# Chapter 2:

# Anti-Derivatives

Chapter 2 Overview: Anti-Derivatives

As noted in the introduction, Calculus is essentially comprised of four operations.

- Limits
- Derivatives
- Indefinite Integrals (or Anti-Derivatives)
- Definite Integrals

There are two kinds of Integrals--the Definite Integral and the Indefinite Integral. The Definite Integral was explored first as a way to determine the area bounded by a curve rather than bounded by a polygon. $A = \sum_{i=1}^{n} f\left(x_i\right) \cdot \Delta x$ was used to denote the sum of an infinite number of rectangles, and the symbol $\int_a^b f(x)dx$ was the exact amount, with $\int$ being an elongated *s* for sum.

Newton and Leibnitz made the connection between the definite integral and the anti-derivative, showing that the process of reversing the derivative is the infinite sum.

The Anti-Derivative and the indefinite integral, because, as operations, inverses of one another—just as squares and square roots, or exponential and log functions. In this chapter, we will consider how to reverse the differentiation process. In a later chapter, we will explore the definite integral.

## 2.1: Anti-Derivatives--the Power Rule

As we have seen, we can deduce things about a function if its derivative is know. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called Anti-differentiation, or Integration.

Symbol: $\int (f(x))\ dx$ = "the integral of *f* of *x*, *d-x*"

The *dx* is called the differential. For now, we will just treat it as part of the integral symbol. It tells us the independent variable of the function (usually, but not always, x) and, in a sense, is where the increase in the exponent comes from. It does have meaning on its own, but we will explore that later.

Looking at the integral as an anti-derivative, that is, as an operation that reverses the derivative, we should be able to figure out the basic process.

Remember:

$$\frac{d}{dx}x^n = n\ x^{n-1} \qquad \frac{d}{dx}[\ln x] = \frac{1}{x}$$

(or, multiply the power in front and subtract one from the power). If we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by the new power. Also, we do not know, from the derivative, if the original function had a constant that became zero, let alone what the constant was.

The Anti-Power Rule

$$\int (x^n)\ dx = \frac{x^{n+1}}{n+1} + c \text{ for } n \neq -1$$

The "+ c" is to account for any constant that might have been there before the derivative was taken. NB. This Rule will not work if n = -1, because it would require that we divide by zero. But we know from the Derivative Rules what yields $x^{-1}$ $\left(or\ 1/x\right)$ as the derivative--ln *x*. So, we can complete the anti-Power Rule as:

The Anti-Power Rule

$$\int\left(x^{n}\right)\,dx=\frac{x^{n+1}}{n+1}+c\ \text{ if }\ n\neq-1$$
$$\int\frac{1}{x}\,dx=\ln\,|\,x\,|+c$$

Since $D_x\,[f(x)+g(x)] = D_x\,[f(x)] + D_x\,[g(x)]$ and $D_x\,[cx^n] = cD_x\,[x^n]$, then

$$\int\left(f(x)+g(x)\right)\,dx=\int f(x)\,dx+\int g(x)\,dx$$
$$\int c\left(f(x)\right)\,dx=c\int f(x)\,dx$$

These allows us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the anti-derivative of a polynomial.
Integrate functions involving Exponential Functions.

Ex 1 $\int \left(3x^2 + 4x + 5\right) dx$

$$\int \left(3x^2 + 4x + 5\right) dx = 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + c$$

$$= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + c$$

$$= x^3 + 2x^2 + 5x + c$$

Ex 2 $\int \left(x^4 + 4x^2 + 5 + \frac{1}{x} - \frac{1}{x^5}\right) dx$

$$\int \left(x^4 + 4x^2 + 5 + \frac{1}{x} - \frac{1}{x^5}\right) dx = \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + Ln|x| - \frac{x^{-5+1}}{-5+1} + c$$

$$= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 5x + Ln\ |\ x\ | + \frac{1}{-4x^4} + c$$

Ex 3 $\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right) dx$

$$\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x}\right) dx = \int \left(x^2 + x^{1/3} - \frac{4}{x}\right) dx$$

$$= \frac{x^{2+1}}{2+1} + \frac{x^{1/3+1}}{1/3+1} - 4Ln|\ x\ | + c$$

$$= \frac{1}{3}x^3 + \frac{3}{4}x^{4/3} - 4Ln\ |\ x\ | + c$$

As we saw with the derivatives, **the Power Rule is often confused with the Exponential Rules**.

| The Power Rule: | The Exponential Rules: |
|---|---|
| $\frac{d}{dx}x^n = n\,x^{n-1}$ | $\frac{d}{dx}\left[e^x\right]=e^x$ <br> $\frac{d}{dx}\left[a^x\right]=a^x \cdot Ln\,a$ |

Just as there is an Anti-derivative Power Rule, there are two Anti-derivative exponential Rules.

| The AntiPower Rule: | The Exponential Rules: |
|---|---|
| $\int\left(x^n\right)\,dx=\frac{x^{n+1}}{n+1}+c \text{ if } n\neq-1$ | $\int\left(e^x\right)\,dx=e^x+c$ |
| $\int\frac{1}{x}\,dx=\ln\mid x\mid+c$ | $\int\left(a^x\right)\,dx=\frac{a^x}{\ln a}+c$ |

Ex 4 $\int\left(4x^5+\sqrt[3]{x^7}-7^x+e^x\right)\,dx$

$$\int\left(4x^5+\sqrt[3]{x^7}-7^x+e^x\right)\,dx=\int\left(4x^5+x^{7/3}-7^x+e^x\right)\,dx$$

$$=4\frac{x^6}{6}+\frac{x^{10/3}}{10/3}-\frac{7^x}{\ln 7}+e^x+c$$

$$=\frac{2}{3}x^6+\frac{3}{10}x^{10/3}-\frac{7^x}{\ln 7}+e^x+c$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial. A product can be FOILED out and a quotient might be able to be divided.

Ex 5 $\int\left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx$

$$\int\left(x^2+\sqrt[3]{x}\right)(2x+1)dx=\int\left(2x^3+2x^{4/3}+x^2+x^{1/3}\right)dx$$
$$=\frac{2x^4}{4}+\frac{2x^{7/3}}{7/3}+\frac{x^3}{3}+\frac{x^{4/3}}{4/3}+c$$
$$=\frac{1}{2}x^4+\frac{6}{7}x^{7/3}+\frac{1}{3}x^3+\frac{3}{4}x^{4/3}+c$$

Ex 6 $\int\left(\frac{x^2+\sqrt[3]{x}-4}{x}\right)dx$

$$\int\left(\frac{x^2+\sqrt[3]{x}-4}{x}\right)dx=\int\left(\frac{x^2}{x}+\frac{x^{1/3}}{x}-\frac{4}{x}\right)dx$$
$$=\int\left(x+x^{-2/3}-\frac{4}{x}\right)dx$$
$$=\frac{x^{1+1}}{1+1}+\frac{x^{-2/3+1}}{-2/3+1}-4\ln|x|+c$$
$$=\frac{1}{2}x^2+3x^{1/3}-4\ln|x|+c$$

## 2.1 Free Response Homework

Perform the Anti-differentiation.

1. $\int\left(6x^2-2x+3\right)dx$

2. $\int\left(x^3+3x^2-2x+4\right)dx$

3. $\int\left(\sqrt[3]{t^2}+2\sqrt{t^3}\right)dt$

4. $\int\left(8x^4-4x^3+9x^2+2x+1\right)dx$

5. $\int\left(\frac{2}{x^3}-\frac{3}{x}+\frac{4}{\sqrt[3]{x}}\right)dx$

6. $\int\left(4x^6-2x^2-\frac{7}{x}+\sqrt[7]{x^4}+\frac{1}{\sqrt{x^5}}\right)dx$

7. $\int\left(7x^6-3x^2-\frac{8}{x}+\sqrt[5]{x^4}+\frac{1}{\sqrt{x^7}}\right)dx$

8. $\int\left(5x^3-2x-\frac{6}{x}+\sqrt[6]{x^7}+\frac{1}{\sqrt{x^9}}\right)dx$

9. $\int\left(\sqrt{x}+3\sqrt[2]{x^3}-\frac{6}{\sqrt{x}}\right)dx$

10. $\int\left(\sqrt{x}-\frac{6}{\sqrt{x}}\right)dx$

11. $\int\left(-3x^4+7^x-\frac{3}{\sqrt[5]{x^6}}-\frac{1}{12x}\right)dx$

12. $\int\left(x^7-4\sqrt[8]{x^7}+7^x-\frac{1}{\sqrt[7]{x^4}}+\frac{1}{5x}\right)dx$

