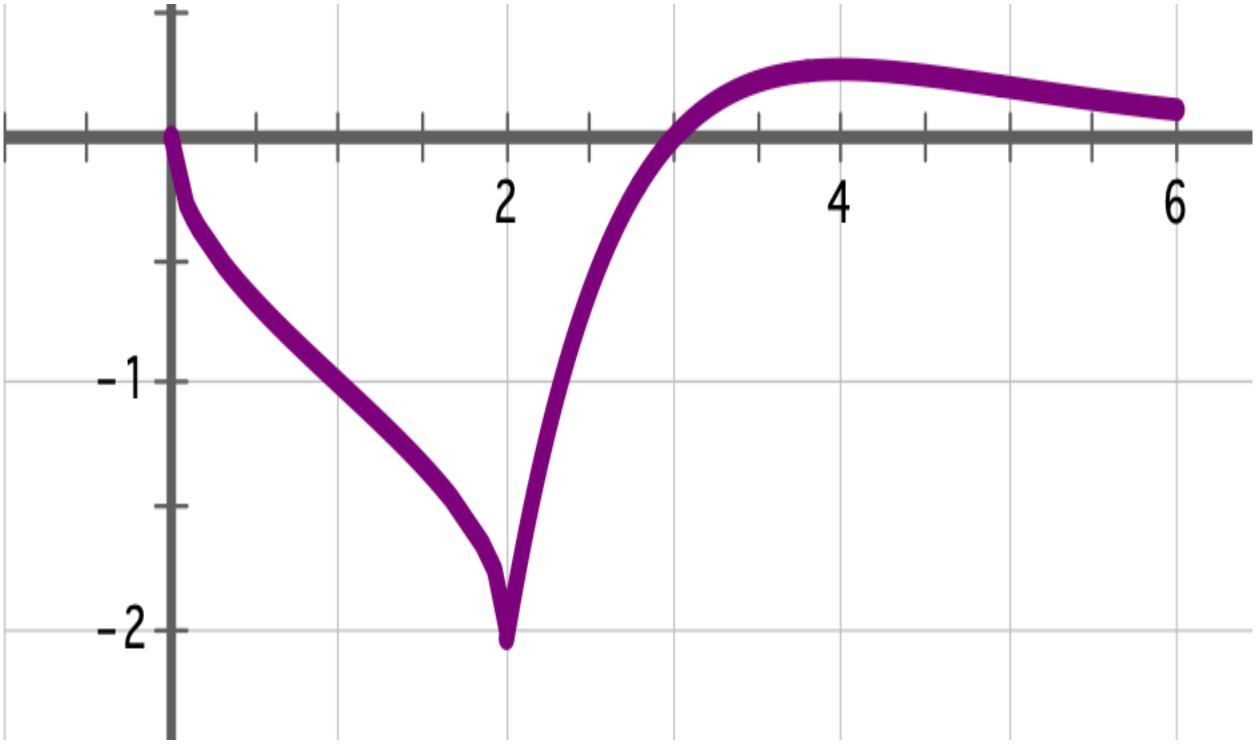
2. Let $g(x)$ be a continuous function on $x \in [0,\ 6]$ where the graph of $g'(x)$ is the function shown below.



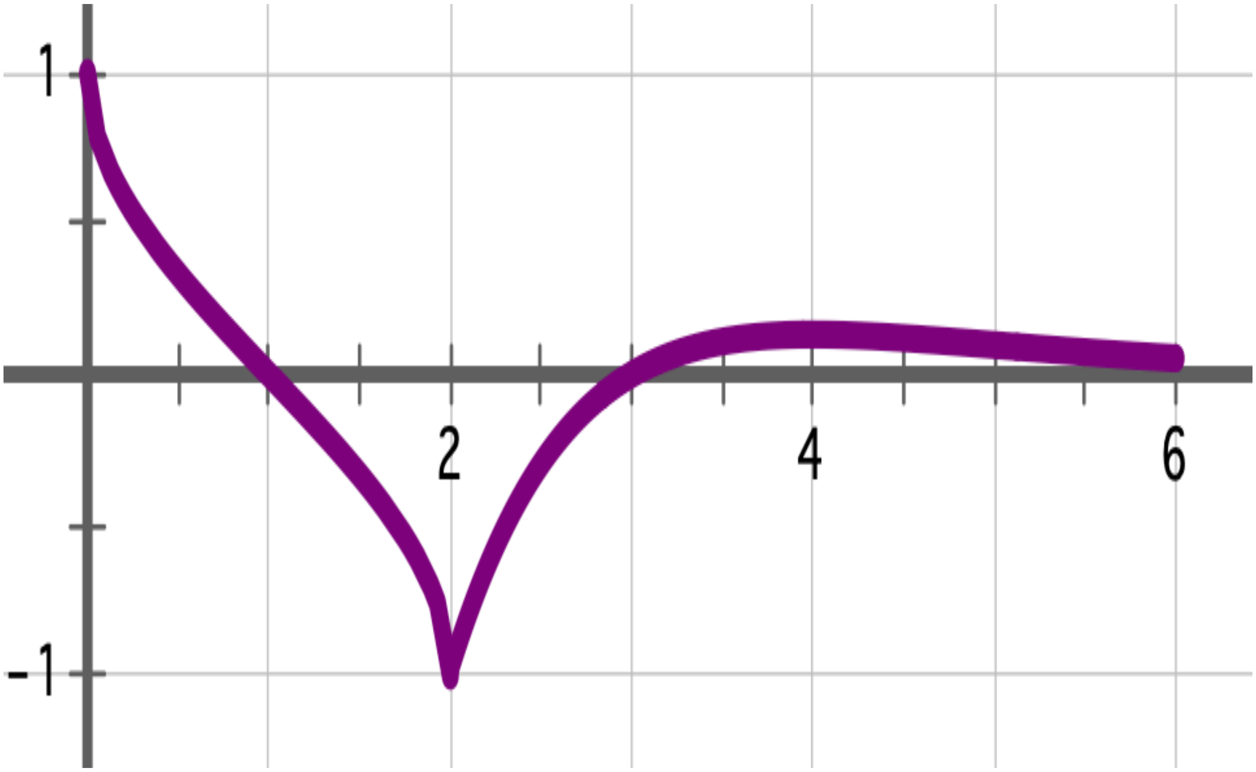
a) Identify the x-value(s) of the relative maximums of $y = g(x)$ on the interval $x \in [0,\ 6]$. Justify your answer.

b) Identify the x-value(s) of the relative minimums $y = g(x)$ on the interval $x \in [0,\ 6]$. Justify your answer.

c) Where are the points of inflection on $y = g(x)$? Justify your answer.

---

3. Let $g(x)$ be a continuous function on $x \in [0,\ \ 6]$ where the graph of $g'(x)$ is the function shown below.



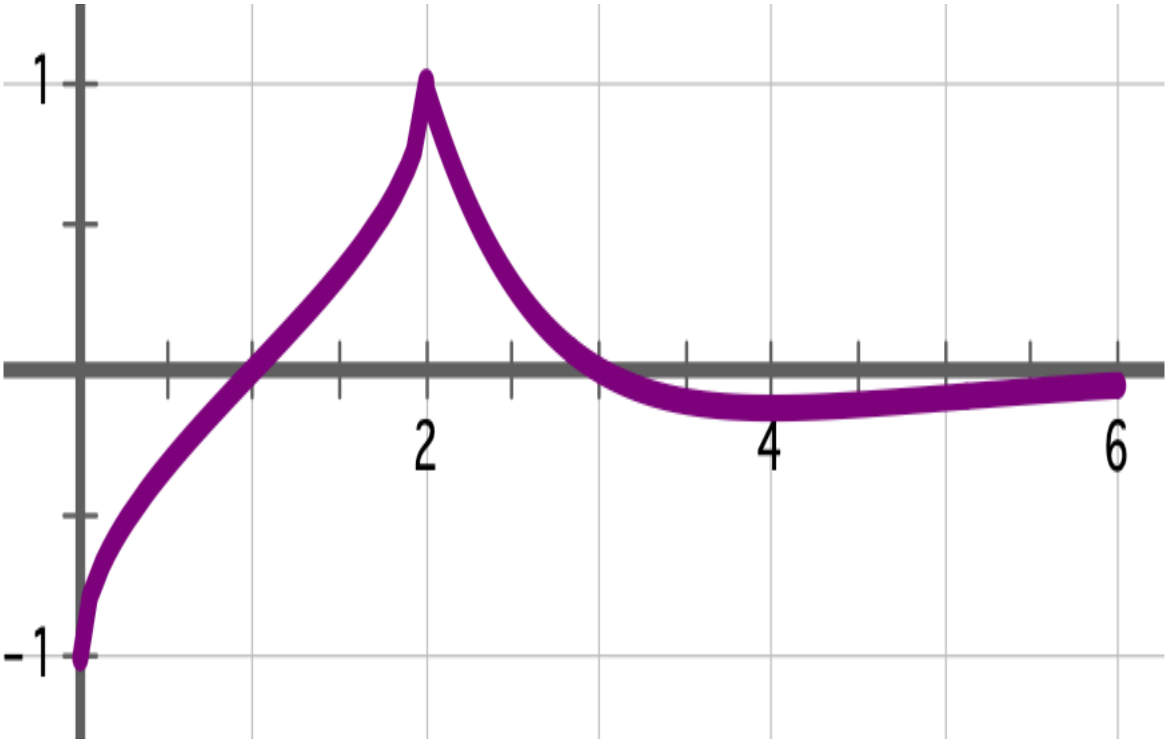
a) Identify the x-value(s) of the relative maximums of $y = g(x)$ on the interval $x \in [0,\ \ 6]$. Justify your answer.

b) Identify the x-value(s) of the relative minimums $y = g(x)$ on the interval $x \in [0,\ \ 6]$. Justify your answer.

c) Where are the points of inflection on $y = g(x)$? Justify your answer.

---

4. Let $g(x)$ be a continuous function on $x \in [0,\ 6]$ where the graph of $g'(x)$ is the function shown below.



a) Identify the x-value(s) of the relative maximums of $y = g(x)$ on the interval $x \in [0,\ 6]$. Justify your answer.

b) Identify the x-value(s) of the relative minimums $y = g(x)$ on the interval $x \in [0,\ 6]$. Justify your answer.

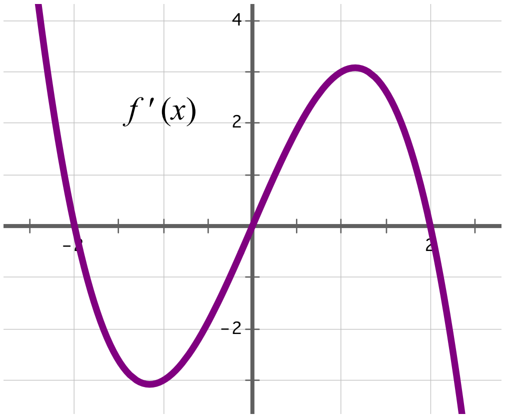
c) Where are the points of inflection on $y = g(x)$? Justify your answer.

For Problem 5-11, the curve below is $y = f'(x)$. Show the sign patterns of the first and second derivatives. Then find:

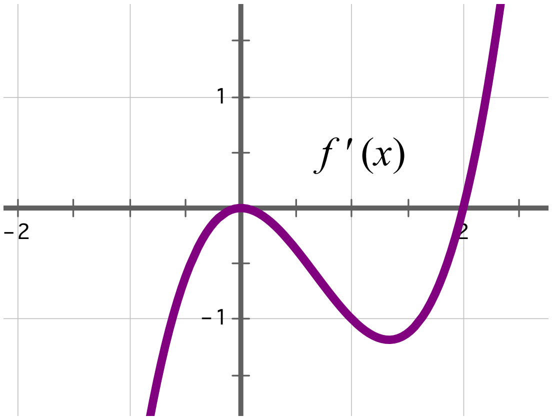
- a) the critical values for the maximum and minimum points,
- b) x-coordinates of the POIs,
- c) intervals of increasing,
- d) the intervals where $y = f(x)$ is concave down, and
- e) sketch a possible curve for $f(x)$ with y-intercept $(0,0)$.

Be careful to pay attention to the x- and y-scales

5.



6.



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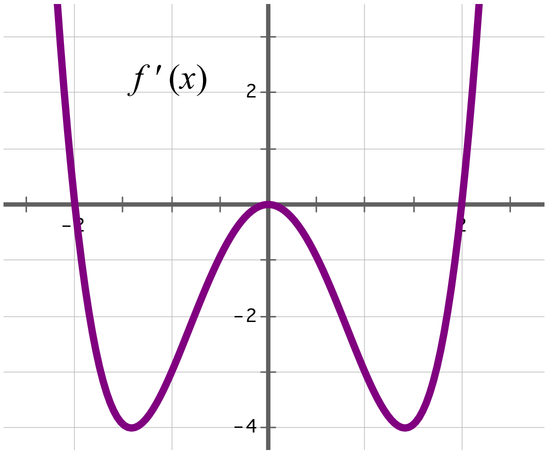
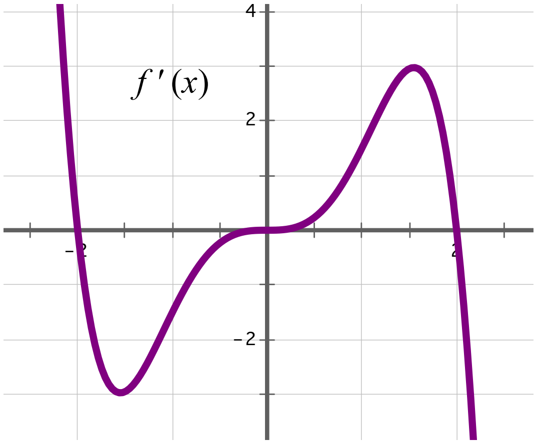
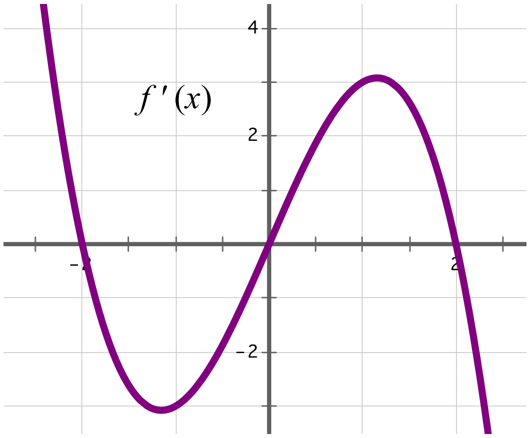


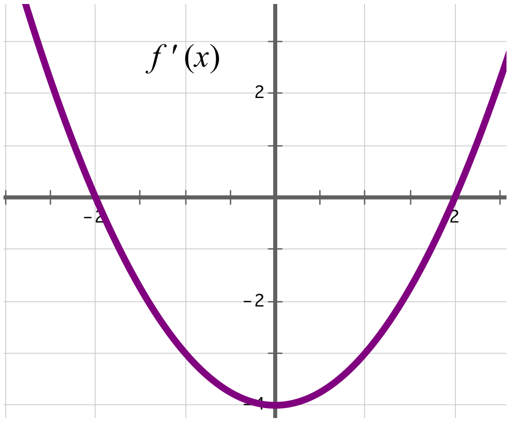
8.