13. $\int\left(x^2+\frac{1}{x^2}-5e^x+3\right)dx$

14. $\int\left(x^4-14\sqrt[7]{x^9}+8^x-\frac{1}{\sqrt[3]{x^7}}+\frac{1}{8x}\right)dx$

15. $\int\left(x^6-5^x-\frac{1}{\sqrt[3]{x^5}}+\frac{1}{2x}\right)dx$

16. $\int\left(x^2+5x+6\right)dx$

17. $\int x^3\left(4x^2+5\right)dx$

18. $\int(4x-1)(3x+8)\,dx$

19. $\int\left(\frac{x^2+4x+3}{\sqrt{x}}\right)dx$

20. $\int\left(\frac{x^2+\sqrt{x}+3}{x}\right)dx$

21. $\int(4t^2+1)(3t^3+7)\,dt$

22. $\int(4x-3)^2\,dx$

23. $\int\left(\frac{4x^3+\sqrt{x}+3}{x^2}\right)dx$

24. $\int\left(\frac{x^2-4x+7}{x}\right)dx$

25. $\int\frac{x^5-7x^3+2x-9}{2x}dx$

26. $\int\frac{x^3+3x^2+3x+1}{x+1}dx$

27. $\int(y^2+5)^2\,dy$

28. $\int(x+1)^3\,dx$

## 2.1 Multiple Choice Homework

1. $\int\frac{1}{x^2}\,dx =$

a) $\ln x^2 + C$ b) $-\ln x^2 + C$ c) $x^{-1} + C$

d) $-x^{-1} + C$ e) $-2x^{-3} + C$

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2. $\int x(10+8x^4)\,dx =$

a) $5x^2+\frac{4}{3}x^6+C$ b) $5x^2+\frac{8}{5}x^5+C$ c) $10x+\frac{4}{3}x^6+C$

d) $5x^2+8x^6+C$ e) $5x^2+\frac{8}{7}x^6+C$

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3. $\int x\sqrt{3x}\ dx =$

a) $\frac{2\sqrt{3}}{5}x^{5/2}+C$ b) $\frac{5\sqrt{3}}{2}x^{5/2}+C$ c) $\frac{\sqrt{3}}{2}x^{1/2}+C$

d) $2\sqrt{3x}+C$ e) $\frac{5\sqrt{3}}{2}x^{3/2}+C$

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4. $\int (x-1)\sqrt{x}\ dx =$

a) $\frac{3}{2}\sqrt{x}-\frac{1}{\sqrt{x}}+c$ b) $\frac{2}{3}x^{3/2}+\frac{1}{2}x^{1/2}+c$ c) $\frac{1}{2}x^2-x+c$

d) $\frac{2}{5}x^{5/2}-\frac{2}{3}x^{3/2}+c$ e) $\frac{1}{2}x^2+2x^{3/2}+c$

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## 2.2: <u>Initial Conditions</u>

Anti-derivatives are often referred to as Indefinite Integrals. One of the things that makes them indefinite is that we do not know the value of the constant of integration (the +*c*). If we had more information, we could find the value of *c*.

*Vocabulary*:
*General Solution* – All of the *y*-equations that would have the given equation as their derivative. Note the +*C* which gives multiple equations.
*Initial Condition* – Constraint placed on a differential equation; sometimes called an initial value.
*Particular Solution* – Solution obtained from solving a differential equation when an initial condition allows you to solve for C.

OBJECTIVES

Use an initial condition to solve for a particular solution to an antiderivative.
Use the derivative to make conclusions about motion.
Relate the position, velocity, and acceleration functions through derivatives and antiderivatives.

Ex 1 $f'(x) = 4x^3 - 6x + 3$. Find $f(x)$ if $f(0) = 13$.

$$\begin{aligned} f(x) &= \int (4x^3 - 6x + 3)dx \\ &= x^4 - 3x^2 + 3x + c \end{aligned}$$

$$f(0) = 0^4 - 3(0)^2 + 3(0) + c = 13$$
$$\therefore c = 13$$

$$f(x) = x^4 - 3x^2 + 3x + 13$$

It is tempting to think that the *c* is always the *y*-value of the initial condition, but it is not that simple.

Ex 2 $g'(x) = 4x^3 - 6x + 3$. Find $g(x)$ if $g(2) = 13$.

$$g(x) = \int \left(4x^3 - 6x + 3\right) dx$$
$$= x^4 - 3x^2 + 3x + c$$

$$f(2) = 2^4 - 3(2)^2 + 3(2) + c = 10 + c = 13$$
$$\therefore c = 3$$

$$g(x) = x^4 - 3x^2 + 3x + 3$$

**2.2 Free Response Homework**

Solve the initial value problems.

1. $\dfrac{dy}{dx} = 6x^2 - 12x + 5$, if $y(1) = 8$.

2. $\dfrac{dy}{dx} = 12x^2 - 36x + 5$, if $y(1) = 1$.

3. $\dfrac{dy}{dx} = 8 - 2x$, if $y(-3) = 0$.

4. $\dfrac{dy}{dx} = 36x^2 - 12x + 8$, if $y(1) = 1$.

5. $f'(x) = 3x^2 - 6x + 3$. Find $f(x)$, if $f(0) = 2$.

6. $f'(x) = x^3 + x^2 - x + 3$. Find $f(x)$, if $f(1) = 0$.

7. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find $f(x)$, if $f(4) = 1$.

8. $f'(x) = (2 - x)^3$ Find $f(x)$, if $f(-3) = 1$.

**2.2 Multiple Choice Questions**

1. If $f'(x) = x^2 + 2x - 3$ and $f(3) = 1$, then $f(x) =$

a) $f(x) = x^3 + 2x^2 - 3x + 9$

b) $f(x) = \dfrac{1}{3}x^3 + x^2 + 3x - 3$

c) $f(x) = \dfrac{1}{3}x^3 + x^2 - 3x + 9$

d) $f(x) = \dfrac{1}{3}x^3 + x^2 - 3x - 8$

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2. If $f'(x)=\sqrt{x}$ and $f(0)=0$, then $f(1)=$

a) $\frac{1}{3}$ b) $2$ c) $\frac{3}{2}$ d) $-2$ e) $\frac{2}{3}$

---

3. If $f'(x)=\sqrt{x}-\frac{2}{\sqrt{x}}$ and $f(4)=3$, then

a) $f(x)=\frac{2}{3}\sqrt{x^3}-4\sqrt{x}+\frac{1}{3}$ b) $f(x)=\frac{2}{3}\sqrt{x^3}-4\sqrt{x}+\frac{17}{3}$

c) $f(x)=\frac{2}{3}\sqrt{x^3}+\sqrt{x}-\frac{19}{3}$ d) $f(x)=\frac{2}{3}\sqrt{x^3}+\sqrt{x}+\frac{1}{3}$

---

4. If $f'(x)=8x^3-9x^2$ and $f(-1)=0$, then $f(x)=$

a) $f(x)=2x^4-3x^3-5$ b) $f(x)=2x^4-3x^3-1$

c) $f(x)=2x^4-3x^3-1$ d) $f(x)=8x^4-9x^3-17$

e) $f(x)=8x^4-9x^3-1$

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5. If $f''(x)=12x^2-2$, $f'(0)=2$, and $f(0)=3$, then $f(1)=$

a) 1 b) 2 c) 3 d) 4 e) 5

---

6. If $f''(x)=\sqrt{x}$, $f\,'(1)=1$, and $f(1)=1$, then

a) $f(x)=\frac{4}{3}\sqrt{x^3}+\frac{1}{3}x-\frac{2}{3}$

b) $f(x)=\frac{4}{15}\sqrt{x^5}+\frac{1}{3}x+\frac{2}{5}$

c) $f(x)=\frac{4}{3}\sqrt{x^3}-\frac{1}{3}$

d) $f(x)=\frac{4}{15}\sqrt{x^5}+\frac{11}{15}$

## 2.3: Rectilinear Motion

OBJECTIVES

Use the derivative to make conclusions about motion.
Relate the position, velocity, and acceleration functions through derivatives and antiderivatives.

A key application of the derivative as a rate of change is the application to motion. The basics about derivatives and motion were explored in PreCalculus.

**Remember:**

Position = $x(t)$ *or* $y(t)$

Velocity = $x'(t)$ *or* $y'(t)$

Acceleration = $x''(t)$ *or* $y''(t)$

**Therefore:**

Position = $\int x'(t)dt$ *or* $\int y'(t)dt$

Velocity = $\int x''(t)dt$ *or* $\int y''(t)dt$

Typically, we refer to horizontal position in terms of $x(t)$ and vertical position in terms of $y(t)$.