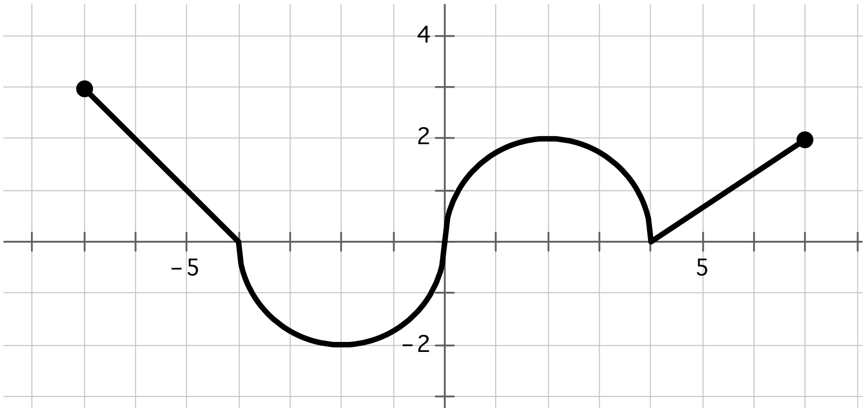


9.



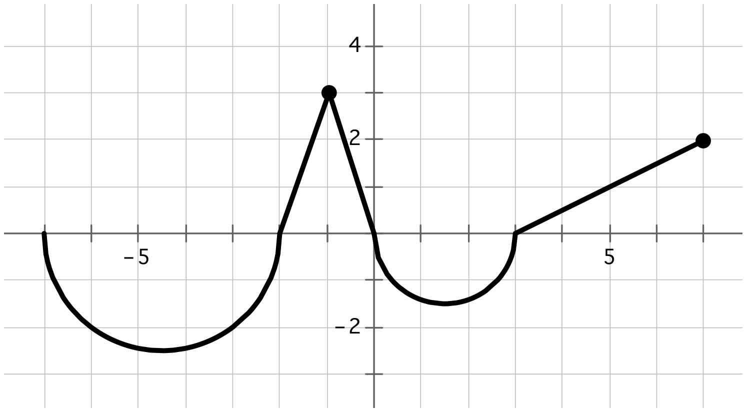
10.





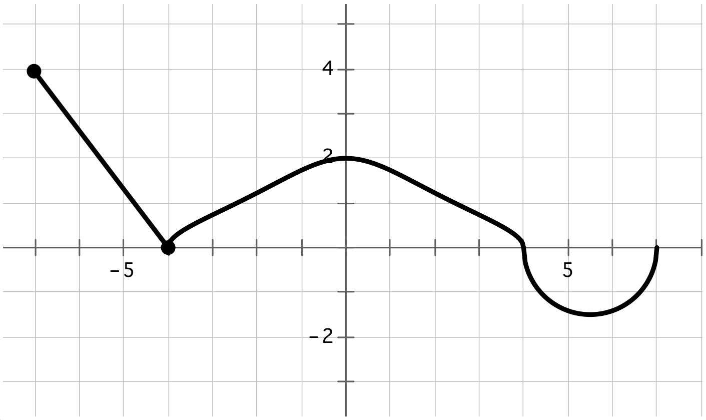
The graph of $f'(x)$ on $x \in [-7,\ 7]$ is shown above.

11a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.
11b) Where are the points of inflection of $y = f(x)$? Justify your answer.
11c) Find the equation of the line tangent to $y = f(x)$ at $x = 2$ if $f(2) = -3$.
11d) On what interval(s) is $f(x)$ increasing and concave up? Explain your reasoning.



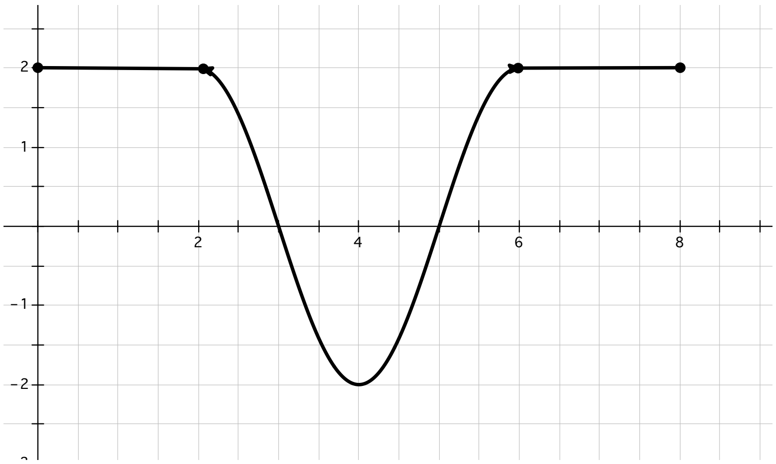
The graph of $f'(x)$ on $x \in [-7,\ 7]$ is shown above.

12a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.
12b) Where are the points of inflection of $y = f(x)$? Justify your answer.
12c) Find the equation of the line tangent to $y = f(x)$ at $x = -1$ if $f(-1) = 1$.
12d) On what interval(s) is $f(x)$ increasing and concave down? Explain your reasoning.



The graph of $f'(x)$ on $x \in [-7,\ 7]$ is shown above.

13a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.
13b) Where are the points of inflection of $y = f(x)$? Justify your answer.
13c) Find the equation of the line tangent to $y = f(x)$ at $x = 0$ if $f(0) = -5$.
13d) On what interval(s) is $f(x)$ decreasing and concave up? Explain your reasoning.



The graph of $f'(x)$ on $x \in [0,\ 8]$ is shown above.

14a) Where are the relative maximums and relative minimums of $y = f(x)$? Justify your answer.

14b) Where are the points of inflection of $y = f(x)$? Justify your answer.

14c) Find the equation of the line tangent to $y = f(x)$ at $x = 0$ if $f(0) = 0$.

14d) On what interval(s) is $f(x)$ decreasing and concave down? Explain your reasoning.

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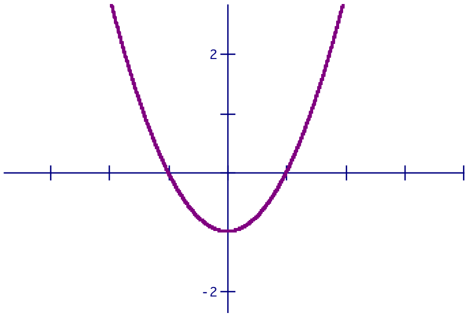
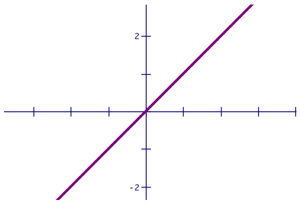
15. AP Handout: BC2003#4, AB2000#3, AB1996 # 1

4.7 Multiple Choice Homework

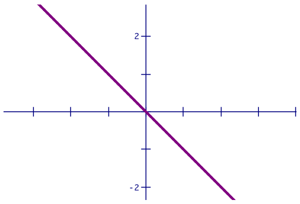


1. Suppose the derivative of *f* has the graph shown above. Which of the following could be the graph of *f*?

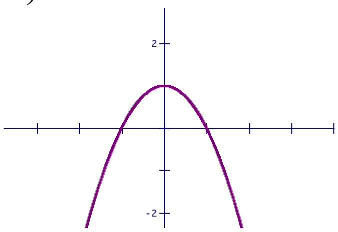
A)



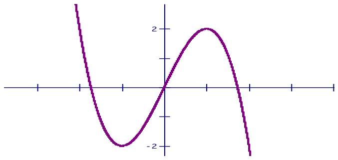
B)



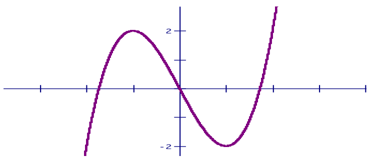
C)



D)

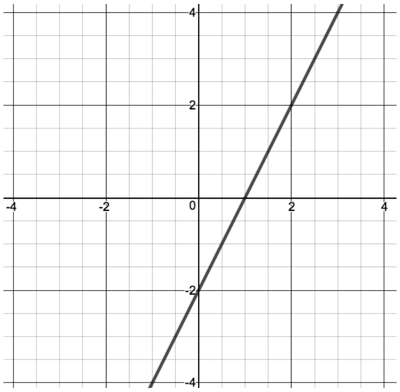


E)



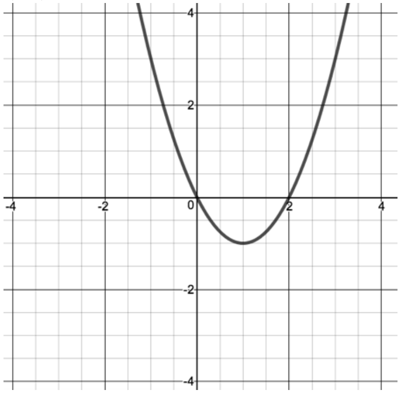
The correct answer is E

2. $g(x)$ is a twice-differentiable function whose second derivative $g''(x)$ is graphed below.

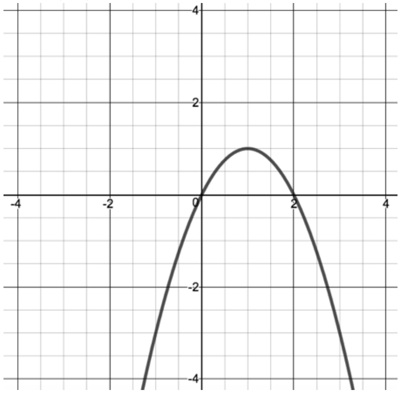


Which of the following could be the graph of $g(x)$?

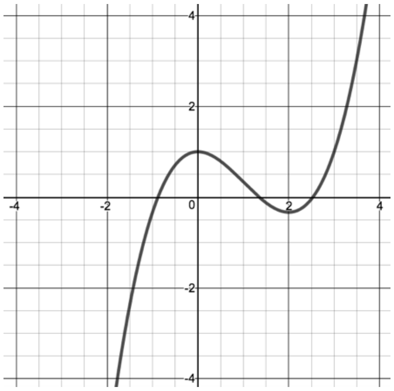
(A)



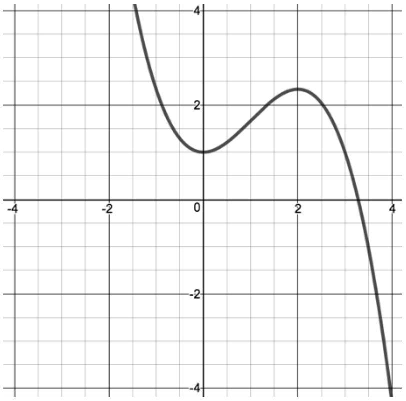
(B)



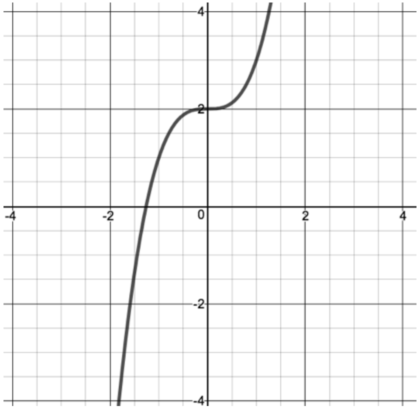
(C)



(D)



(E)



The correct answer is C

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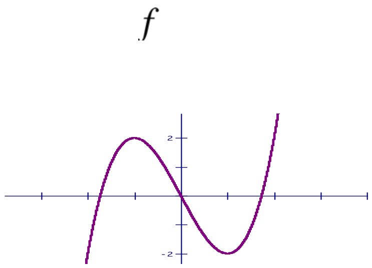
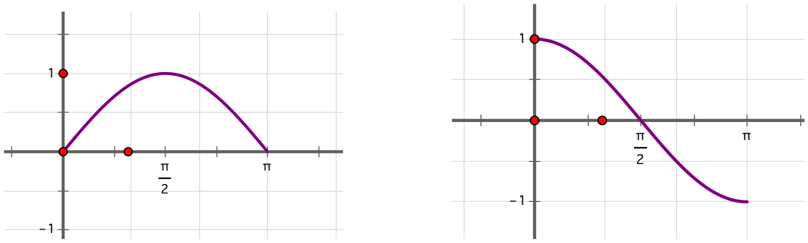
3. Which of the following pairs of graphs represent the graph of *f* and its derivative f'?

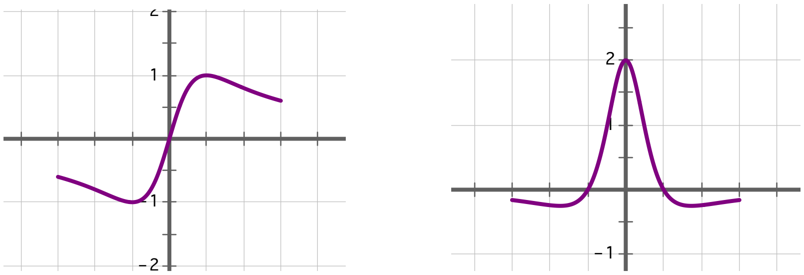
I.



II.



III.



a) I only

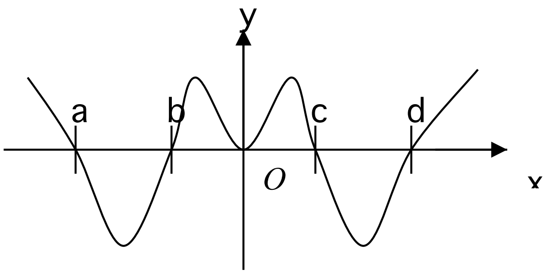
b) II only

c) III only

d) I and II

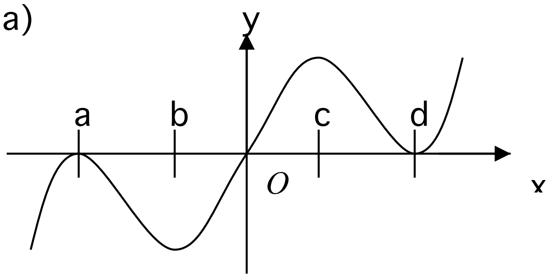
e) II and III only

The correct answer is A

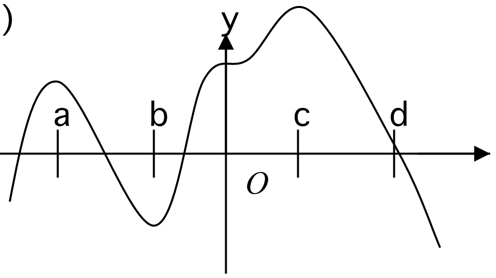


4. Let $f(x) = \int_{a}^{x} h(t)\,dt$, where h has the graph shown above. Which of the following could be the graph of f?

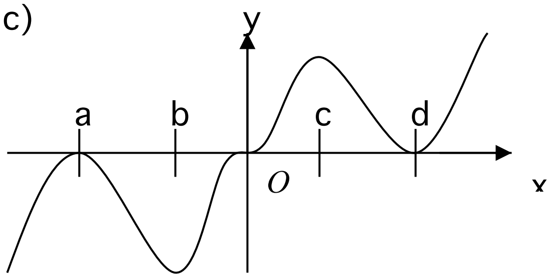
a)



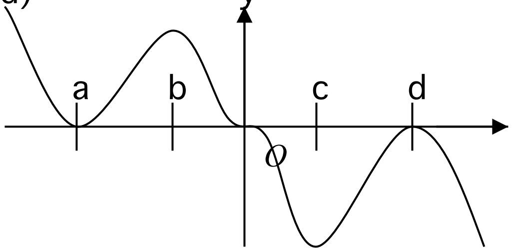
b)



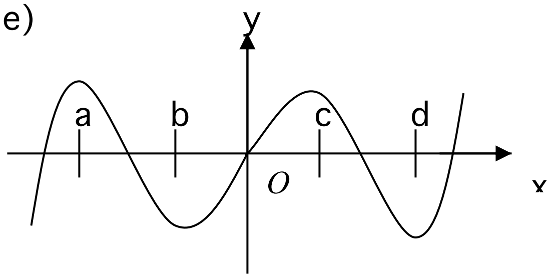
c)



d)



e)



The correct answer is C

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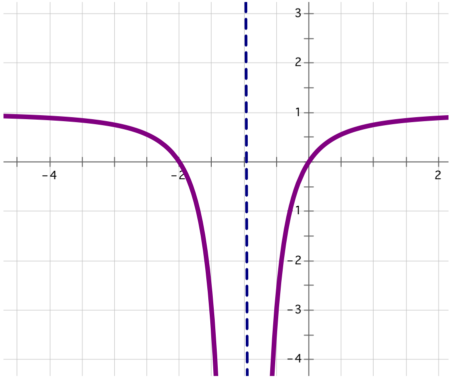
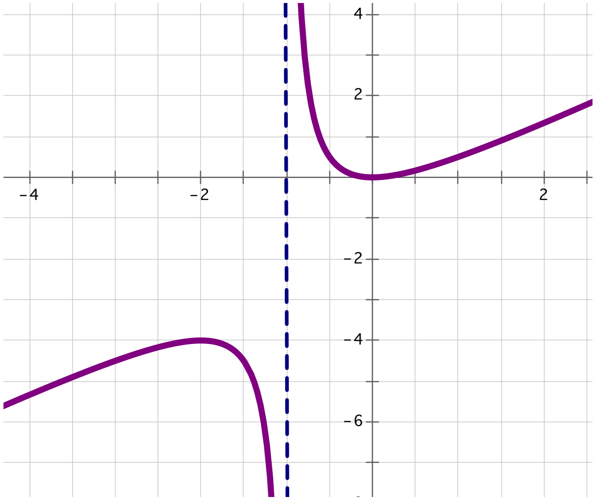
5. This is the graph of $f'(x)$.

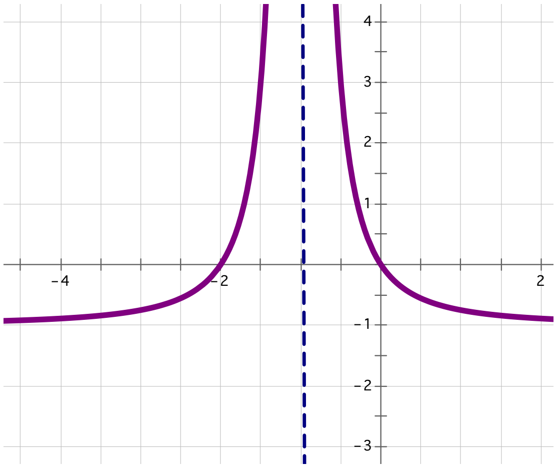


Which of the following is the graph of $f(x)$?

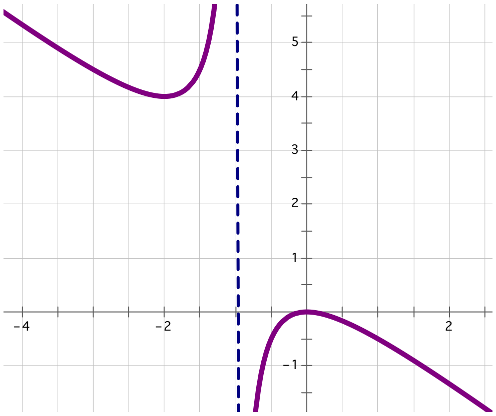
a)



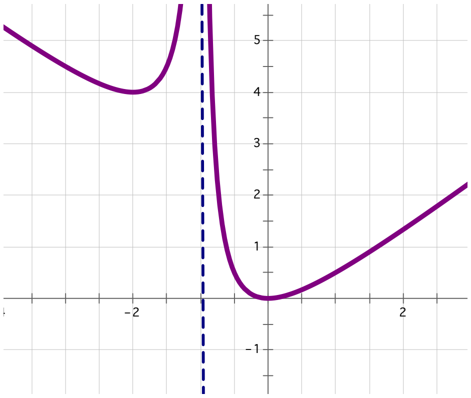
b)

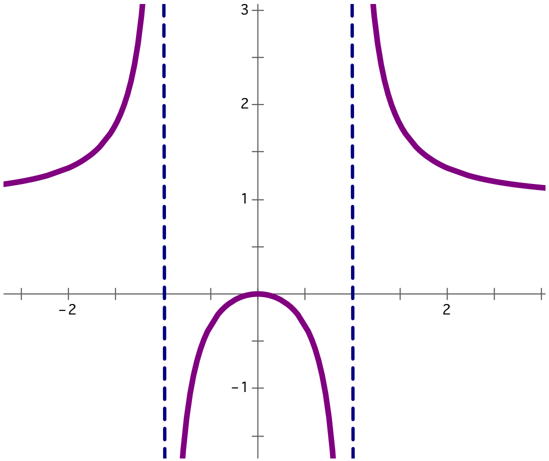


c)



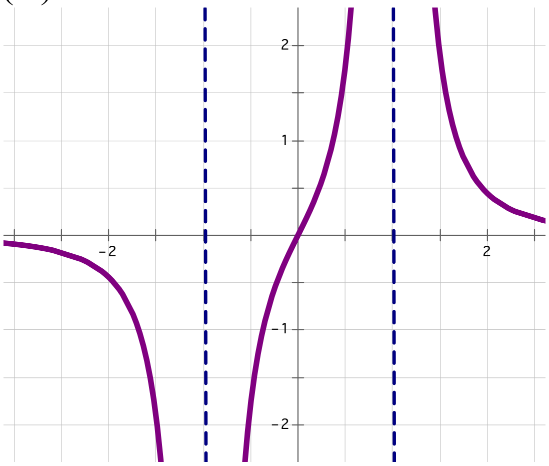
d)



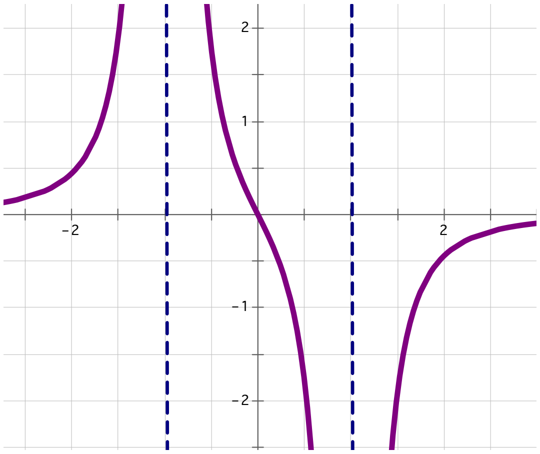


6. The graph of $f(x)$ is shown above. Which graph below is most likely to be $f'(x)$?

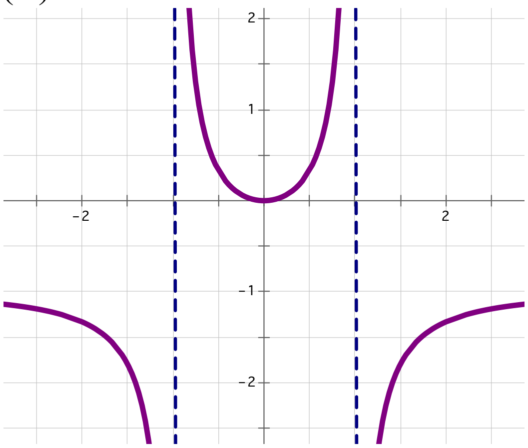
(A)



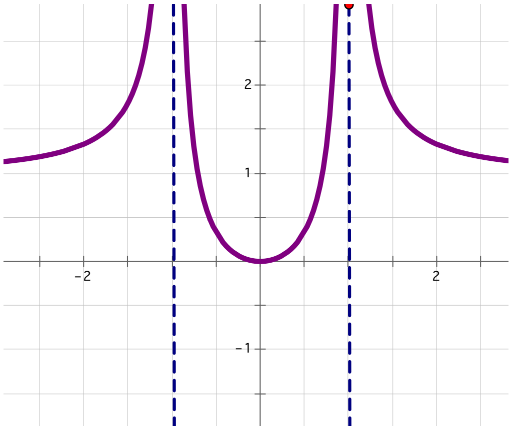
(B)



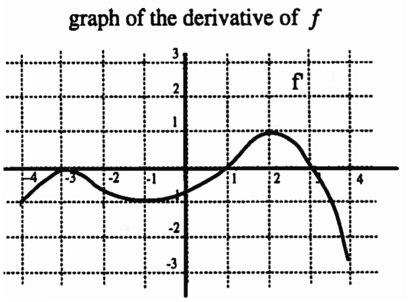
(C)



(D)



7. The figure shows the graph of f', the *derivative* of a function *f*. The domain of is the interval $-4 \le x \le 4$. Which of the following are true about the graph of *f*?

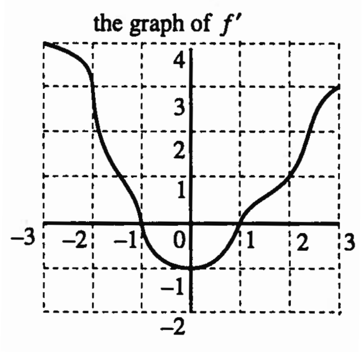


I. At the points where $x = -3$ and $x = 2$ there are horizontal tangents.
II. At the point where $x = 1$ there is a relative minimum point.
III. At the point where $x = -3$ there is an inflection point.

a) None b) II only c) III only d) II and III only e) I, II, and III

---

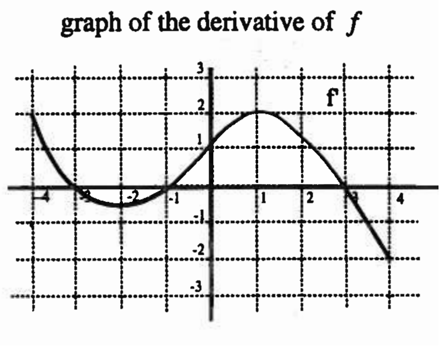
8. The graph of f', the derivative of a function *f*, is shown below. Which of the following statements are true about the function *f?*



I. *f* is increasing on the interval $(-2,-1)$.
II. *f* has an inflection point at $x = 0$.
III. *f* is concave up on the interval $(-1,0)$.

a) I only
b) II only
c) III only
d) I and II only
e) II and III only

9. The graph of the **derivative** of a function is *f* shown below. Which of the following is true about the function *f*?

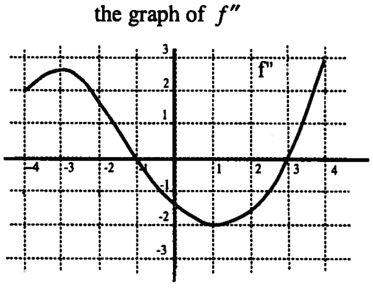

graph of the derivative of f

I. *f* is increasing on the interval $(-2,1)$.
II. *f* is continuous at $x=0$.
III. *f* has an inflection point at $x=-2$.

a) I only
b) II only
c) III only
d) II and III only
e) I, II, and III

---

10. The graph of the **second derivative** of a function *f* is shown below.


the graph of f''

Which of the following is true?

I. The graph of *f* has an inflection point at $x=-1$.
II. The graph of *f* is concave down on the interval $(-1,3)$.
III. The graph of the derivative function f' is increasing at $x=1$.

a) I only b) II only c) III only

d) I and II only e) I, II, and III

---

11. The derivative of the function g is $g'(x) = \cos(\sin x)$. At the point where $x = 0$ the graph of g

I. is increasing
II. is concave down
III. attains a relative maximum point

(a) I only (b) II only (c) III only

(d) I and III only (e) I, II, and III

4.8 Graphical Analysis II

An important part of calculus is being able to read information from graphs. Earlier, we looked at graphs of first derivatives and answered questions concerning the traits of the "original" function. This section will answer the same questions, even with similar graphs – but this time the "original" function in will be defined as an integral in the form of the FTC. **This is a major AP topic.**

Objectives

- Use the graph of a function to answer questions concerning maximums, minimums, and intervals of increasing and decreasing
- Use the graph of a function to answer questions concerning points of inflection and intervals of concavity.
- Use the graph of a function to answer questions concerning the area under a curve.

As we saw before, the graphs of a function and its derivatives are related. We summarized that relationship in a table that we will now expand.

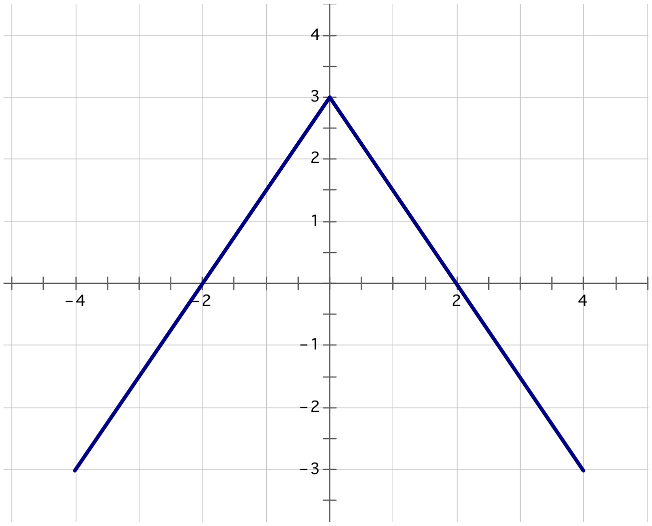
G(x)= $\int_0^x F(t)\,dt$	The *y*-values of G(x)	Increasing EXTREME Decreasing	Concave up POI Concave down
F(x)	Area under *F(x)*	Positive ZERO Negative	Increasing EXTREME Decreasing
F'(x)			Positive ZERO Negative

NB. The AP Test will always require the statement that "$g(x) = \int_0^x f(t)\,dt$ means that $g'(x) = f(x)$," and the arguments about extremes, concavity and increasing and decreasing be made from the point of view of the derivatives.

REMEMBER:

Position: $f(b) = f(a) + \int_a^b f'(t)\,dt$

Ex 1 The graph of *f* is shown below. *f* consists of three line segments.



Let $g(x) = \int_0^x f(t)dt$.

a) Find $g(-2)$, $g'(-2)$, and $g''(-2)$.
b) For what values on $x \in (-4,\ 4)$ is $g(x)$ increasing? Explain your reasoning.
c) For what values on $x \in (-4,\ 4)$ is $g(x)$ concave up? Explain your reasoning.
d) Sketch a graph of $g(x)$ on $x \in [-4,\ 4]$.

a) $g(-2) = -3$. $g(-2)$ would equal the area of the triangle from -2 to 0, but the integral goes from 0 to -2, therefore the integral equals the negative of the area. $g'(-2) = f(-2) = 0$.