Ex 1 The acceleration of a particle is described by $a(t)=3t^2+8t+1$. Find the particular velocity equation for if $v(0) = 3$.

$$
\begin{aligned}
v(t) &= \int\left(a(t)\right)\,dt = \int\left(3t^2+8t+1\right)\,dt \\
&= t^3+4t^2+t+c_1 \\
3 &= (0)^3+4(0)^2+(0)+c_1 \\
3 &= c_1 \\
v(t) &= t^3+4t^2+t+3
\end{aligned}
$$

EX 2 Consider the velocity equation with the initial value:

$$v(t) = 4e^t + 3 \text{ and } y(0) = 3$$

Find (a) the acceleration at $t = 3$, and (b) the particular position equation.

a) $$a(t) = \frac{d}{dt}\left(4e^t + 3\right) = 4e^t$$
$$a(3) = 4e^6 = 1613.715$$

b) $$y(t) = \int\left(4e^t + 3\right)dt = 4e^t + 3t + c$$
$$y(0) = 3 \rightarrow 3 = 4e^0 + 0 + c = 4 + c \rightarrow c = -1$$
$$y(t) = 4e^t + 3t - 1$$

Ex 3 The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

$$v(t) = \int\left(a(t)\right)\, dt = \int\left(12t^2 - 6t + 4\right)\, dt$$
$$= 4t^3 - 3t^2 + 4t + c_1$$
$$0 = 4(1)^3 - 3(1)^2 + 4(1) + c_1$$
$$-5 = c_1$$
$$v(t) = 4t^3 - 3t^2 + 4t - 5$$

$$x(t) = \int\left(v(t)\right)\, dt = \int\left(4t^3 - 3t^2 + 4t - 5\right)\, dt$$
$$= t^4 - t^3 + 2t^2 - 5t + c_2$$
$$3 = (1)^4 - (1)^3 + 2(1)^2 - 5(1) + c_2$$
$$6 = c_2$$
$$x(t) = t^4 - t^3 + 2t^2 - 5t + 6$$

On the AP Calculus Exam, there is a tendency to start with velocity and use both the derivative and the antiderivative to consider the other two aspects of motion.

## 2.3 Free Response Homework

For the given velocity equations and initial values, answer the following questions:

a) For what values is the particle moving right.

b) What is the acceleration at $t=3$?

c) Find the particular position equation.

1. $v(t)=t^2-4t-12;\ x(1)=0$

2. $v(t)=t^2-5t+4;\ x(3)=-1$

3. $v(t)=t^3-9t^2-4t+36;\ x(0)=4$

4. $v(t)=t^3-6t^2+9t-54;\ x(1)=3$

5. $v(t)=2t^3-5t^2-8t+20;\ x(0)=-4$

6. $v(t)=27+18t-3t^2-2t^3;\ x(1)=-1$

7. $v(t)=t^5-16t^3; x(0)=2$

8. $v(t)=t^4-5t^2-36; x(4)=-1$

Find the particular position equations for the following acceleration equations and initial conditions.

9. $a(t)=36t^2-12t+8$, with $v(1)=1$ and $x(1)=3$.

10. $a(t)=t^2-2t+4$, with $v(0)=2$ and $x(0)=4$.

11. $a(t)=\left(t^2-1\right)^2$, with $v(0)=2$ and $x(0)=-4$.

12. $a(t)=t^2-\frac{1}{\sqrt{t}}$, with $v(1)=5$ and $x(4)=-1$.

## 2.3 Multiple Choice Questions

1. The acceleration function for a particle moving along a line is $a(t) = 3t^2 + 4t + 6$ with initial conditions $v(0) = 10$ and $x(0) = 2$. Find the position at time *t*.

a) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 12$

b) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 10t + 2$

c) $x(t) = 3t^4 + t^3 + t^2 + 10t + 2$

d) $x(t) = \frac{1}{4}t^4 + \frac{2}{3}t^3 + 3t^2 + 2$

e) $x(t) = 3t^4 + t^3 + t^2 + 2$

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2. The acceleration function for a particle moving along a line is $a(t) = t + 4$ with initial conditions $v(0) = 3$ and $x(0) = 1$. Find the position at time *t*.

a) $x(t) = \frac{t^2}{2} + 4t + 3$

b) $x(t) = \frac{t^3}{6} + 2t^2 + 3t + 1$

c) $x(t) = \frac{t^3}{6} + 2t^2 + 1$

d) $x(t) = \frac{t^3}{6} + 2t^2 + t + 3$

e) $x(t) = \frac{t^3}{6} + 4t^2 + 4$

---

3. A particle is moving upward along the y-axis until it reaches the origin and then it moves downward such that $v(t)=8-2t$ for $t \geq 0$. The position of the particle at time t is given by

a) $y(t)=-t^2+8t-16$ b) $y(t)=-t^2+8t+16$

c) $y(t)=2t^2-8t-16$ d) $y(t)=8t-t^2$

e) $y(t)=8t-2t^2$

---

5. If a particle's acceleration is given by $a(t)=12t+4$ and $v(1)=5$ and $y(0)=2$, then $y(2)=$

a) 20 b) 10 c) 4 d) 16 e) 12

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5. A particle moves along the $x$-axis with acceleration at any time $t$ given as $a(t)=3t^2+4t+6$. If the particle's velocity is 10 and its position is 2 when $t=0$, what is the position function?

a) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+12$

b) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+10t+2$

c) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+2$

d) $x(t)=3t^4+t^3+t^2+10t+2$

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6. If a particle's acceleration is given by $a(t)=12t+4$ and $v(1)=5$ and $y(0)=2$, then $y(2)=$

a) 20 b) 10 c) 4 d) 16 e) 12

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## 2.4: Integration by Substitution: Reversing the Chain Rule

Reversing the Power Rule was fairly easy. The other three core derivative rules—the Product Rule, the Quotient Rule, and the Chain Rule—are a little more complicated to undo. This is because they yield a more complicated function as a derivative, one which usually has several algebraic simplification steps. The integral of a Rational Function is particularly difficult to unravel because, as we saw, a rational derivative can be obtained by differentiating a composite function with a log or a radical, or by differentiating another rational function. Reversing the Product Rule is as complicated, though for other reasons.

**Key Idea: There is no single product or quotient rule for integrals.**

Instead, there are several rules which apply in different situations, and it is not always clear at the start which to use. It all depends on what algebraic manipulations let to the product or quotient.

Products can be a result of:

- The Chain Rule
- Differentiating a product.
- Differentiating some trig functions

Quotients can be the result of:

- Common denominators
- Differentiating a quotient.
- Differentiating a log with a composite function.
- Differentiating some trig inverse functions

Composite functions are among the most pervasive functions in math. Therefore, we will start with undoing products and quotients that involve composites.

Remember:

The Chain Rule: $$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)\cdot g'(x)$$

The derivative of a composite turns into a product of a composite and a non-composite. So, if we have a product to integrate, it might be that the product came from the Chain Rule. The integration is not done by a formula so much as a process that might or might not work. We make an educated guess and hope it works out. You will learn other processes in Calculus for when it does not work.

This is one of those mathematical processes that makes little sense when first seen. But after seeing several examples, the meaning suddenly becomes clear. **Be Patient.**

| OBJECTIVE |
| --- |
| Use Integration by Substitution to integrate composite, product expressions. |

Ex 1 $\int\left(3x^2\left(x^3+5\right)^{10}\right)dx$

$\left(x^3+5\right)^{10}$ is the composite function.

$u = x^3 + 5$

$du = 3x^2 dx$

$$\int\left(3x^2\left(x^3+5\right)^{10}\right)dx = \int\left(u^{10}\right)du$$

$$= \frac{u^{11}}{11} + c$$

$$= \frac{1}{11}\left(x^3+5\right)^{11} + c$$

Ex 2 $\int\left(2x\sqrt{x^2-7}\right)dx$

$\sqrt{x^2-7}$ is the composite function. So $\quad u = x^2-7$

$du = 2xdx$

$$\int\left(2x\sqrt{x^2-7}\right)dx = \int\left(x^2-7\right)^{1/2}(2xdx)$$
$$= \int\left(u^{1/2}\right)du$$
$$= \frac{u^{3/2}}{3/2}+c$$
$$= \frac{2}{3}u^{3/2}+c$$
$$= \frac{2}{3}\left(x^2-7\right)^{3/2}+c$$

One of the rules in the last section allows us some flexibility in with the *du*:

> Since $\int c\left(f(x)\right)\,dx = c\int f(x)\,dx$, then $\int f(x)dx = \frac{1}{c}\int f(x)\left(c\cdot dx\right)$.