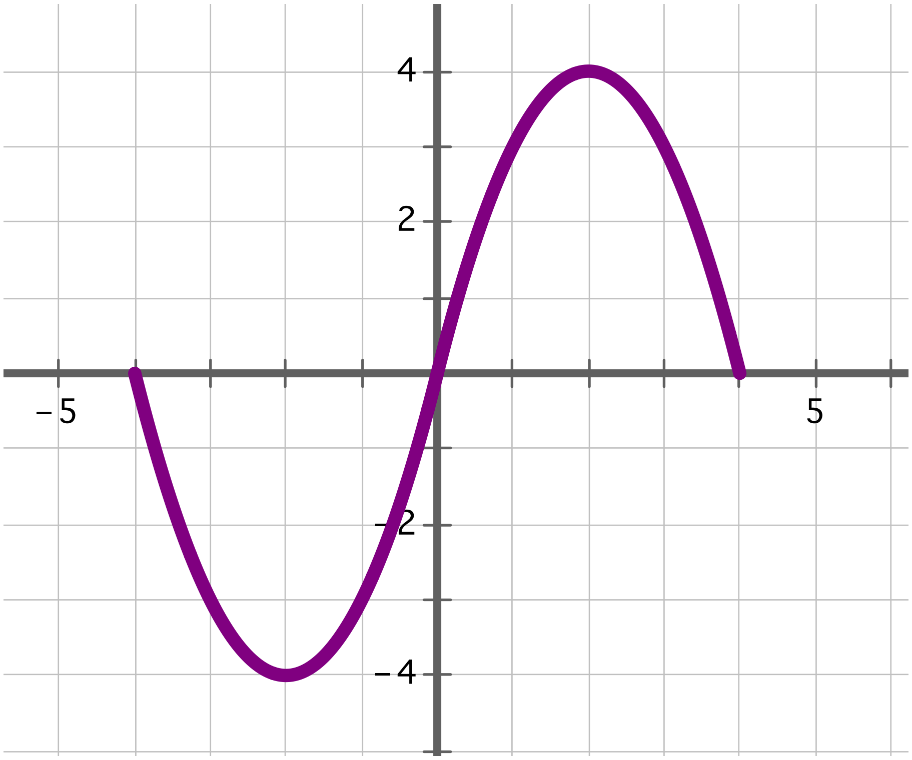
$g''(-2) = f'(-2) = \frac{3}{2}$. This is the slope of $f(x)$ at x=-2.

b) $g(x)$is increasing when $g'(x)$ is positive. $g'(x) = f(x)$. Therefore, $g(x)$is increasing on $x \in [-2,\ 2]$

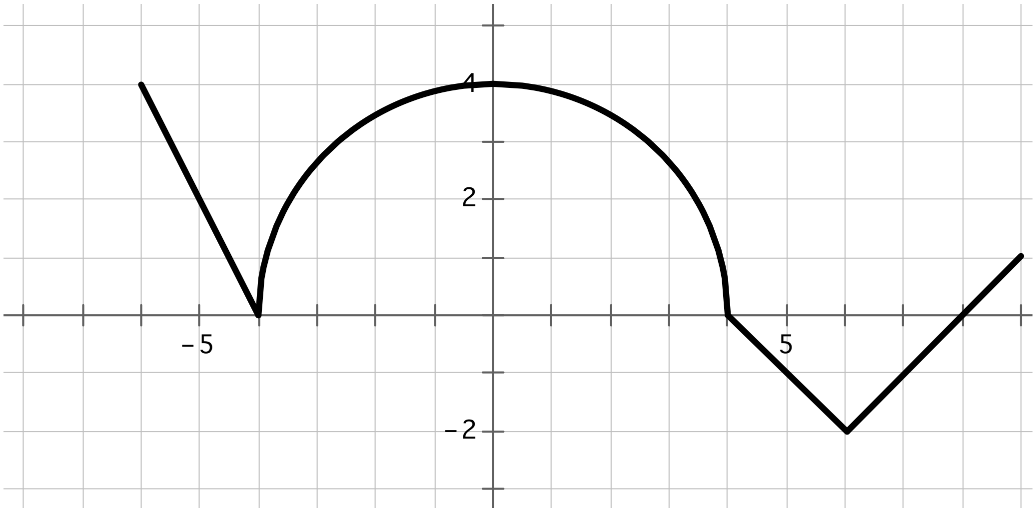
c) $g(x)$ concave up when $f(x)$ is increasing. Therefore, $g(x)$is concave up on $x \in (-4,\ 0)$

d) Because of the equal triangle areas, $g(-4) = g(0) = g(4) = 0$
$g(x)$ has a relative max at $x = 2$ and a relative min at $x = -2$.
$g(x)$is increasing on $x \in [-2,\ 2]$ and decreasing on $x \in [-4,\ -2] \cup [2,\ 4]$
$g(x)$ is concave up on $x \in (-4,\ 0)$ and concave down on $x \in (0,\ 4)$.

The graph looks something like this:



Ex 2 Let $h(x) = 1 + \int_{-4}^{x} g(t)dt$. The graph of g is shown below.



(a) Find $h(4)$, $h'(4)$, and $h''(4)$.
(b) What is the instantaneous rate of change of *h(x)* at $x = 1$?
(c) Find the absolute minimum value of *h* on $[-6,\ 8]$.
(d) Find the *x* – coordinate of all points of inflection of *h* on $(-6,\ 8)$.
(e) Let $F(x) = \int_{x}^{0} g(t)dt$. When is *F* increasing?

(a) $h(4) = 1 + \int_{-4}^{4} g(t)dt$ would equal the area from 0 to 4. This is a semi-circle of radius 4. So, $h(4) = 1 + \int_{-4}^{4} g(t)dt = 1 + 8\pi$.

$$h'(4) = \frac{d}{dx}\left[1 + \int_{0}^{x} g(t)dt\right] = g(4) = 0$$

$$h''(0) = g'(0) = dne$$

(b) The instantaneous rate of change of *h(x)* at $x = 1$ would be $g(1)$. To find $g(1)$, we would need the equation of the circle. Since I t has its center at the origin and radius = 4, the equation is $x^2 + y^2 = 16$ or $y = \sqrt{16 - x^2}$.

$$g(1) = \sqrt{16 - 1^2} = \sqrt{15}$$

(c) Since $g(x) = h'(x)$, the critical values of h would be the zeros and endpoints of g, namely, x = -6, -4, 4, 8, 9. The graph of *g(x)* yields a sign pattern:

$$\begin{array}{cccccccccc} g & & + & 0 & + & 0 & - & 0 & + & \\ & \longleftarrow & & & & & & & & \longrightarrow \\ x & -6 & & -4 & & 2 & & 8 & & 9 \end{array}$$

From this, we can see that x = -4 and 2 cannot be minimums. x = 2 is a maximum because the sign of $g(x) = h'(x)$ switches from + to - and x = -4 is not an extreme because the sign does not change. Now all we need are *h(-6)* and *h(-9)*. Whichever is smaller is the absolute minimum.

$$h(-6) = \int_0^{-6} g(t)dt = -\int_{-6}^{0} g(t)dt = -(4 + 4\pi)$$

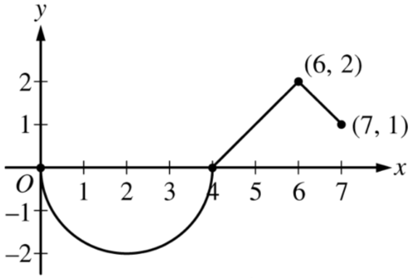
$$h(9) = \int_0^{9} g(t)dt = 4\pi - 4$$

h(-6) is the smaller value so the absolute minimum is $-(4 + 4\pi)$

(d) $g'(x) = h''(x)$ so the points of inflection on h would be where the slope of g changes sign, namely at x = -4, 0 and 6.

(e) $F(x) = \int_x^{0} g(t)dt \rightarrow F'(x) = -g(x)$. So, *F* is increasing when g is negative, namely on $x \in [4,\ 8]$.

Ex 3 AP Calculus AB 2022 #3



Graph of f'

3. Let f be a differentiable function with $f(4) = 3$. On the interval $0 \le x \le 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

(a) Find $f(0)$ and $f(5)$.

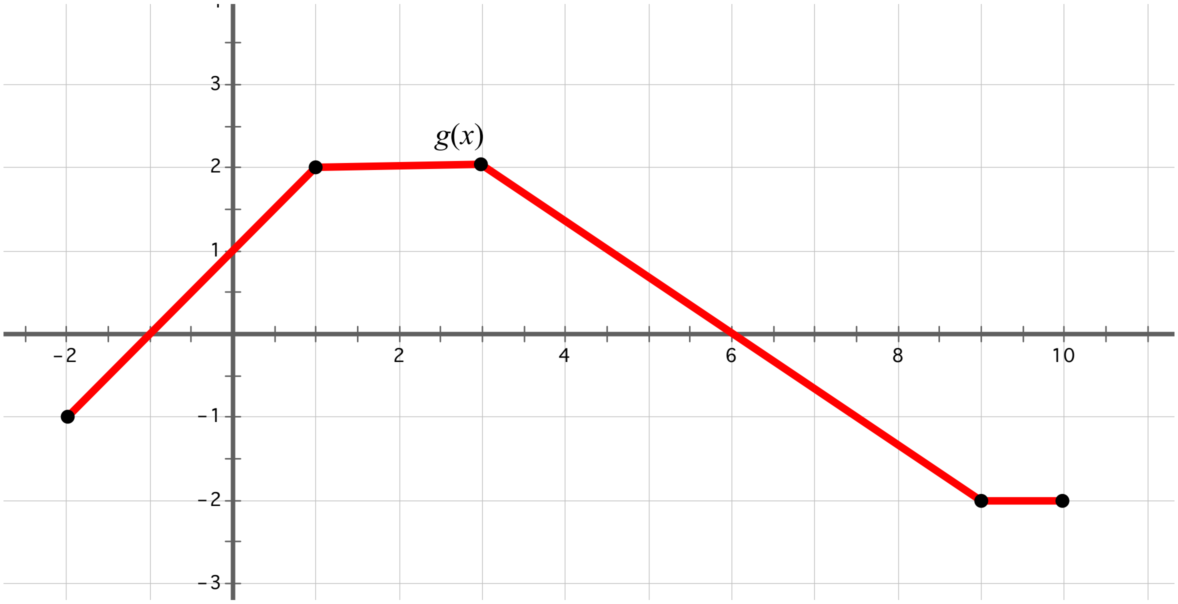
(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \le x \le 7$? Show the analysis that leads to your answer.

(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \le x \le 7$. Justify your answer.

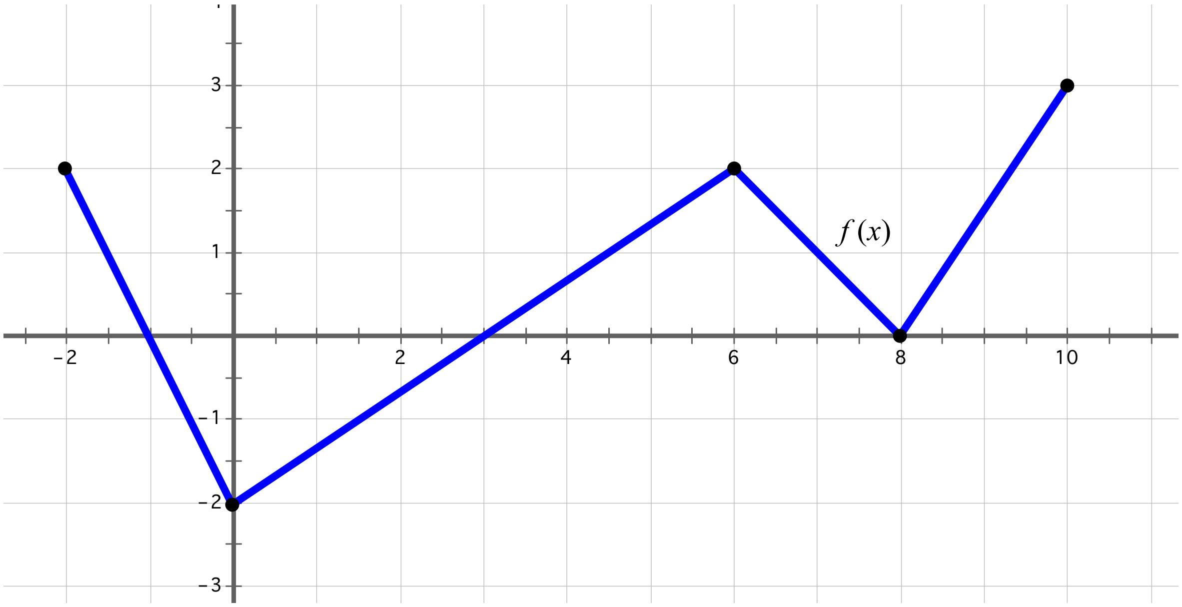
4.8 Free Response Homework

1. Let $f(x)=\int_3^x g(t)dt$ for $-2\leq t\leq 10$, where the graph of the differentiable function *f* is shown below.



a) Find $f(0)$, $f'(0)$, and $f''(0)$.
b) Find the average rate of change of $f(x)$ on $-2\leq t\leq 10$?
c) At what *x*-values is $f(x)$ decreasing and concave up? Justify your answer.
d) Find the *x*-coordinate of the absolute minimum of $f(x)$. Justify your

2. Let $g(x)=\int_{-2}^{x} f(t)dt$ for $-2 \leq t \leq 10$, where the graph of the differentiable function f is shown below.



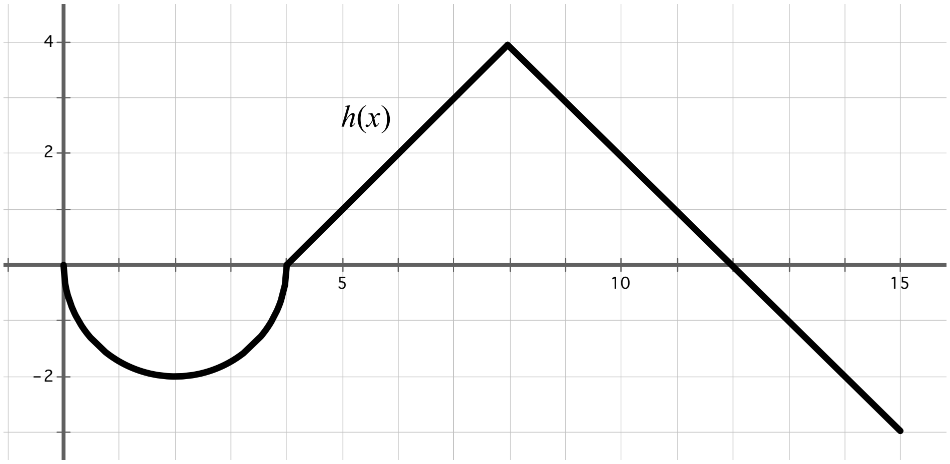
a) Find $g(2)$, $g'(2)$, and $g''(2)$.

b) Find the average rate of change of $g(x)$ on $-2 \leq t \leq 10$?

c) At what x-values is $g(x)$ increasing and concave down? Justify your answer.

d) Find the x-coordinate of the absolute minimum of $g(x)$. Justify your

3. Let $k(x) = \int_0^x h(t)dt$ for $0 \le t \le 15$, where the graph of the differentiable function *f* is shown below.



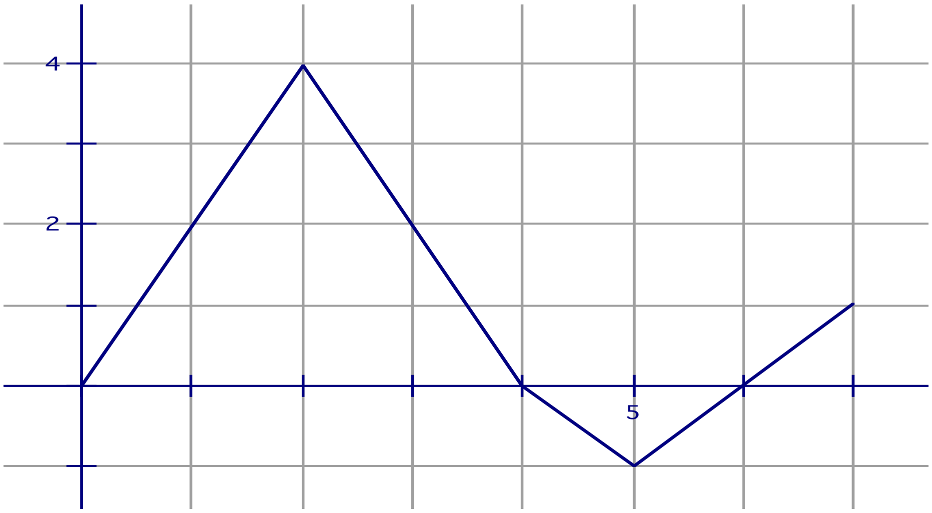
a) Find $k(4)$, $k'(4)$, and $k''(4)$.

b) Find the average rate of change of $k(x)$ on $0 \le t \le 15$?

c) At what *x*-values is $k(x)$ decreasing and concave down? Justify your answer.

d) Find the *x*-coordinate of the absolute minimum of $k(x)$. Justify your

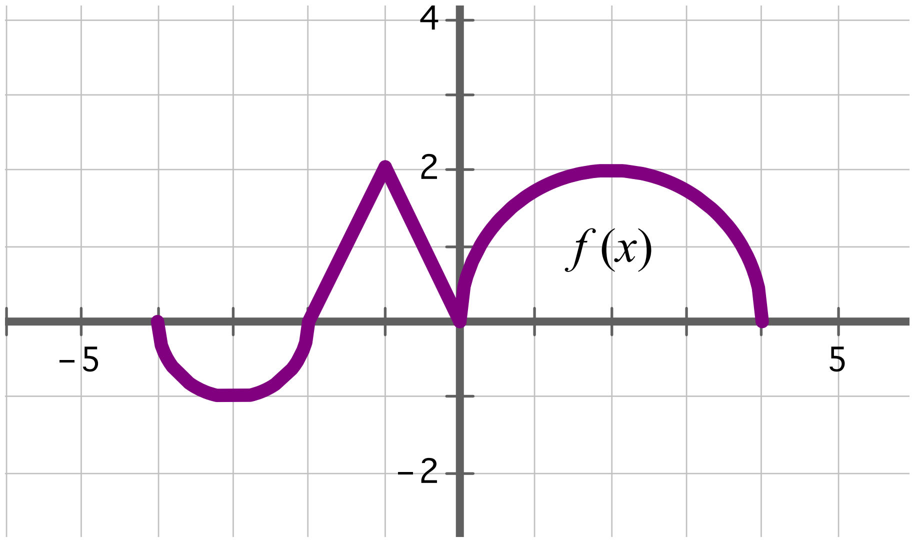
4. Let $g(x) = \int_{2}^{x} f(t)dt$ for $0 \leq t \leq 7$, where the graph of the differentiable function *f* is shown below.



a) Find $g(3)$, $g'(3)$, and $g''(3)$.
b) Find the average rate of change of $g(x)$ on $0 \leq c \leq 3$?
c) For how many values of *c*, where $0 < c < 3$, is $g'(c)$ equal to the average rate found in part (b)? Justify your answer.

d) Find the *x*-coordinate of each point of inflection of the graph of *g* on the interval $0 < t < 7$. Justify your answer.