Ex 3 $\int\left(x\left(x^2+5\right)^3\right)dx$

$\left(x^2+5\right)^3$ is the composite function. So $\quad u = x^2+5$

$du = 2xdx$

$$\int\left(x\left(x^2+5\right)^3\right)dx = \frac{1}{2}\int\left(x^2+5\right)^3(2xdx)$$
$$= \frac{1}{2}\int\left(u^3\right)du$$
$$= \frac{1}{2}\cdot\frac{u^4}{4}+c$$
$$= \frac{1}{8}\left(x^2+5\right)^4+c$$

Integration by Substitution ($u$-subs)

0) Notice that you are trying to integrate a product (or quotient).
1) Identify the inside function of the composite and call it $u$.
2) Find $du$ from $u$.
3) If necessary, multiply a constant inside the integral to create $du$, and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
4) Substitute $u$ and $du$ into the equation.
5) Perform the integration by the Anti-Power Rule (or Transcendental Rules, in next section.)
6) Resubstitute the $x$-equivalent for $u$.

Ex 4 $\int\left(\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\right)dx$

$\sqrt[4]{x^4+2x^2-5}$ is the composite function. So

$$u = x^4+2x^2-5$$

$$du = \left(4x^3+4x\right)dx = 4\left(x^3+x\right)dx$$

$$\int\left(\left(x^3+x\right)\sqrt[4]{x^4+2x^2-5}\right)dx = \frac{1}{4}\int\left(\sqrt[4]{x^4+2x^2-5}\right)4\left(x^3+x\right)dx$$

$$= \frac{1}{4}\int\left(\sqrt[4]{u}\right)du$$

$$= \frac{1}{4}\int\left(u^{1/4}\right)du$$

$$= \frac{1}{4}\frac{u^{5/4}}{5/4}+c$$

$$= \frac{1}{5}\left(x^4+2x^2-5\right)^{5/4}+c$$

| The Anti-Power Rule: | The Exponential Rules: |
|---|---|
| $\int (u^n) du = \frac{u^{n+1}}{n+1} + c$ if $n \neq -1$ | $\int (e^u) du = e^u + c$ |
| $\int \frac{1}{u} du = \ln\lvert u\rvert + c$ | $\int (a^u) dx = \frac{a^u}{\ln a} + c$ |

Ex 5 $\int \left( \frac{3x^2 + 4x - 5}{(x^3 + 2x^2 - 5x + 2)^3} \right) dx$

$$u = x^3 + 2x^2 - 5x + 2$$
$$du = (3x^2 + 4x - 5)dx$$

$$\int \left( \frac{3x^2 + 4x - 5}{(x^3 + 2x^2 - 5x + 2)^3} \right) dx = \int (x^3 + 2x^2 - 5x + 2)^{-3}((3x^2 + 4x - 5)dx)$$
$$= \int (u^{-3}) du$$
$$= \frac{u^{-2}}{-2} + c$$
$$= \frac{-1}{2(x^3 + 2x^2 - 5x + 2)^2} + c$$

Ex 6 $\int \left( x^3 e^{x^4} \right) du =$

$$u = x^4 \rightarrow du = 4x^3 dx$$

$$\int \left( x^3 e^{x^4} \right) dx = \frac{1}{4} \int e^{x^4} (4x^3) dx = \frac{1}{4} \int (e^u) du = \frac{1}{4} e^u + c = \frac{1}{4} e^{x^4} + c$$

**2.4 Free Response Homework**

Perform the Anti-differentiation.

1. $\int (5x+3)^3 dx$

2. $\int \left(x^3\left(x^4+5\right)^{24}\right) dx$

3. $\int \left(1+x^3\right)^2 dx$

4. $\int (2-x)^{2/3} dx$

5. $\int \left(x\sqrt{2x^2+3}\right) dx$

6. $\int \frac{dx}{(5x+2)^3}$

7. $\int \frac{x^3}{\sqrt{1+x^4}}\, dx$

8. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}} dx$

9. $\int \frac{v^2}{5-v^3} dv$

10. $\int \left((2x+3)\sqrt[5]{2x^2+6x+1}\right) dx$

11. $\int (2x+5)\left(x^2+5x+6\right)^6 dx$

12. $\int 3t^2\left(t^3+1\right)^5 dt$

13. $\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}}\, dm$

14. $\int \frac{3x^2}{\left(1+x^3\right)^5} dx$

15. $\int (4s+1)^5 ds$

16. $\int \frac{5t}{t^2+1} dt$

17. $\int \frac{3m^2}{m^3+8} dm$

18. $\int (181x+1)^5 dx$

19. $\int \frac{x^2-3e^{x^3}}{e^{x^3}}\, dx =$

20. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

21. $\int\left(2e^{2x}-5^{3x}\right)\,dx$

22. $\int\left(x^2e^{x^3}-x5^{3x^2}\right)\,dx$

---

**2.4 Multiple Choice Homework**

1. $\displaystyle\int\frac{x}{x^2-4}\,dx=$

a) $\displaystyle\frac{-1}{4\left(x^2-4\right)^2}+C$

b) $\displaystyle\frac{1}{2\left(x^2-4\right)}+C$

c) $\displaystyle\frac{1}{2}\ln\left|x^2-4\right|+C$

d) $2\ln\left|x^2-4\right|+C$

e) $\displaystyle\frac{1}{2}\arctan\left(\frac{x}{2}\right)+C$

---

2. $\displaystyle\int\frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx=$

a) $\ln\sqrt{x}+C$

b) $x+C$

c) $e^x+C$

d) $\displaystyle\frac{1}{2}e^{2\sqrt{x}}+C$

e) $e^{\sqrt{x}}+C$

---

3. $\int (t-4)(t^2-8t)^5 dt =$

a) $\frac{(t^2-8t)^6}{6}+C$ b) $\frac{(t^2-8t)^6}{12}+C$ c) $\frac{(t^2-8t)^6}{3}+C$

d) $\frac{(t-4)^6}{6}+C$ e) $\frac{(t-4)^6}{3}+C$

---

4. $\int \frac{3x^2}{\sqrt{x^3+3}} dx$

a. $2\sqrt{x^3+3}+c$ b. $\frac{3}{2}\sqrt{x^3+3}+c$ c. $\sqrt{x^3+3}+c$

d. $\ln\sqrt{x^3+3}+c$ e. $\ln(x^3+3)+c$

---

5. $\int x(x^2-1)^4 dx =$

a) $\frac{1}{10}x^2(x^2-1)^5+C$ b) $\frac{1}{10}(x^2-1)^5+C$ c) $\frac{1}{5}(x^3-x)^5+C$

d) $\frac{1}{5}(x^2-1)^5+C$ e) $\frac{1}{5}(x^2-x)^5+C$

---

6. $\int 4x^2\sqrt{3+x^3}\,dx$

a. $\dfrac{16(3+x^3)^{3/2}}{9}+c$ b. $\dfrac{8(3+x^3)^{3/2}}{9}+c$ c. $\dfrac{8(3+x^3)^{3/2}}{3}+c$

d. $\dfrac{4}{3(3+x^3)^{1/2}}+c$ e. $\dfrac{8}{3(3+x^3)^{1/2}}+c$

---

7. $\int\left(\dfrac{3x^2+6x-4}{(x^3+3x^2-4x+2)^2}\right)dx$

a. $\ln\left|x^3+3x^2-4x+2\right|+c$ b. $\ln\left|x^3+3x^2-4x+2\right|^2+c$

c. $(x^3+3x^2-4x+2)^{-1}+c$ d. $-(x^3+3x^2-4x+2)^{-1}+c$

---

8. $\int\left((x^3+x)\sqrt[4]{x^4+2x^2-5}\right)dx=$

a. $\dfrac{1}{5}(x^4+2x^2-5)^{5/4}+c$ b. $\dfrac{4}{5}(x^4+2x^2-5)^{5/4}+c$

c. $\dfrac{1}{4}(x^4+2x^2-5)^{5/4}+c$ d. $4(x^4+2x^2-5)^{1/4}+c$

---

## 2.5: Integration by Substitution and the Transcendental Functions

The proof of the Transcendental Integral Rules can be left to a more formal Calculus course. But since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Derivative Rules

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$

$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = \left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right] = \left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right] = a^u \cdot \ln a \frac{du}{dx} \qquad \frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot Lna}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{u^2+1} \cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{u^2+1} \cdot D_u$$

Transcendental Integral Rules

$$\int(\cos u)du = \sin u + c$$

$$\int(\sin u)du = -\cos u + c$$

$$\int(\sec^2 u)du = \tan u + c$$

$$\int(e^u)du = e^u + c$$

$$\int(a^u)du = \frac{a^u}{\ln a} + c$$

$$\int(\csc u \cot u)du = -\csc u + c$$

$$\int(\sec u \tan u)du = \sec u + c$$

$$\int(\csc^2 u)du = -\cot u + c$$

$$\int\left(\frac{1}{u}\right)du = Ln|u| + c$$

$$\int\frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C$$

$$\int\frac{du}{1+u^2} = \tan^{-1}u + C$$

$$\int\frac{du}{u\sqrt{u^2-1}} = \sec^{-1}u + C$$

Note that there are only three integrals that yield inverse trig functions where there were six inverse trig derivatives. This is because the other three rules derivative rules are just the negatives of the first three. As we will see later, these three rules are simplified versions of more general rules, but for now we will stick with the three.

Ex 1 $\int(\sin x + 3\cos x)dx$

$$\int(\sin x + 3\cos x)dx = \int(\sin x)dx + 3\int(\cos x)dx$$
$$= -\cos x + 3\sin x + c$$

Ex 2 $\int(e^x + 4 + 3\csc^2 x)dx$

$$\int(e^x + 4 + 3\csc^2 x)dx = \int(e^x)dx + 4\int dx + 3\int(\csc^2 x)dx$$

$$= e^x + 4x - 3\cot x + c$$

Trig Inverse Integral Rules

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

One more rule that looks like it should be a Trig Inverse Rule, but it is NOT:

$$\int \frac{du}{u^2 - a^2} = \frac{1}{2a} ln\left|\frac{u - a}{u + a}\right| + C$$

The proof of this rule is beyond the scope of this course, but it will be proven in BC Calculus. **For now, this will be a rule to memorize and use**.

Ex 3 If $\int \frac{dx}{u^2 + 4}$

$$\int \frac{dx}{u^2 + 4} = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

Ex 4 If $\frac{dy}{dx} = \sec x(\sec x + \tan x)$, find $y(x)$ if $y(0) = 0$.

$$y = \int(\sec x(\sec x + \tan x))dx = \int(\sec^2 x)dx + \int(\sec x \tan x)dx$$

$$= \tan x + \sec x + c$$

$$0 = \tan 0 + \sec 0 + c$$

$$0 = 0 + 1 + c$$

$$c = -1$$

$$y = \tan x + \sec x - 1$$

OBJECTIVE

Use $u$-substitution to integrate composite and product expressions involving the transcendental functions.

Ex 5 $\int(\sin 5x)dx$

$$u = 5x$$

$$du = 5dx$$

$$\int(\sin 5x)dx = \frac{1}{5}\int(\sin 5x)5dx$$

$$= \frac{1}{5}\int(\sin u)du$$

$$= \frac{1}{5}(-\cos u) + c$$

$$= -\frac{1}{5}\cos 5x + c$$

Ex 6 $\int(\sin^6 x\cos x)dx$

$$u = \sin x$$

$$du = \cos x dx$$

$$\int(\sin^6 x\cos x)dx = \int(u^6)du$$

$$= \frac{1}{7}u^7 + c$$

$$= \frac{1}{7}\sin^7 x + c$$

Ex 7 $\int(x^5\sin x^6)dx$

$$u = x^6$$

$$du = 6x^5 dx$$

$$\int(x^5\sin x^6)dx = \frac{1}{6}\int(\sin x^6)(6x^5 dx)$$

$$= \frac{1}{6}\int(\sin u)du$$

$$= -\frac{1}{6}\cos u + c$$

$$= -\frac{1}{6}\cos x^6 + c$$

Ex 8 $\int \left(\cot^3 x \csc^2 x\right) dx$

$$u = \cot x$$

$$du = -\csc^2 x dx$$

$$\begin{aligned}\int \left(\cot^3 x \csc^2 x\right) dx &= -\int \left(\cot^3 x\right)\left(-\csc^2 x dx\right) \\ &= -\int \left(u^3\right) du \\ &= -\frac{1}{4}u^4 + c \\ &= -\frac{1}{4}\cot^4 x + c\end{aligned}$$

Note that integrals distribute over multiplication. This means we can break up an integral into several integrals and use different *u*-subs in each case:

Ex 9 $\int \left(xe^{x^2} + 4x^2 - 3\sin 5x\right) dx$

$$\begin{aligned}\int \left(xe^{x^2} + 4x^2 - 3\sin 5x\right) dx &= \int \left(xe^{x^2}\right) dx + \int \left(4x^2\right) dx + \int \left(-3\sin 5x\right) dx \\ &= \frac{1}{2}\int e^{x^2}(2x dx) + 4\int x^2 dx - \frac{3}{5}\int \sin 5x (5dx)\end{aligned}$$

$$u_1 = x^2 \qquad\qquad u_2 = 5x$$

$$du_1 = 2x dx \qquad\qquad du_2 = 5dx$$

$$\begin{aligned}&= \frac{1}{2}\int e^{u_1} du_1 + 4\int x^2 dx - \frac{3}{5}\int \sin u_2 du_2 \\ &= \frac{1}{2}e^{u_1} + 4\left(\frac{x^3}{3}\right) - \frac{3}{5}\left(-\cos u_2\right) + c \\ &= \frac{1}{2}e^{x^2} + \frac{4}{3}x^3 + \frac{3}{5}\cos 5x + c\end{aligned}$$

Ex 10 $\int \tan x dx$

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}dx \\ &= -\int \frac{1}{u}du \\ &= -\ln|\cos x| + c \\ &= \ln|\cos x|^{-1} + c \\ &= \ln|\sec x| + c\end{aligned}$$

This gives us two more integral rules:

$$\int \tan u\,du = \ln|\sec u| + c \qquad\qquad \int \cot u\,du = \ln|\sin u| + c$$

The last two trig integrals, $\int \sec u\,du$ and $\int \csc u\,du$, are of the same kind as these two. We will give the rules here and prove them later.

$$\int \sec u\,du = \ln|\sec u + \tan u| + c \qquad\qquad \int \csc u\,du = \ln|\csc u - \cot u| + c$$

Ex 11 $\int \left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right)dx$

$$u = \sqrt{x} = x^{1/2}$$

$$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2x^{1/2}}dx$$

$$\int\left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right)dx = 2\int\left(\cos\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}}dx\right)$$

$$= 2\int(\cos u)du$$

$$= 2\sin u + c$$

$$= 2\sin\sqrt{x} + c$$

Ex 12 $\int \frac{x}{\sqrt{1-x^4}}dx$

$$u = x^2$$

$$du = 2xdx$$

$$\int \frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(x^2\right)^2}}(2xdx)$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$$

$$= \frac{1}{2}\sin^{-1}u + c$$

$$= \frac{1}{2}\sin^{-1}x^2 + c$$

Summary of Integrating Quotients

There are essentially four ways to integrate fractions:

1. Simplify the Algebra by performing the division; then integrate.

2. Chose a $u$-sub that eliminates the division; then integrate.

3. Chose a $u$-sub that leads to $\int \frac{1}{u} du = \ln|u| + C$

4. Chose a $u$-sub that leads to $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$

## 2.5 Free Response Homework

Perform the Anti-differentiation.

1. $\int \left(x^4 \cos x^5\right) dx$

2. $\int (\sin(7x+1)) dx$

3. $\int \left(\sec^2(3x-1)\right) dx$

4. $\int \left(\dfrac{\sin\sqrt{x}}{\sqrt{x}}\right) dx$

5. $\int \left(\tan^4 x \sec^2 x\right) dx$

6. $\int \left(\sqrt{\cot x}\,\csc^2 x\right) dx$

7. $\int \left(e^{6x}\right) dx$

8. $\int \dfrac{\cos 2x}{\sin^3 2x} dx$

9. $\int \sec(2x) dx$

10. $\int \dfrac{\csc^2\left(e^{-x}\right)}{e^x} dx$

11. $\int \dfrac{e^{\sqrt{x}}}{\sqrt{x}} dx$

12. $\int \dfrac{\ln x}{x} dx$

13. $\int \dfrac{x}{1+x^4} dx$

14. $\int \dfrac{\cos x}{\sqrt{1-\sin^2 x}} dx$

15. $\int \dfrac{\sec(\ln x)\tan(\ln x)}{3x} dx$

16. $\int e^x \csc e^x \cot e^x dx$

17. $\int \dfrac{\cos x}{1+\sin x} dx$

18. $\int t \sec^2\left(4t^2\right)\sqrt{\tan\left(4t^2\right)}\, dt$

19. $\int \dfrac{x \ln\left(x^2+1\right)}{x^2+1} dx$

20. $\int \dfrac{1}{x^2}\left(\sin\dfrac{1}{x}\right)\left(\cos\dfrac{1}{x}\right) dx$