5. Let $h(x)=\int_0^t f(t)\,dt$ on $x\in[-4,\ 4]$. Let the graph of be comprised of two semicircles and a line segment as shown below.

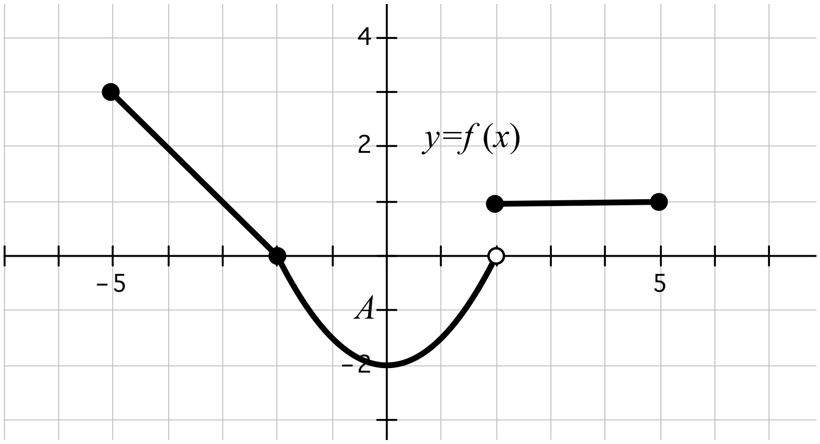


(a) Find $h(2)$, $h'(2)$, and $h''(2)$.

(b) Find the average rate of change of $h(x)$ on $x\in[0,\ 2]$.

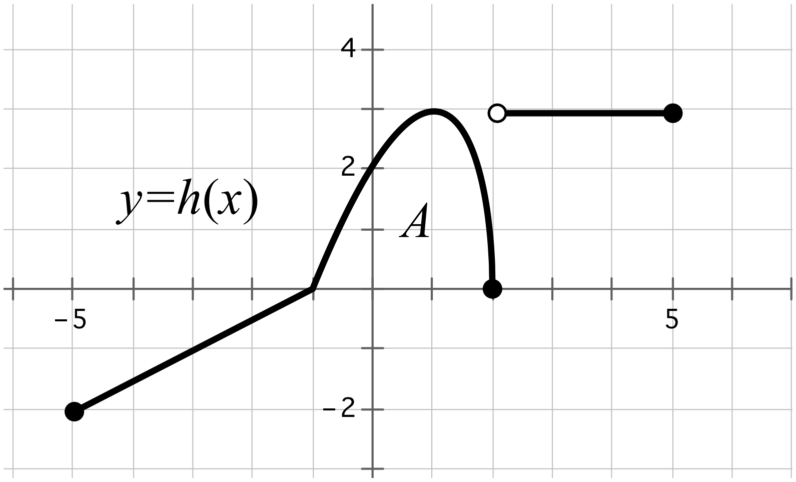
(c) At what *x*-values is $h(x)$ decreasing and concave up? Justify your answer.

(d) What is the absolute maximum value of $h(x)$on the interval $x\in[-4,\ 4]$?



6. The graph above, $f(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a parabola. Let $g(x) = 4 + \int_{-2}^{x} f(t)\, dt$, and let Area A equal 5.7.

(a) Find $g(-5)$ and $g'(-5)$.
(b) Find $g(2)$.
(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute maximum? Explain.
(d) On what interval(s) is $g(x)$ both decreasing and concave down? Explain why.



7. The graph above, $h(t)$ on $-5 \le x \le 5$, is comprised of two line segments and the graph of a radial function. Let $g(x) = 4 + \int_{-1}^{x} h(t)\, dt$. The area A is 5.

(a) Find $g(-5)$ and $g'(-5)$.
(b) Find $g(2)$.
(c) At what x-value, on $-5 \le x \le 5$, does $g(x)$ have the absolute minimum? Explain.
(d) On what interval(s) is $g(x)$ both increasing and concave down? Explain why.

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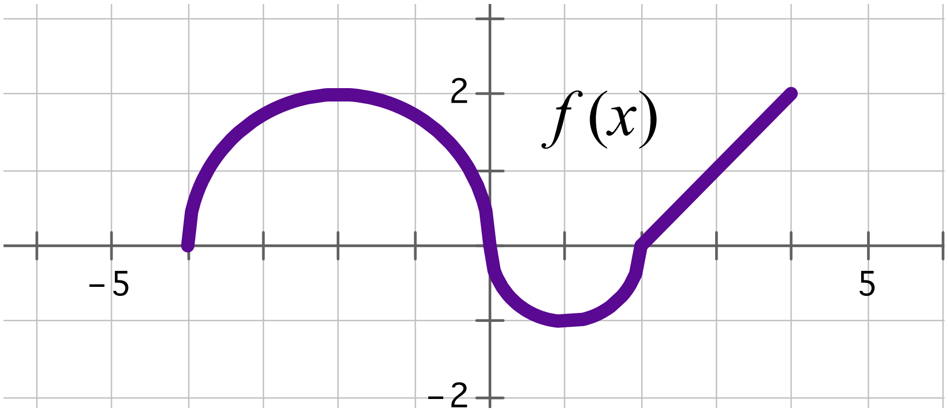
8. Let $h(x) = \int_0^t f(t)\, dt$ on $x \in [-4,\ 4]$. Let the graph of be comprised of two semicircles and a line segment as shown below.



(a) Find $h(2)$, $h'(2)$, and $h''(2)$.

(b) Find the average rate of change of $h(x)$ on $x \in [0,\ 2]$.

(c) At what x-values is $h(x)$ decreasing and concave up? Justify your answer.

(d) What is the absolute maximum value of $h(x)$on the interval $x \in [-4,\ 4]$?

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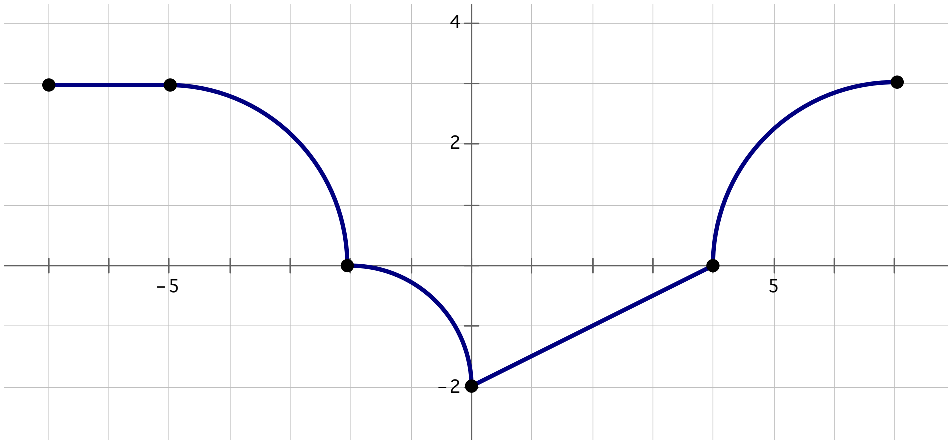
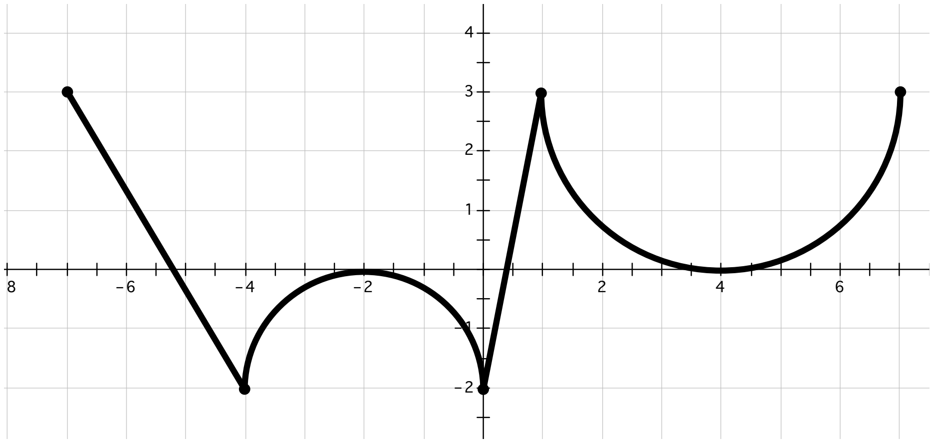
9. Let $g(x)=\int_{-2}^{x} f(t)dt$ for $-7 \le t \le 7$, where the graph of the differentiable function *f* is shown below.



a) Find $g(4)$, $g'(2)$, and $g''(4)$.
b) Find the average rate of change of $g(x)$ on $-7 \le t \le 0$?
c) At what *x*-values is $g(x)$ decreasing and concave up? Justify your answer.
d) Find the *x*-coordinate of the absolute minimum of $g(x)$. Justify your answer.

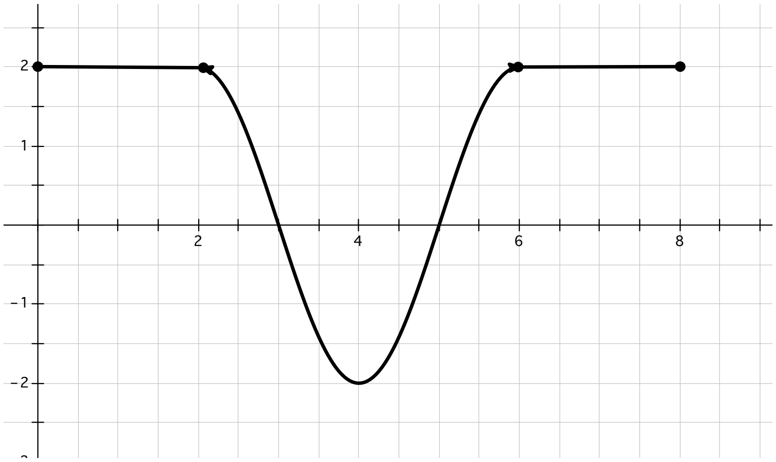
10. Let $g(x)=\int_{-4}^{x} f(t)dt$ for $-7 \le t \le 7$, where the graph of the differentiable function *f* is shown below.



a) Find $g(0)$, $g'(0)$, and $g''(0)$.
b) Find the equation of the line tangent to $g(x)$ at $x=0$.
c) At what *x*-values is $g(x)$ decreasing and concave up? Justify your answer.

d) Find the x-coordinate of the absolute maximum of $g(x)$. Justify your answer.

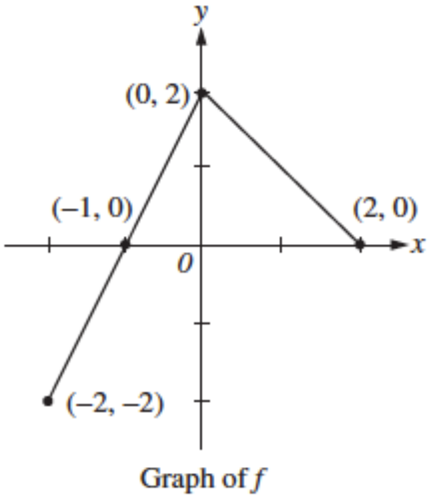
11. Let $g(x) = 10 + \int_0^x f(t)dt$ for $0 \le t \le 8$ and let $f(t)$ be the differentiable function (shown below) comprised of two horizontal line segments and one cycle of the cosine wave $y = 2\cos\left[\frac{\pi}{2}(t-2)\right]$.



a) Find $g(4)$, $g'(4)$, and $g''(4)$.
b) Find the average rate of change of $g(x)$ on $0 \le t \le 8$?
c) Find the average value of $f(x)$ on $0 \le t \le 8$.
d) Find the absolute minimum of $g(x)$. Justify your answer.

12. Handout of AP Questions: BC 2002B #4, BC 1999 #5, BC 2003 #5, BC 2002 #4, BC 2009B #5

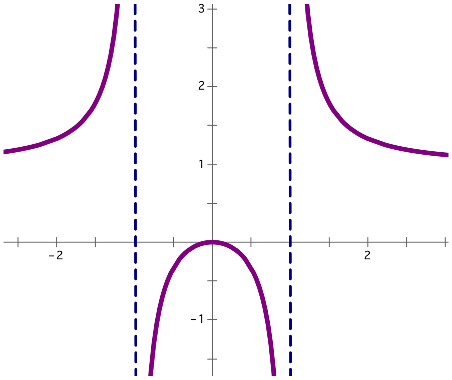
4.8 Multiple Choice Homework



Graph of *f*

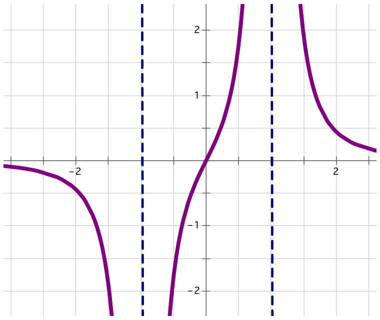
1. The graph of the function *f* shown above consists of two line segments. If g is the function defined by $g(x) = \int_0^x f(t)\,dt$, then $g(-1) =$

a) -2 b) -1 c) 0 d) 1 e) 2

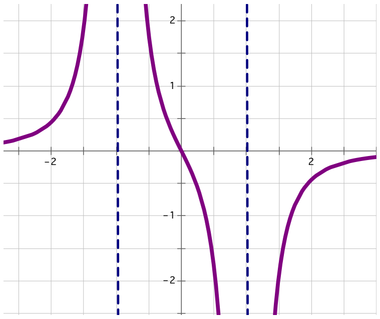


2. Let the graph above be $g(x)=\int_0^x f(t)\,dt$. Which of the following could be the graph of f ?

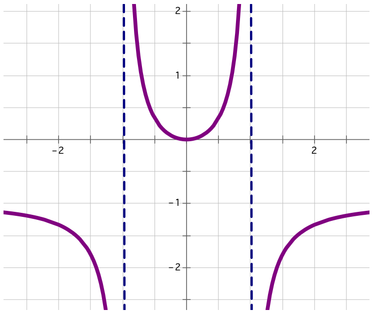
a)



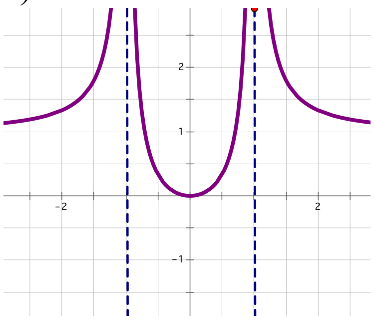
b)



c)



d)

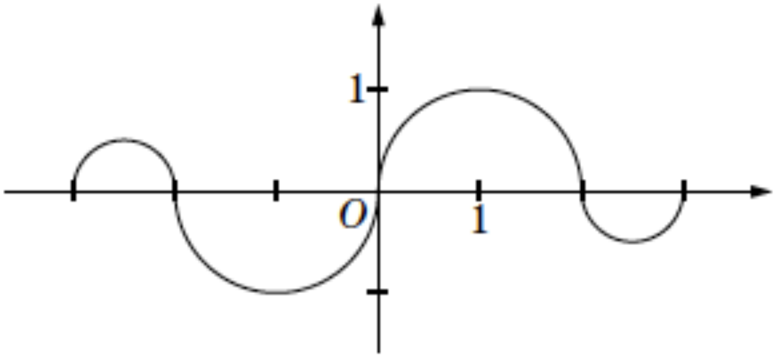


3. Let g be the function given by $g(x) = \int_1^x 100(t^2 - 3t + 2)\, dt$. Which of the following statements about g must be true?

I. g is increasing on $x \in (1, 2)$.
II. g is increasing on $x \in (2, 3)$.
III. $g(3) < 0$

a) I only b) II only c) III only

d) II and III only e) I, II, and III



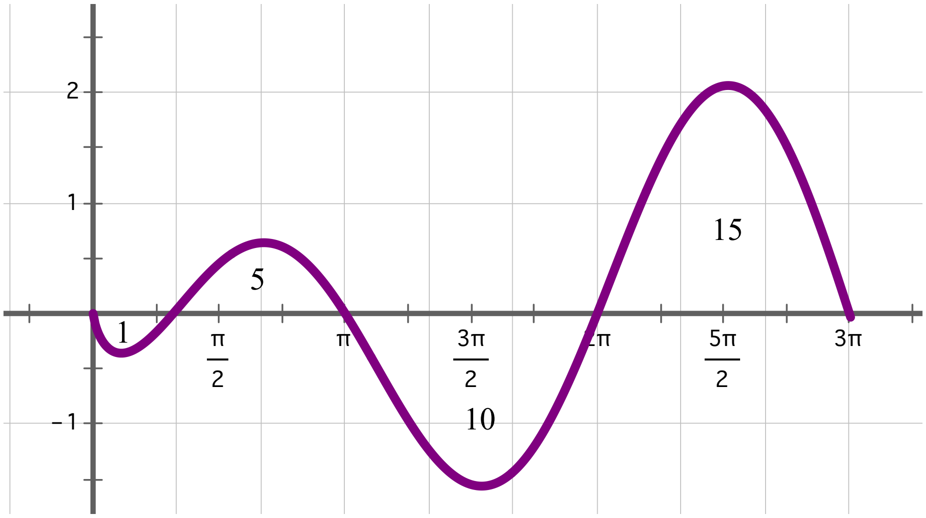
Graph of f

4. The graph of the function f above consists of four semicircles. If $g(x) = \int_0^x f(t)\, dt$, where is $g(x)$ nonnegative?

a) $x \in [-3, 3]$ only
b) $x \in [-3, -2] \cup [0, 2]$ only
c) $x \in [0, 3]$ only
d) $x \in [0, 2]$ only
e) $x \in [-3, -2] \cup [0, 3]$ only

---

5. The graph of f', the derivative of f, is shown below, for $0 \le x \le 3\pi$.



The area of the region between the graph of f' and the x-axis are 1, 5, 10 and 15, respectively. If $f(0) = 2$, what is the maximum value of f on the closed interval $0 \le x \le 3\pi$?

a) 9 b) 10 c) 11 d) 20 e) 31

Derivative Applications I Test

1. If $g'(x) = 3xe^{2x} + 5$, then $g(x)$ has a point of inflection at:

a) $x = -1$ b) $x = 1/2$ c) $x = -1/2$

d) $x = 1$ e) Nowhere

---

2. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that they have values given on the table below.

x	$f'(x)$	$f''(x)$	$g'(x)$	$g''(x)$
2	0	-2	8	0
4	8	0	0	3
8	0	-12	0	4

Then at $x = 8$, $g(x)$ has a:

a) Relative Maximum b) Relative Minimum

c) Point of Inflection d) Zero

e) None of these

---

3. Given the following function, with $x > 0$, on which interval is the function decreasing?

$$f(x) = \frac{x}{\ln(5x)}$$

a) $(1, 5e)$

b) $\left(0, \frac{1}{5}\right) \cup \left(\frac{1}{5}, \frac{1}{5}e\right)$

c) $\left(0, \frac{1}{5}\right)$

d) $(1, 5)$

e) $\left(1, \frac{1}{5}e\right)$

---

4. If $f'(x) = -6(x-3)^2(x-9)$ which of the following is true about $y = f(x)$?

a) f has a point of inflection at $x = 3$ and a local minimum at $x = 9$.

b) f has a local maximum at $x = 3$ and a local minimum at $x = 9$.

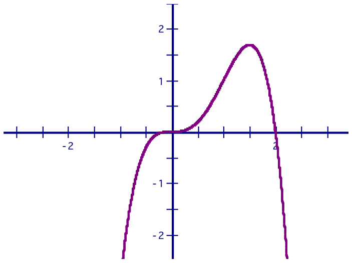
c) f has a local minimum at $x = 3$ and a local maximum at $x = 9$.

d) f has a local minimum at $x = 3$ and a point of inflection at $x = 9$.

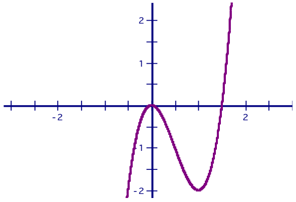
e) f has a point of inflection at $x = 3$ and a local maximum at $x = 9$.

---

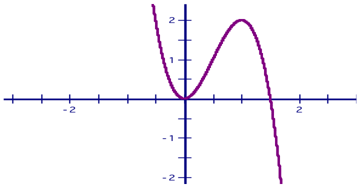
5. Suppose the function $f(x)$ has the graph shown below. Which of the following could be the graph of $f'(x)$?



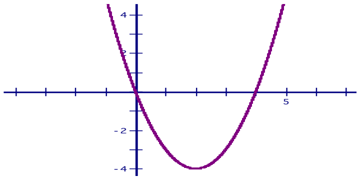
a)



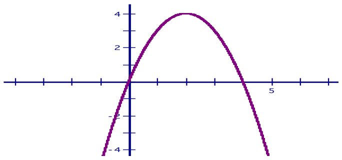
b)



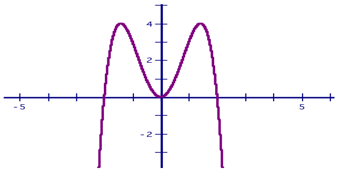
c)



d)



e)



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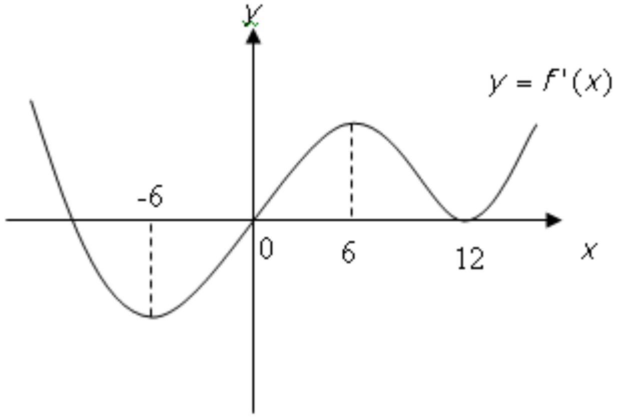
6. The graph of the derivative of *f* is shown below. Which of the following must be true?



a) *f* is concave down on [0, 12].

b) *f* is increasing on [-6, 6].

c) *f* has a local maximum at $x = 0$.

d) *f* has a local minimum at $x = -6$.

e) *f* has a point of inflection at $x = 12$.

---

7. The sum of two positive integers x and y is 150. Find the value of x that minimizes $P = x^3 - 150xy$

a) $x = 25$

b) $x = 75$

c) $x = 50$

d) $x = 125$

e) $x = 100$

---

8. The function *f* is defined as $f(x) = \dfrac{(x-4)^2}{x-7}$ for $x \neq 7$

Which of the following is **false**?

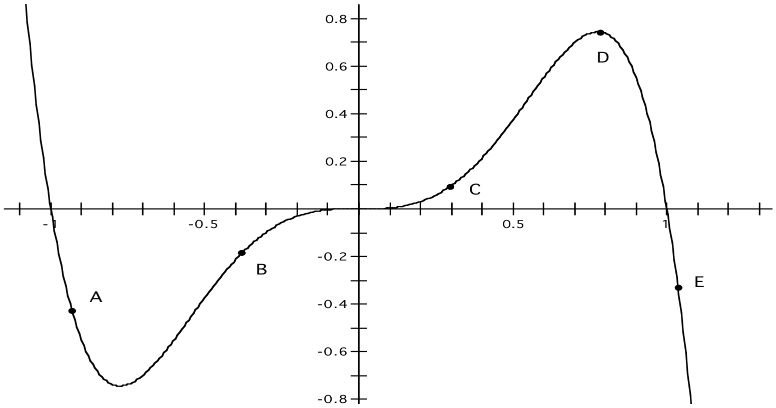
a) *f* is concave up for $x > 7$.

b) *f* is decreasing on [4, 7].

c) *f* has a local maximum at $x = 4$.

d) *f* has a horizontal asymptote at $y = 1$.

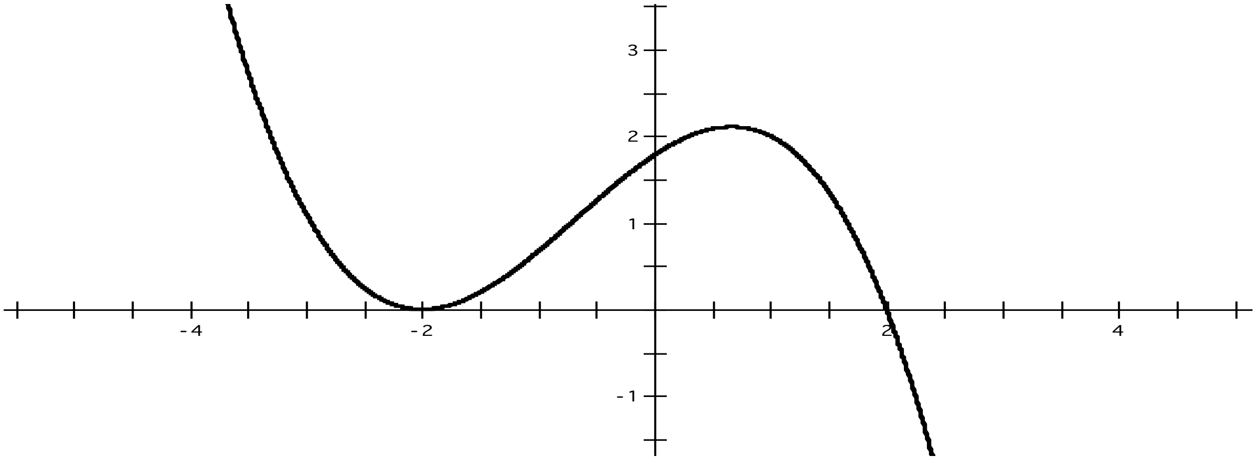
e) *f* has a vertical asymptote at $x = 7$.



9. The graph of the function $f(x)$ is shown above. At which point on the graph of $f(x)$ is $f'(x) < 0$ and $f''(x) > 0$?

a) A b) B c) C d) D e) E

10. The graph of the **second** derivative of f is shown below.



Which of the following statements are true about f?

I. The graph of f has a point of inflection at $x = -2$.
II. The graph of f is concave down on $x \in (0, 4)$
III. If $f'(0) = 0$, then f is increasing at $x = 2$.

a) I only b) II only c) III only

d) I and II only e) I, II and III

---

11. Which of the following statements is true about the function $f(x)$ if its derivative $f'(x)$is defined by $f'(x) = x(x-a)^3$ for $a > 0$.

I. The graph of $f(x)$ is increasing at $x = 2a$.
II. The graph of $f(x)$ has a local maximum at $x = 0$.
III. The graph of $f(x)$ has a point of inflection at $x = a$.

a) I, II and III b) II and III only c) I and III only

d) I and II only e) I only

---

12. The table below gives values of the derivative of a function f.

x	0.998	0.999	1.000	1.001	1.002
$f'(x)$	0.980	0.995	1.000	0.995	0.980

Based on this information, it appears that in the interval covered in the table,

a) f is increasing and concave up everywhere.

b) f is increasing and concave down everywhere.

c) f has a point of inflection.

d) f is decreasing and concave up everywhere.

e) f is decreasing and concave down everywhere.

Free Response:

1. $y = f(x)$ is a continuous function with the following information:

	$x < -1$	$x = -1$	$-1 < x < 4$	$x = 4$	$4 < x < 9$	$x = 9$	$9 < x$
$f'(x)$	Positive	DNE	Positive	DNE	Positive	0	Positive
$f''(x)$	Positive	DNE	Positive	DNE	Negative	0	Positive

a) What is/are the x-coordinates of the relative extremes of $f(x)$? Justify your answer.

b) What is/are the x-coordinates of the points of inflection of $f(x)$? Justify your answer.

c) What is happening at $x = -1$? Justify your answer.

d) Sketch the graph of $y = f(x)$

2. 2. One innovative approach to global warming is to capture carbon dioxide that is created as a byproduct of producing concrete and storing the CO_2 in the sandstone in depleted natural gas fields. At a particular site on a particular day, the CO_2 is injected into the sandstone at a rate of $I(t) = 121\sin\left[\frac{\pi}{65}t^2\right]$. The CO_2 stabilizes the ground and forces remaining natural gas upward. After two hours, the pressure has built enough that the natural gas can be extracted at a rate of $E(t) = 50 - 50\cos\left[\frac{\pi}{6}(t-2)\right]$. $I(t)$ and $E(t)$ are measured in metric tons per hour and *t* is measured in hours where $0 \le t \le 8$ for $I(t)$ and $2 \le t \le 8$ for $E(t)$.

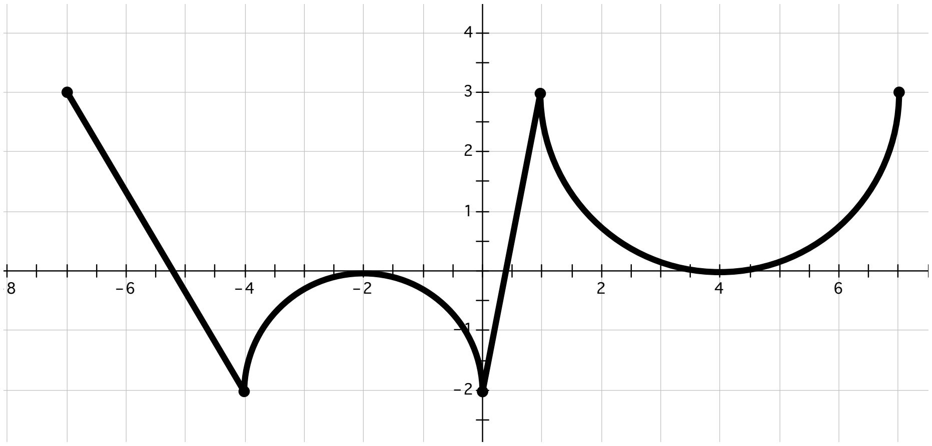
(a) How many metric tons of natural gas are extracted from the field over $2 \le t \le 8$?