21. $\int\left(x^2\sec^2\left(x^3\right)+\frac{\ln^3 x}{x}\right)dx$

22. $\int x^2\sec^2\left(x^3\right)+2xe^{x^2}dx$

23. $\int\left(x^5-\sin(3x)+xe^{x^2}\right)dx$

24. $\int\left(\frac{2x}{x^2+5}-\sec^2(3x)+xe^{x^2}-\pi\right)dx$

25. $\int\left(x^3-x^2\sec\left(2x^3\right)+x^3e^{x^4}\right)dx$

26. $\int\left(3\sqrt{x}-\tan(2x)+\frac{x}{e^{3x^2}}\right)dx$

27. $\int\left(5\sqrt{x^3}-\sec(3x)+\frac{4x}{e^{4x^2}}\right)dx$

28. $\int\left(3\sqrt{x^3}-\sec(2x)+\frac{x}{e^{4x^2}}\right)dx$

29. $\int\left(sec(5x)\tan(5x)\ +\ sec^2(3x)\ +\ sec(7x)\right)\,dx$

30. $\int\left(\cos 3x\ +\ \cos^2 5x\ +\ \sin 7x\,\sqrt{\cos 7x}\right)\,dx$

31. $\int\left(\frac{1}{(9+x)^2}\ +\ \frac{1}{9+x^2}\ +\ \frac{x}{9+x^2}\right)dx$

32. $\int\left(\frac{1}{\sqrt{3-x^2}}\ +\ \frac{x}{\sqrt{3-x^2}}\ +\ \frac{1}{3-x}\right)dx$

33. $\int\left(\frac{1}{x\sqrt{x^2-4}}\ +\ \frac{x}{\sqrt{x^2-4}}\ +\ \frac{1}{x^2-4}\right)dx$

34. $\int\left(\frac{1}{(1-x)^2}+\frac{x}{1-x^2}+\frac{1}{1+x^2}\right)dx$

35. $\mathrm{a(t)} = -4\sin 2\mathrm{t}$ describes the acceleration of a particle. Find $v(t)$ and $x(t)$ if $v(0)=0$ and $x(0) = -3$.

36. $a(t)=\cos(3t)$ describes the acceleration of a particle. Find $v(t)$ and $x(t)$if $v(\pi) = -1$ and $x(0)=2$

37. The acceleration of a particle is described by $a(t)=e^t-\sin 2t$. Find the distance equation for $x(t)$ if $v(0)=0$ and $x(0)=3$.

38. The acceleration of a particle is described by $a(t)=e^{2t}-\cos t$. Find the distance equation for $x(t)$ if $v(0)=0$ and $x(0)=3$.

39. The acceleration of a particle is described by $a(t)=e^{3t}-3t^2$. Find the distance equation for $x(t)$ if $v(0)=\frac{2}{3}$ and $x(0)=1$.

## 2.5 Multiple Choice Homework

1. $\int\left(x^3+2+\frac{1}{x^2+1}\right)dx=$

a) $\frac{x^4}{4}+2x+\tan^{-1}x+C$

b) $x^4+2+\tan^{-1}x+C$

c) $\frac{x^4}{4}+2x+\frac{3}{x^3+3}+C$

d) $\frac{x^4}{4}+2x+\tan^{-1}2x^2+C$

e) $4+2x+\tan^{-1}x+C$

---

2. $\int\cos(3-2x)dx=$

a) $\sin(3-2x)+C$

b) $-\sin(3-2x)+C$

c) $\frac{1}{2}\sin(3-2x)+C$

d) $-\frac{1}{2}\sin(3-2x)+C$

e) $-\frac{1}{5}\sin(3-2x)+C$

---

3. $\int\left(2-\sin\frac{t}{5}\right)^2\cos\frac{t}{5}\,dt =$

a) $\frac{1}{3}\left(2-\sin\frac{t}{5}\right)^3+c$ b) $\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3+c$

c) $-\frac{5}{3}\left(2-\sin\frac{t}{5}\right)^3+c$ d) $5\left(2-\sin\frac{t}{5}\right)^3+c$

e) $-\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3+c$

---

4. $\int\frac{e^{x^2}-2x}{e^{x^2}}\,dx =$

a) $x-e^{x^2}+C$ b) $x-e^{-x^2}+C$ c) $x+e^{-x^2}+C$

d) $-e^{x^2}+C$ e) $-e^{-x^2}+C$

---

5. $\int 6\sin x\cos^2 x\,dx =$

a) $2\sin^3 x+C$ b) $-2\sin^3 x+C$ c) $2\cos^3 x+C$

d) $-2\cos^3 x+C$ e) $3\sin^2 x\cos^2 x+C$

---

6. $\int \frac{4x}{1+x^2}\,dx =$

a) $4\arctan x + C$ b) $\frac{4}{x}\arctan x + C$ c) $\frac{1}{2}\ln\left(1+x^2\right) + C$

d) $2\ln\left(1+x^2\right) + C$ e) $2x^2 + 4\ln|x| + C$

---

7. $\int \frac{x}{4+x^2}\,dx$

a) $\tan^{-1}\frac{x}{2} + c$ b) $\ln\left(4+x^2\right) + c$ c) $\tan^{-1}x + c$

d) $\frac{1}{2}\ln\left(4+x^2\right) + c$ e) $\frac{1}{2}\tan^{-1}\frac{x}{2} + c$

---

8. $\int\left(2^x - 4e^{2\ln x}\right)dx$ $\int\left(2^x - 4e^{2\ln x}\right)dx$

a) $2^x\ln 2 - \frac{4}{3}e^{2\ln x} + c$ b) $x2^{x-1} - \frac{4}{3}x^3 + c$ c) $\frac{2^x}{\ln 2} - \frac{4}{3}e^{2\ln x} + c$

d) $x2^{x-1} - \frac{4}{3}e^{2\ln x} + c$ e) $\frac{2^x}{\ln 2} - \frac{4}{3}x^3 + c$

---

9. If $\frac{dy}{dx} = \cos x \sin^2 x$ and if $y = 0$ when $x = \pi$, what is the value of $y$ when $x =$ 0?

a) –1 b) $-\frac{1}{3}$ c) 0 d) $\frac{1}{3}$ e 1

---

10. The anti-derivative of $2\tan x$

a) $2\ln|\sec x| + c$ b) $2\sec^2 x + c$ c) $\ln|\sec^2 x| + c$

d) $2\ln|\cos x| + c$ e) $\ln|2\sec x| + c$

---

11. Which of the following statements are true?

I. $\int (x^5 \sin x^6) dx = -\frac{1}{6}\cos x^6 + c$ II. $\int \tan x dx = \sec^2 x + c$

III. $\int \left((x^3 + x)\sqrt[4]{x^4 + 2x^2 - 5}\right) dx = \frac{1}{5}(x^4 + 2x^2 - 5)^{5/4} + c$

a) I only b) II only c) III only d) I and II only e) II and III only

ab) I and III only ac) I, II, and III ad) None of these

---

12. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

a) $2e^{\sqrt{x}} + c$ b) $\frac{1}{2}e^{\sqrt{x}} + c$ c) $e^{\sqrt{x}} + c$

d) $2\sqrt{x}\, e^{\sqrt{x}} + c$ e) $\frac{e^{\sqrt{x}}}{2\sqrt{x}} + c$

---

13. If $x'(t) = 2t\cos t^2$, find $x(t)$ when $x\left(\sqrt{\frac{\pi}{2}}\right) = 3$

a) $x(t) = -4t^2\sin t^2$
b) $x(t) = -4t^2\sin t^2 + 2\cos t^2$
c) $x(t) = \sin t^2 + 3$
d) $x(t) = -\sin t^2 + 4$
e) $x(t) = \sin t^2 + 2$

---

14. A particle moves along the y-axis so that at any time $t \geq 0$, it velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \frac{\pi}{2}$ is $y = 3$, the particle's position at time $t = 0$ is

a) $-4$ b) 2 c) 3 d) 4 e) 6

---

15. If $\frac{dy}{dx} = \sin x \cos^3 x$ and if $y = 1$ when $x = \pi$, what is the value of $y$ when $x = 0$?

a) 0 b) 1 c) 1.5 d) 2 e) 2.5

---

16. If $\frac{dy}{dx} = \cos x \sin^2 x$ and if $y = 0$ when $x = \pi$, what is the value of $y$ when $x =$ 0?

a) -3 b) -2 c) 0 d) 2 e) 3

---

17. A particle moves along the $x$-axis so that at any time $t \geq 0$, it acceleration is given $a(t) = -4\sin(2t)$. If $v(0) = 7$ and $x(0) = 0$, then the particle's position equation is

a) $x(t) = \sin(2t) + 5t$ b) $x(t) = \sin(2t) + 7t$

c) $x(t) = \sin(2t) + 9t$ d) $x(t) = 16\sin(2t) + 7t$

---

18. An object moves with velocity $v(t) = \sec^2(2t)$. It is known that the particle's position at time 0 is 2. What is the particle's position function?

a) $s(t) = \tan(2t) + 2$
b) $s(t) = \frac{1}{2}\tan(2t) + 2$
c) $s(t) = \sec^2(2t)\tan^2(2t) + 2$
d) $s(t) = \ln|\sec(2t)| + 2$
e) $s(t) = \frac{1}{2}\ln|\sec(2t)| + 2$

---

19. The acceleration of a particle is given by $a(t) = 4e^{2t}$. When $t = 0$, the position of the particle is $x = 2$ and $v = -2$. Determine the position of the particle at $t = \frac{1}{2}$.

a) $e - 3$ b) $e - 2$ c) $e - 1$ d) $e$ e) $e + 1$

---

## 2.6 Separable Differential Equations

Up until now, we have been somewhat cavalier with our use of notation because we have been emphasizing the concept of the integral as an antiderivative. Now we want to be more attentive to how the symbols – especially $\frac{dy}{dx}$ – work. An equation with $\frac{dy}{dx}$ in it is called a differential equaion. What we hde been doing with the antiderivative was isolating *y* in the differential equation. Let us look at Example 1 from 2.1 as an illustration:

Ex 1 (2.1) $\int\left(3x^2+4x+5\right)dx=$

Our interpretation then was that $\frac{dy}{dx}=3x^2+4x+5$ and we needed to solve for *y*.

How that works is

$$\frac{dy}{dx}=3x^2+4x+5$$
$$dy=\left(3x^2+4x+5\right)dx$$
$$\int dy=\int\left(3x^2+4x+5\right)dx$$
$$y=x^3+2x^2+5x+c$$

Notice that the Leitniz symbol $\frac{dy}{dx}$ is a fraction and cross-multiplication can be performed. The symbols $dy$ and $dx$ are called *differentials*.

What will be explored next is differential equations containing both ***x* and *y***, as well as $\frac{dy}{dx}$. Some can be solved easily and some cannot. The ones that can be solved easily are called *Separable Differential Equations.*

**Vocabulary:**

*Differential Equation* – an equation that contains a derivative.
*General Solution* – All of the *y*-equations that would have the given equation as their derivative. Note the +*C* which gives multiple equations.
*Initial Condition* – Constraint placed on a differential equation; sometimes called an initial value.
*Particular Solution* – Solution obtained from solving a differential equation when an initial condition allows you to solve for C.
*Separable Differential Equation* – A differential equation in which all terms with *y*'s can be moved to the left side of an equals sign ( = ), and in which all terms with *x*'s can be moved to the right side of an equals sign ( = ), by multiplication and division only.

OBJECTIVES

Given a separable differential equation, find the general solution.
Given a separable differential equation and an initial condition, find a particular solution.

Ex 1 Find the general solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

| | |
|---|---|
| $\frac{dy}{dx} = -\frac{x}{y}$ | Start here. |
| $ydy = -xdx$ | Separate all the *y* terms to the left side of the equation and all of the *x* terms to the right side of the equation. |
| $\int ydy = \int -xdx$ | Integrate both sides. |
| $\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$ | You only need *C* on one side of the equation and we put it on the side containing the *x*. |
| $y^2 = -x^2 + C$ | Multiply both sides by 2. Note: 2*C* is still a constant, so we'll continue to note it just by *C*. |

$x^2 + y^2 = C$ — This equation should seem familiar. It's the family of circles centered at the origin with radius $\sqrt{C}$.

$y = \pm\sqrt{C - x^2}$ — Isolate *y*.

Also to note, we could check our solution by taking the derivative of our solution.

$x^2 + y^2 = C \Rightarrow 2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$.

Ex 2 Find the particular solution to $\frac{dy}{dx} = x^2y^3$, given $y(0) = 1$.

$$\frac{dy}{dx} = x^2y^3$$

$$y^{-3}dy = x^2dx$$

$$\frac{y^{-2}}{-2} = \frac{x^2}{2} + c$$

$$y(0) = 1 \to -\frac{1}{2} = 0 + c \to -\frac{1}{2} = c$$

$$\frac{y^{-2}}{-2} = \frac{x^2}{2} - \frac{1}{2}$$

Now isolate *y:*

$$\frac{1}{y^2} = -x^2 + 1$$

$$y^2 = \frac{1}{1 - x^2}$$

$$y = \frac{1}{\pm\sqrt{1 - x^2}}$$

Since $x = 0$ gave us $y = positive\ 1$, $y = \dfrac{1}{\sqrt{1 - x^2}}$

Ex 3 Find the particular solution to $\frac{dy}{dx} = x\sec y$, given $y(0) = \frac{\pi}{2}$.

$$\frac{dy}{dx} = x\sec y$$
$$\frac{1}{\sec y}dy = xdx$$
$$\int \cos y\, dy = \int xdx$$
$$\sin y = \frac{1}{2}x^2 + c$$

$$y(0) = \frac{\pi}{2} \rightarrow \sin\frac{\pi}{2} = \frac{1}{2}(0)^2 + c \rightarrow 1 = c$$

$$\sin y = \frac{1}{2}x^2 + 1$$
$$y = \sin^{-1}\left(\frac{1}{2}x^2 + 1\right)$$

Ex 4 Find the particular solution to $\frac{dy}{dx} = x^2 y$, given $y(0) = -2$.

$$\frac{dy}{dx} = x^2 y$$
$$\frac{dy}{y} = \left(x^2\right)dx$$
$$\ln|y| = \frac{1}{3}x^3 + c$$
$$|y| = e^{\left(\frac{1}{3}x^3 + c\right)} = e^{\left(\frac{1}{3}x^3\right)}e^c$$
$$y = Ke^{\left(\frac{1}{3}x^3\right)}$$

$$y(0) = -2 \rightarrow -2 = Ke^0 \rightarrow K = -2$$

$$y = -2e^{\left(\frac{1}{3}x^3\right)}$$

Note that, while $e^c$ is always positive, hence the absolute values on $y$, $K$ and be a negative number. This is why we solve for $K$ rather than solving for $c$.

**Steps to Solving a Differential Equation:**

1. Separate the variables, **using multiplication and division only**. Note: Leave constants on the right side of the equation.
2. Integrate both sides of the equation.
3. Don't forget the + *C*. Note: only write the +*C* on the right side of the equation.
4. Plug in the initial condition to solve for *C* and put it in the equation. If your integration resulted in a natural log, do step 5 before step 4.
5. Isolate *y*.

Note: Solve for *C* immediately if the left integral does not result in a ln. Simplify before solving if there is a ln.

Ex 5 Find the particular solution to $y' = 2xy - 3y$, given $y(3) = 2$.

$$\frac{dy}{dx} = (2x-3)y$$
$$\frac{dy}{y} = (2x-3)dx$$
$$\ln|y| = x^2 - 3x + C$$
$$y = e^{x^2-3x+C} = e^{x^2-3x}e^C = Ke^{x^2-3x}$$
$$y(3) = 2 \rightarrow 2 = Ke^0$$
$$K = 2$$
$$y = 2e^{x^2-3x}$$

Ex 6 Find the particular solution to $\frac{dr}{dt} = \frac{3t^2 - \sin t}{4r}$ given that $r(0) = 3$.

$$\frac{dr}{dt} = \frac{3t^2 - \sin t}{4r}$$
$$4r\,dr = \left(3t^2 - \sin t\right)dt$$
$$2r^2 = t^3 + \cos t + C$$
$$2 \cdot 3^2 = 0^3 + \cos 0 + C \Rightarrow C = 17$$
$$2r^2 = t^3 + \cos t + 17$$
$$r = \pm\sqrt{\frac{t^3}{2} + \frac{1}{2}\cos t + \frac{17}{2}}$$
$$r = \sqrt{\frac{t^3}{2} + \frac{1}{2}\cos t + \frac{17}{2}}$$

Again, you can check your solution by taking the derivative of your solution.

Ex 7   AP 1998 BC #4

## 2.6 Free Response Homework

Find the general solution each of the following differential equations.

1. $\frac{dy}{dx} = \frac{y}{x}$

2. $\frac{dy}{dx} = xy^2$

3. $\left(x^2+1\right)\frac{dy}{dx} = xy$

4. $\frac{dy}{dx} = \left(3y^2\right)\frac{1}{1+x^2}$

5. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3-3}}{y^2}$

6. $\frac{dy}{dt} = \frac{\sec^2 t}{ye^{5y^2}}$

7. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

8. $\frac{dy}{dx} = \frac{x^2+1}{\sec y \tan y}$

9. $\frac{dy}{dx} = 4xy^3$

10. $\frac{dy}{dx} = y^2 \cos x$

11. $\frac{dv}{dt} = 2+2v+t+tv$

12. $\frac{dy}{dt} = \frac{t}{y\sqrt{y^2+1}}$

13. $\frac{d\theta}{dr} = \frac{1+\sqrt{r}}{\sqrt{\theta}}$

Find the solution of the differential equation that satisfies the given initial condition.