(b) Is the total amount of CO_2 and natural gas decreasing or increasing at $t = 7$? Indicate the correct units.

(c) Find the time, if any, when the rate of injection of CO_2 is equal the rate of extraction of natural gas.

(d) Find the total change of gasses in the sandstone during this 8-hour day. Using the correct units, explain the result.

(e) If there were 10,000 metric tonnes of natural gas and CO_2 in the sandstone at the start of the day, what is the maximum amount of natural gas and CO_2 in the sandstone duriong this day?



3. The graph above is $f'(x)$ on $x \in [-7,\ 7]$.

a) Identify the *x*-value(s) of the relative maximums of $y = f(x)$? Justify your answer.

b) Identify the *x*-value(s) of the points of inflection of $y = f(x)$? Justify your answer.

c) On what interval(s) is *f* deceasing and concave up? Justify your answer.

Chapter 4 Answer Key

4.1 Free Response Answer Key

1. -400, 931 2. -45.063

3. .606, -.606 4. .046, -.602

5. 0, 5.151 6. 0 7. 0, 8.845 8. -.722, .466

9. 0 10. $y = -6,\ 1,\ -8,\ 3$

11. Max pt. @ $x = 0.131$; Min pt. @ $x = 2.535$

12. Max pt. @ $x = -0.869$; Min pt. @ $x = 1.535$

13. Max pt. @ $x = 1$; Min pt. @ $x = \frac{11}{3}$

14. Max pt. @ $x = 2$; Min pt. @ $x = \frac{20}{9}$

15. Max pt. @$x = \frac{1}{2}$; Neither @ $x = 0$

16. Neither @ $x = -2$; Min pt. @ $x = 0$

17. Max pt. @ $x = 3$; Min pt. @ $x = 0.179,\ \ 2.488$

18. Neither @ $x = -1$; Min pt. @ $x = 2$; Max pt. @ $x = 4$

19. Max pt. @ $x = -\sqrt{\frac{3}{2}}\ \ and\ \ 6$; Min pt. @ $x = -2\ \ and\ \ \sqrt{\frac{3}{2}}$

20. Max pt. @ $x = 0\ \ and\ \ 2$; Min pt. @ $x = -1\ \ and\ \ \sqrt{3}$

21. Max pt. @ $x = -3,\ \ 1.023$; Min pt. @ $x = -1.467$

22. Max pt. @ $x = -1$; Min pt. @ $x = 2$; Neither at $x = 0$

23. $f(0)=0 \therefore$ *abs* min, $f(1)={}^{1}/_{2} \therefore$ *abs* max, $f(2)={}^{2}/_{5} \therefore$ *rel* min

24. $f(0)=0 \therefore$ *abs* min, $f(2)=7.560 \leftarrow \therefore$ *abs* max, $f(8)=0 \therefore$ *abs* min

25. $f(0)=0 \therefore$ *abs* min, $f(1)={}^{1}/_{e} \therefore$ *abs* max, $f(2)={}^{1}/_{e^2} \therefore$ *rel* min

26. $f(1)=0 \therefore$ *abs* min, $f(e)={}^{1}/_{e} \therefore$ *abs* max, $f(3)=.366 \therefore$ *rel* min

27. $f(0)=0 \therefore$ *abs* min, $f(\ln 2)=.25 \therefore$ *abs* max, $f(1)=.233 \therefore$ *rel* min

4.1 Multiple Choice Answer Key

1. C 2. E 3. A 4. C 5. E 6. B

7. E 8. A 9. B 10. E 11. A 12. A

13. A

4.2 Free Response Answer Key

1a. $r=2$ 1b. $\$3.02$ 2. 119.687 in^2 3. $x=y=\sqrt{110}$

4. $x=1$ 5. $A=63.234$ 6. $M=C$

7. $V=854.033\ in^3$ 8. $14{,}062.5ft^2$ 9. $4000cm^3$

10. $x=\dfrac{-28}{17},\ y=\dfrac{7}{17}$ 11. $(\pm.510, .794)$ 12. 12

13. $\dfrac{4000\pi}{3\sqrt{3}} \approx 2418.418$ 14. max is all the square, min is $x=4.350$cm

4.2 Multiple Choice Answer Key

1. C 2. B 3. D 4. B 5. D 6. C

7. D

4.3 Free Response Answer Key

1. $x = .423,\ 1.577$ 2. $\Rightarrow x = {}^{1}\!/_{4},\ {}^{3}\!/_{4}$ 3. $t = -2.150$

4. The MVT does not apply.

5. $\frac{f(8)-f(1)}{8-1} \approx f'(3.2)$ 6. $\frac{f(9)-f(1)}{9-1} \approx f'(2.4) = f'(4.6) = f'(6.9)$

7. $y \in [5,\ 29]$ 8. The IVT does not apply. 9. $y \in \left[-4\sqrt{2},\ 0\right]$

10. $y \in \left[\frac{1}{\sqrt{2}},\ \frac{8}{\sqrt{14}}\right]$

11a) 0.45 °F/hr^2

b) The total temperature change between midnight and 8am is 9.9°F.

c) Yes.

d) Yes

12a) 0.032

b) Yes.

c) While there might be a time when *SFR* = 0.0056, it is not guaranteed.

13. Yes

14. See APCentral.

4.3 Multiple Choice Answer Key

1. B 2. A 3. D 4. D 5. D

6. C 7. C

4.4 Free Response Answer Key

1a) $P'(9) = 1 - 3e^{-0.2\sqrt{9}} = -.646 < 0$, therefore, the amount of chorine is decreasing.

1b) $P'(t) = 1 - 3e^{-0.2\sqrt{t}} = 0 \rightarrow t = 30.173$ days

1c) $C(t) = 55 + \int_0^{30.173} \left(1 - 3e^{-0.2\sqrt{t}}\right)\, dt = 35.104 < 40 \therefore$ yes, chlorine needs to be added.

2. 2500 people

3a. $L(t) = 3 + \int_0^t \left[P(x) - 5\right] dx$ 3b. 3374

4a. $A(t) = 50 + \int_0^t S(x) - M(x)\ dx$ 4b. Max = 111.48 in; Min = 50 in.

5a. $A(t) = 1{,}000{,}000 + \int_0^t \left[G(x) - L(x)\right] dx$ 5b. 999,987.137 yottatons

6a. $A(t) = 1200 + \int_0^t \left[95\sqrt{x}\sin^2\left(x/6\right) - 275\sin^2\left(x/3\right)\right] dx$ 6b. $t = 6.495$

7a. 13.304 *inches*__________7b. increasing

7c. $H(t) = 15 + \int_0^t \left[A(x) - B(x)\right] dx$

7d. 3.963 inches below the top of the ship.

8a) 1440 *cats*

8b) At $t = 10.3$ *months*, the rate at which healthy cats are being adopted is decreasing by 28.008 cats per month per month.

8c) $C(t) = 131 + \int_0^{12} R(t) - A(t)\,dt = 71\ cats$

d) $t =$ 8.410 months.

9a) 78.057 yds^3. 9b) 71.3 yds^3

9c) $U(t) = 2 + \int_0^t [H(x) - W(x)]\,dx$. 9d) 12.868 yds^3.

10a) $-0.3\ acres/hr^2$. b) 25.4 $acres$ c) 10.674 $acres$

d) 4.613 $acres$ e) 0

11a) 26,456 foot-acres

b) 6202.667 foot-acres.

c) $-9\ \text{ft}-\text{acres}/\text{mo}^2$. The rate of water entering the plant is decreasing at 9 foot-acres per month per month when $\text{t} = 6$months .

d) $\text{t} = 10$ months.

12a) 52 sacks.

b) approximately .0175 sacks per game per game.

c) .896 sacks per game

d) 7.597 more sacks

13a) 24,500 gallons

b) $T'(6.2) = 1864.127$. The rate at which potable water is leaving the system is increasing at a rate of 1,867.127 gallons per hour per hour at time $t = 6.2$ hours.

c) $A(t) = \int_0^t [R(x) - T(x)]dx$

d) 9,707.291 gallons

e) Yes, because $A(8) < 0$

14a) 29,887 gallons

b) 638.421 gallons per hour per hour at time $t = 5.2$ hours.

c) 24, 252 gallons

d) 6,835 gallons

e) 12,031 gallons.

15a) 26499.426 gallons

b) $P'(5.3) = -386.613$. The rate at which potable water is leaving the system is decreasing at a rate of 386.613 gallons per hour per hour at time $t = 5.3$ hours.

c) $A(t) = 4500 + \int_0^t [N(x) - P(x)]dx$

d) $t = 7.1569593$

16a) $4.9 + \int_6^{66} A(t) \approx 4.9 + 12(1.3) + 12(1.4) + 12(2.1) + 12(3.5) + 12(5.7) = 172.9$

b) US aircraft production was increasing at an approximate rate of 0.092 thousand aircraft per month per month at $t = 20$ months.

c) $D(t) = 3.7 + \int_{6}^{66} \left[X(x) - S(x) \right] dx$

d) 56.831 thousand aircraft.

4.5 Free Response Answer Key

1. CD: $x \in (-\infty, -3) \cup (0,3)$

2a. yes 2b. No

3. CU $x \in \left(-1/3,\ \infty\right)$, CD: $x \in \left(-\infty,\ -1/3\right)$, POIs: $\left(-1/3,\ -12.593\right)$

4. CU $x \in (-\infty,\ 1) \cup \left(7/3,\ \infty\right)$, CD: $x \in \left(1,\ 7/3\right)$, POIs:
$(1,\ 5), \left(7/3,\ -4.481\right)$

5. CU $x \in (-\infty,\ -1.464) \cup (0,\ 5.464)$, CD: $x \in (-1.464,\ 0) \cup (5.464,\ \infty)$,

POIs: $(-1.464,\ .953), (0,\ 0), (5.464,\ -.646)$

6. CU $x \in (-\infty,\ -2) \cup (2,\ \infty)$, CD: $x \in (-2,\ 2)$, POIs: None

7. CD: $x \in (-\infty,\ -2) \cup (0,\ 2) \cup (2,\ \infty)$, POIs: None

8. CU $x \in \left(-2\sqrt{2},\ 0\right)$, CD: $x \in \left(0,\ 2\sqrt{2}\right)$, POIs: $(0,\ 0)$

9. CU $x \in (\pi,\ 2\pi)$, CD: $x \in (0,\ \pi)$, POIs: $\left(\pi,\ \dfrac{\pi}{2}\right)$

10. CU $x \in (2,\ \infty)$, CD: $x \in (-\infty,\ 2)$, POIs: $(2,\ .271)$

11. CD $x \in \left(-1/\sqrt{2}, 1/\sqrt{2}\right)$, CU: $x \in \left(-\infty, -1/\sqrt{2}\right) \cup \left(1/\sqrt{2}, \infty\right)$,

POIs: $\left(-1/\sqrt{2}, .607\right), \left(1/\sqrt{2}, .607\right)$

12. CU $x \in (-3,0) \cup (3,\infty)$, CD: $x \in (-\infty,0) \cup (0,\infty)$, POIs: $(0, 0)$

13. CU $x \in (-\infty,0) \cup (0,\infty)$, No POIs

14. $x = \dfrac{11}{3}$ is at a max; $x = 1$ is at a min.

15. $x = 0$ is at a min; $x = 2$ is neither.

16. $x = 2$ is at a max; $x = -2$ is at a min.

17. $x = 3$ is at a min; $x = 0$ is neither.

18. $x = \ln 2$ is at the max.

19. $f\left(e^{-1/2}\right)$ is a minimum

20. $h(-2)$ is the minimum; $h(2)$ is the maximum

21. $f\left(-\dfrac{1}{\sqrt{2}}\right)$ is the minimum; $f\left(\dfrac{1}{\sqrt{2}}\right)$ is the maximum

4.5 Multiple Choice Answer Key

1. A 2. B 3. C 4. D 5. A 6. B

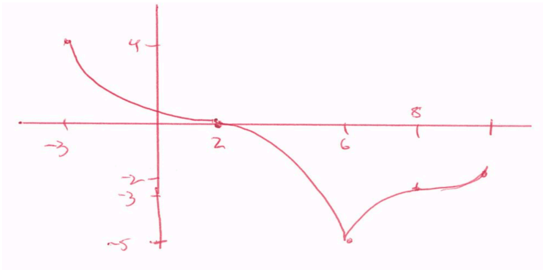
7. E 8. D 9. C 10. D 11. D 12. E

13. C

4.6 Free Response Answer Key

4.6 Free Response Answer Key

1.



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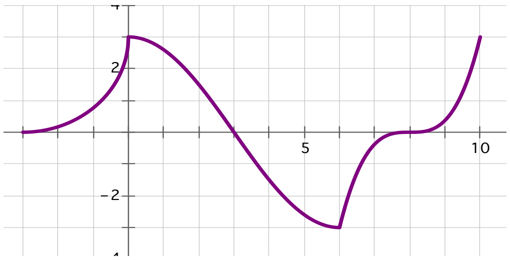

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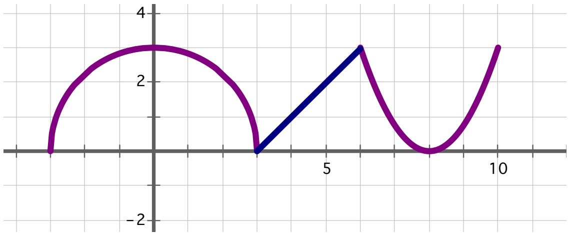

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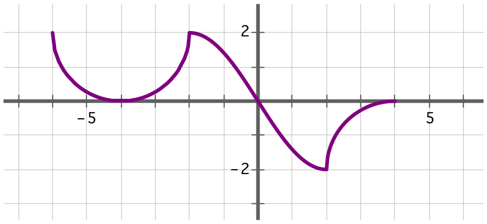

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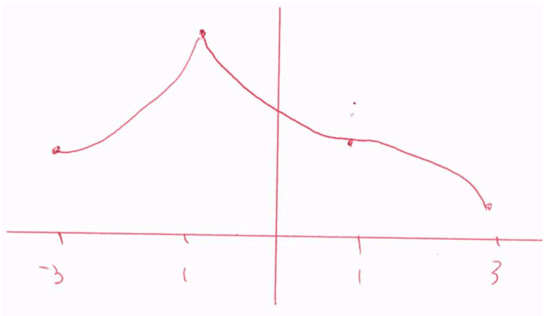

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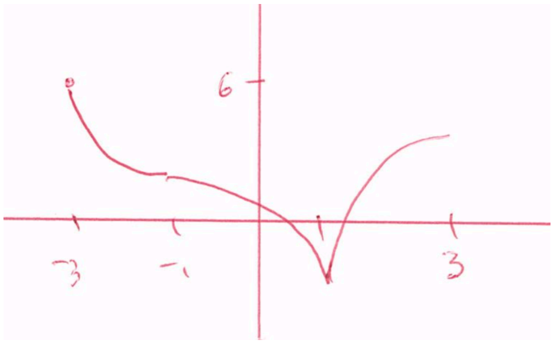

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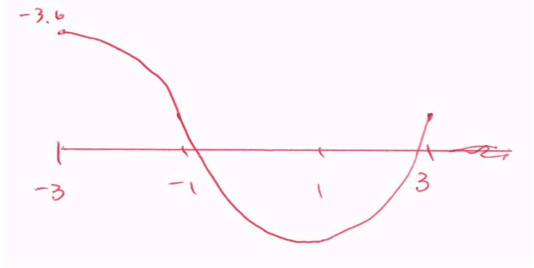
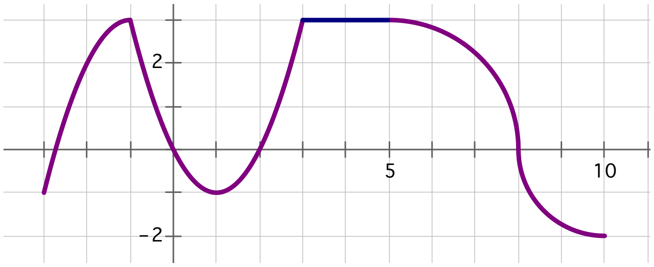


8.



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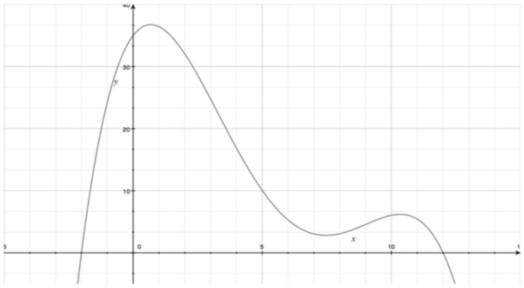

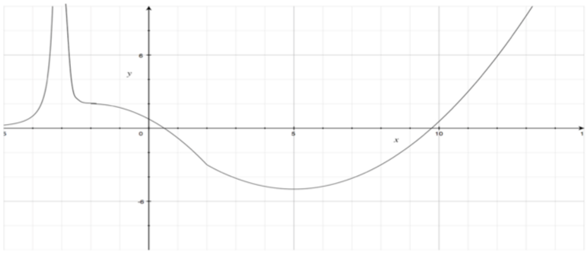

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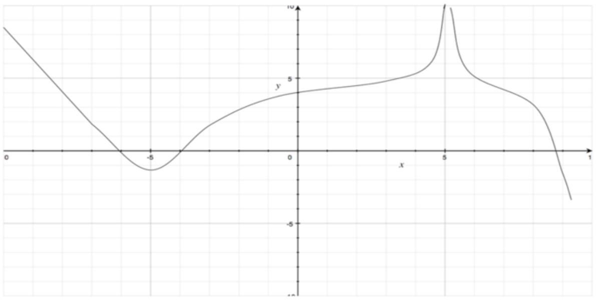

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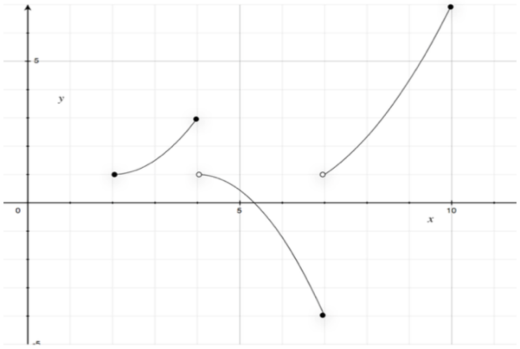


4.6 Multiple Choice Answer Key

1. C 2. A 3. B 4. B 5. A 6. D

7. C 8. B 9. A 10. E

4.7 Free Response Answer Key

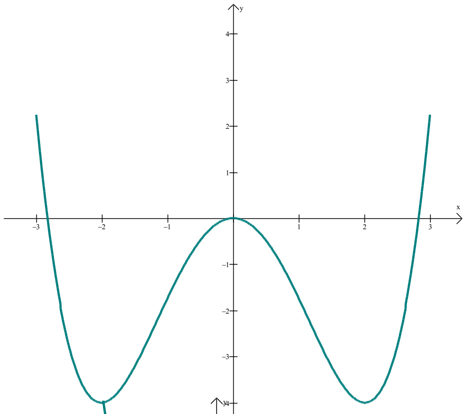
1a. $x=3$ b. $x=0 \ \ and \ \ 6$ c. $x=2 \ \ and \ \ 4$

2a. $x=0 \ \ and \ \ 6$ b. $x=3$ c. $x=2 \ \ and \ \ 4$

3a. $x=1 \ \ and \ \ 6$ b. $x=0 \ \ and \ \ 3$ c. $x=2 \ \ and \ \ 4$

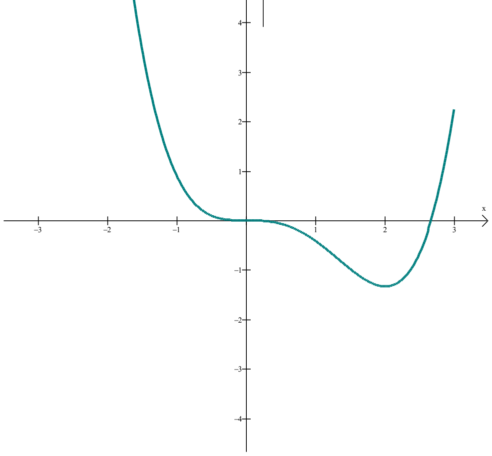
4a. $x=0 \ \ and \ \ 3$ b. $x=1 \ \ and \ \ 6$ c. $x=2 \ \ and \ \ 4$

5a. min pts. @ $x = \pm 2$, max pt. @ $x = 0$

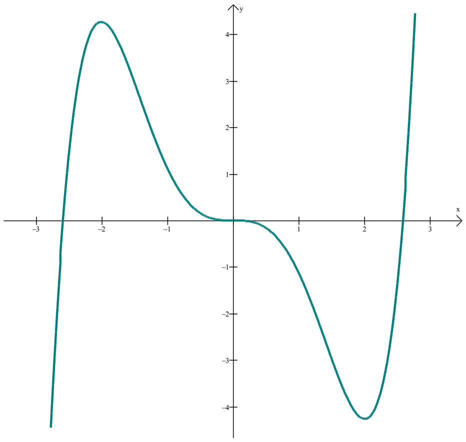


5b. $x = \pm 1$
5c. inc: $x \in (-2,0) \cup (2,3)$
5d. $x \in (-1,1)$

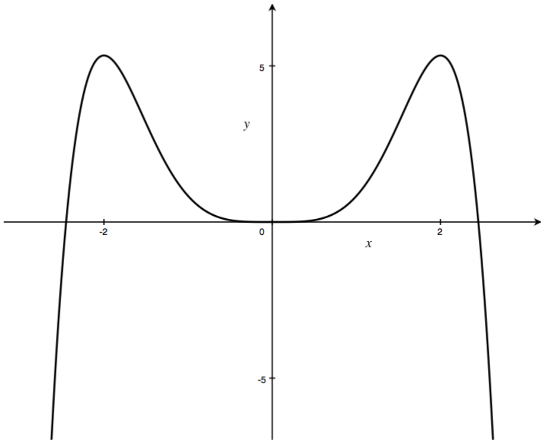
6a. min pt. @ $x = 2$
6b. $x = 0, 1.3$
6c. inc: $x \in (2,3)$
6d. $x \in (0,1.3)$



7a. min pt. @ $x = 2$; pt. @ $x = -2$
7b. $x = \pm 1.4, 0$
7c. $x \in (-3,-2) \cup (2,3)$
7d. $x \in (-3,-1.4) \cup (0,1.4)$

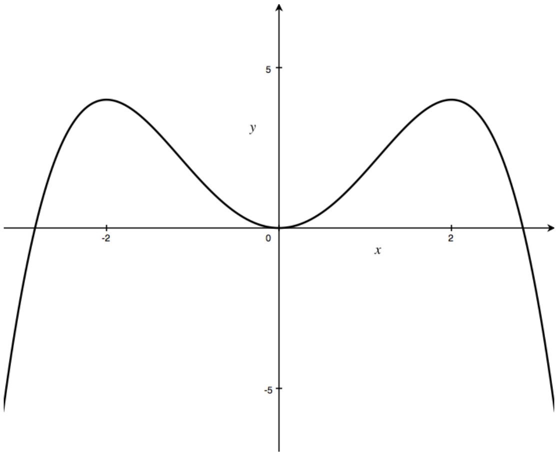


8a. max pts. @ $x = \pm 2$; pt. @ $x = 0$
8b. $x = \pm 1.5$
8c. $x \in (-3, -2) \cup (0, 2)$
8d. $x \in (-3, -1.5) \cup (1.5, 3)$



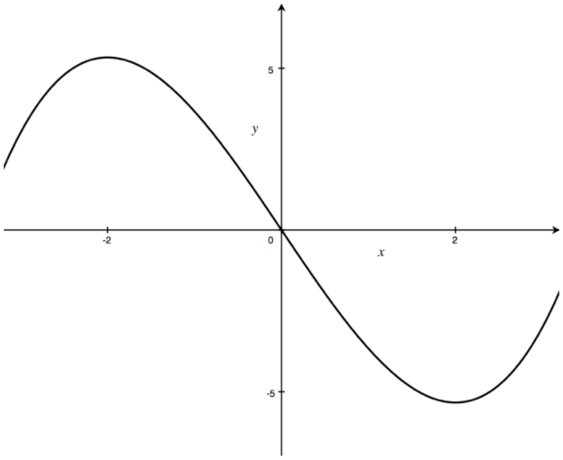
9a. max pt. @ $x = \pm 2$; min pt. @ $x = 0$

9b. $x = \pm 1.3$
9c. $x \in (-3, -2) \cup (0, 2)$
9d. $x \in (-3, -1.3) \cup (1.3, 3)$



10a. min pt. @ $x = 2$; pt. @ $x = -2$

10b. $x = 0$
10c. inc: $x \in (-3, -2) \cup (2, 3)$
10d. $x \in (-3, 0)$



11a. max pt. at $x = -4$ because $f'(x)$ switches from + to – and at $x = 7$ because $f'(x)$is positive and the function stops; min pt. $x = 0$ because $f'(x)$ switches from – to + and at $x = -7$ because the function starts and $f'(x)$is positive.

11b. $f(x)$ has at points of inflection at $x = \pm 2$ and 4 where $f'(x)$ switches from increasing to decreasing or vice versa.

11c. $y + 3 = 2(x - 2)$

12a. max pt. at $x = 0$ because $f'(x)$ switches from + to – , at $x = -7$ because the function starts and $f'(x)$is positive , and at $x = 7$ because $f'(x)$is positive and the function stops; min pt. $x = -2$ and 3 because $f'(x)$ switches from – to +
12b. $f(x)$ has at points of inflection at $x = -4.5, -1,$ and 1.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

12c. $y - 1 = 3(x + 1)$

13a. max pt. at $x = 4$ because $f'(x)$ switches from + to –; min pt. at $x = -7$ because the function $f'(x)$ starts and is positive , and at $x = 7$ because $f'(x)$is negative and the function stops.

13b. $f(x)$ has points of inflection at $x = -4, 0,$ and 5.5 where $f'(x)$ switches from increasing to decreasing or vice versa.

13c. $y + 5 = 2x$

14a. max pt. at $x = 3$ because $f'(x)$ switches from + to – and at $x = 8$ because $f'(x)$is positive and the function stops; min pt. at $x = 5$ because the function $f'(x)$ starts and is positive , and at $x = 0$ because $f'(x)$starts and is positive.

14b. $f(x)$ has a point of inflection at $x = 4$ where $f'(x)$ switches from decreasing to increasing.

14c. $y = -2x$

15. See AP Central

4.7 Multiple Choice Answer Key

1. E 2. E 3. A 4. E 5. A 6. B

7. D 8. D 9. A 10. D 11. A

4.8 Free Response Answer Key

1a. $f(0)=\int_{3}^{0} g(t)dt=-3.5$**;** $f'(0)=g(0)=1$**;** $f''(0)=g'(0)=1$

1b. .225 1c. $x\in(-2,\ -1)$ 1d. $x=-1$

2a. $g(2)=\int_{-2}^{2} f(t)dt=-8$, $g'(2)=f(2)=-\frac{2}{3}$, and $g''(2)=f'(2)=\frac{2}{3}$.

2b. $\frac{5}{12}$ 2c. $x\in(-2,\ -1)\cup(6,\ 8)$ 2d. $x=3$

3a. $k(4)=-2\pi$, $k'(4)=0$, and $k''(4)$ *dne*.

3b. $\frac{11.5-2\pi}{15}$ 3c. $x\in(0,\ 2)\cup(12,\ 15)$ 3d. $x=4$

4a. $g(3)=3$; $g'(3)=2$; $g''(3)=-\frac{1}{2}$ 4b. $\frac{7}{3}$

4c. Two. 4d. $x=2$ and 5

5a. $h(2)=2\pi$; $h'(2)=2$; $h''(2)=0$ 5b. π

5c. $x\in(-3,\ -2)$ 5d. 2π

6a. $g(-5)=-\frac{1}{2}$; $g'(-5)=3$ 6b. $g(2)=-1.7$

6c. $x=5$ 6d. $x\in(-2,\ 0)$

7a. $g(-5)=8$; $g'(-5)=-2$ 7b. $g(2)=9$

7c. $x=-1$ 7d. $x\in(1,\ 2)$

8a. $h(2)=-\frac{\pi}{2}$; $h'(2)=0$; $h''(2)=dne$ 8b. $\frac{\pi}{4}$

8c. $x\in(1,\ 2)$ 8d. $2-\frac{\pi}{2}$

9a. $g(4)=\pi-8$; $g'(2)=0$; $g''(4)=dne$ 9b. $\frac{2}{7}-\frac{13\pi}{28}$

9c. $x\in(2,\ 5)$ 9d. $x=\ -7$

10a. $g(0)=2\pi-8$; $g'(0)=-2$; $g''(0)=dne$ 10b. $y-(2\pi-8)=-2(x-0)$

10c. $x\in(-4,\ -2)$ 10d. $x=7$

11a. $g(4)=14$; $g'(4)=-2$; $g''(4)=0$ 11b. 1

11c. 1 11d. 10

12. See AP Central

4.8 Multiple Choice Answer Key

1. B 2. A 3. B 4. A 5. A

Derivative Applications I Practice Test Key

1. C 2. B 3. B 4. E 5. B 6. E

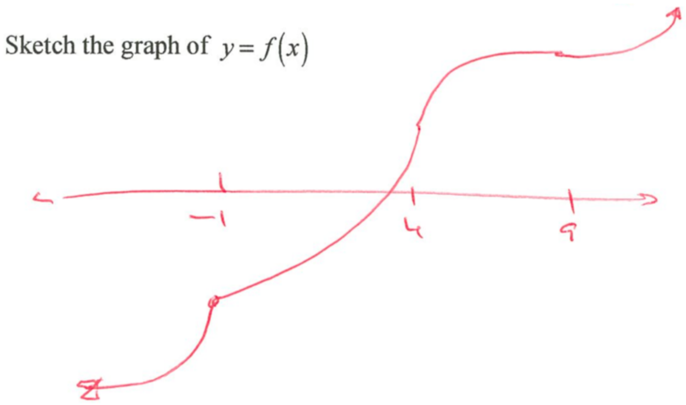
7. C 8. D 9. A 10. C 11. D 12. C

1a. There are no extremes because the first derivative never switches signs and the endpoint are not included in the domain.

b. $x=4$ and 9 because $f'(x)$ switches from + to –

c. At $x=-1$, there is a cusp point, because neither $f'(x)$ nor $f''(x)$exist and neither sign pattern changes.

d.



Sketch the graph of $y = f(x)$

2a) 300 *metric tons*

b) decreasing

c) $t = 6.794$ *hours*

d) 192.320 *metric tons*

e) 10242.255 *metric tons*3a. $x = -7$ *and* 0.4. $x = -7$ because it is the left endpoint and $f' > 0$ after $x = -7$. $x = 0.4$ because f' switches from positive to negative.

b. $x = \pm 4, \ -2, \ 0, \ 1$ because f' switches from increasing to decreasing or vice versa.

c. $x \in (-5.25, \ -4)$ and $x \in (0, \ 0.4)$ because f' is negative and increasing.