14. $\frac{dy}{dx} = xy^2;\ y(0)=5$

15. $\frac{dy}{dx} = \frac{2x}{y};\ y(0)=1$

16. $\frac{dy}{dx} = \frac{2x^3}{3y^2};\ y\left(\sqrt{2}\right)=0$

17. $\frac{dy}{dx} = \left(x^2+1\right)\left(2-y\right);\ y(1)=3$

18. $\frac{dy}{dx} = \left(y^2+1\right);\ y(1)=0$

19. $\frac{dy}{dx} = \frac{xy^2+x}{y};\ y(0)=-1$

20. $\frac{du}{dt}=\frac{2t+\sec^2 t}{2u}$; $u(0)=-5$

21. $\frac{dy}{dx}=yx-y\sin x$; $y(0)=5e$

22. Solve the initial-value problem $\frac{dy}{dx}=\frac{\sin x}{\sin y}$; $y(0)=\frac{\pi}{2}$.

23. Solve the equation $e^{-y}\frac{dy}{dx}+\cos x=0$.

24. Find an equation of the curve that satisfies $\frac{dy}{dx}=4x^3y$ and whose $y$-intercept is 7.

## 2.6 Multiple Choice Homework

1. Find the particular solution to $\frac{dy}{dx}=\frac{x+1}{e^y}$, where $y(0)=2$.

a) $y=2+e^{\left(\frac{1}{2}x^2+x\right)}$  b) $y=2e^{\left(\frac{1}{2}x^2+x\right)}$  c) $y=\ln\left|\left(\frac{1}{2}x^2+x+1\right)\right|+2$

d) $y=\ln\left|\left(\frac{1}{2}x^2+x+e^2\right)\right|$  e) $y=2+\ln\left(\ln\left|\left(\frac{1}{2}x^2+x+1\right)\right|\right)$

---

2. Find the particular solution to $\left(x^2+1\right)\frac{dy}{dx}=y$, where $y(0)=2$.

a) $y=2e^{\tan^{-1}x}$  b) $y=e^{\tan^{-1}x}$  c) $y=e^{\left(\tan^{-1}x\right)\ln 2}$

d) $y=\sqrt{2\tan^{-1}x}$  e) $y=\sqrt{2\tan^{-1}x+4}$

---

3. Identify is the first mistake (if any) in this process:

$$\frac{dy}{dx} = xy + x$$

Step 1: $\frac{1}{y+1}dy = xdx$

Step 2: $\ln|y+1| = x^2 + c$

Step 3: $|y+1| = e^{x^2} + c$

Step 4: $y = e^{x^2} + c$

a) Step 1 b) Step 2 c) Step 3

d) Step 4 e) There is no mistake.

---

4. Identify is the mistake (if any) in this process:

$$\frac{dy}{dx} = 6x^2y^2$$

Step 1: $\frac{1}{y^2}dy = 6x^2dx$

Step 2: $\ln|y^2| = 2x^3 + c$

Step 3: $y^2 = e^{2x^3+c}$

Step 4: $y = \pm\sqrt{ke^{2x^3}}$

a) Step 1 b) Step 2 c) Step 3

d) Step 4 e) There is no mistake.

---

5. The solution to the differential equation $\frac{dy}{dx} = 8xy$ with initial condition $y(0) = 5$ is

a) $\ln(4x^2 + 5)$ b) $e^{4x^2} + 5$ c) $e^{4x^2} + 4$

d) $5\ln(4x^2)$ e) $5e^{4x^2}$

---

6. Find the general solution to $\frac{dy}{dx} = \frac{3y}{2x+1}$.

a) $\frac{3}{2}y^2 = x^2 + x + c$ b) $y = \frac{c}{2}\ln|2x+1|$

c) $y = c(2x+1)^{3/2}$ d) $3y^2 = \frac{1}{2}\ln|2x+1| + c$

e) $y = \frac{3}{2}\ln|2x+1| + c$

---

7. Find the general solution to $\frac{dy}{dx} = y(2x+1)$ where $y(0) = 2$.

a) $y = 2e^{(x^2+x)}$ b) $y = \frac{2\ln y}{x^2 + \ln 2}$ c) $y = 2e^{(2x^2+x)}$

d) $y = 1 + e^{(x^2+x)}$ e) $y = e^{x^2} + e^x$

---

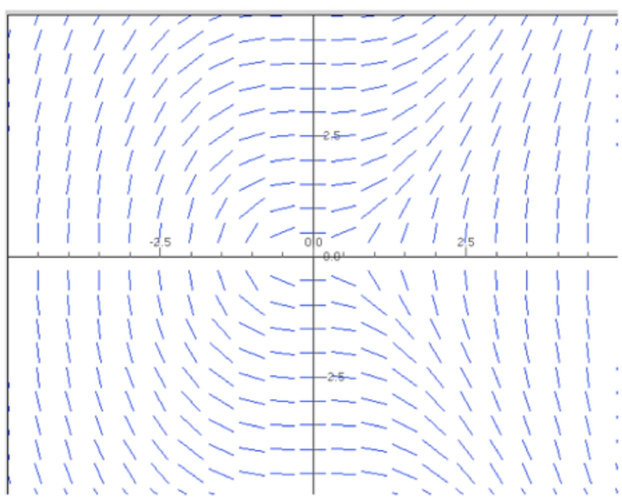
## 2.7 Intro to AP: Slope Fields

This section is very much an AP-driven section. The underlying philosophy is that math should be understood and explained algebraically, graphically, numerically and verbally. The topic of differential equations fits nicely into this paradigm in that the visual is a graphical representation and the connection between the equation and the slopes is a numerical process.

**Vocabulary:**

*Slope field* – Given any function *f*, a slope field is drawn by taking evenly spaced points on the Cartesian coordinate system (usually points having integer coordinates) and, at each point, drawing a small line with the slope of the function.

Here is an example of a slope field:



The line segments represent the slopes of the lines tangent to the solution curve at the specific points where each is drawn.

Objectives:

- Given a differential equation, sketch its slope field.
- Given a slope field, sketch a particular solution curve.
- Given a slope field, determine the family of functions to which the solution curves belong.
- Given a slope field, determine the differential equation that it represents.

There are four ways that the AP Exam usually approaches Slope Fields:

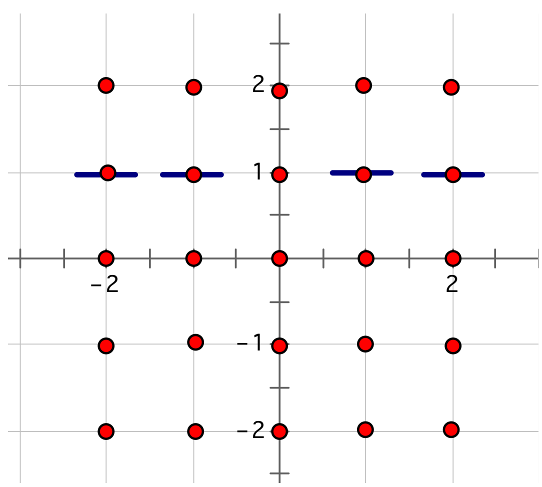
1. Draw a Slope Field (free response)
2. Sketch the solution to a Slope Field (free response)
3. Identify the solution equation to a Slope Field (multiple choice)
4. Identify the differential equation for a Slope Field (multiple choice)

Two of these questions (#2 and #3) are graphically oriented and two (#1 and #4) are numerically based.

## 1. Slope Fields Numerically (FRQs)

Ex 1 Sketch the slope field for $\frac{dy}{dx}=\frac{1-y}{x}$ at the points indicated.

Note that, wherever $y=1$, $\frac{dy}{dx}=0$. So, the segments at $y=1$ will be horizontal. Also, where $x=0$, $\frac{dy}{dx}=\text{dne}$. So, there are not segments on the $x$-axis.



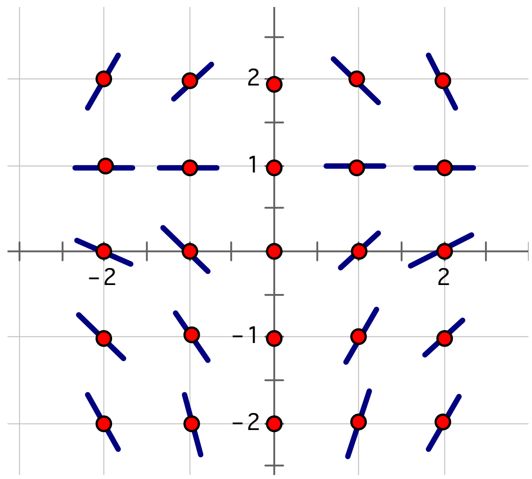

We can then plug the numerical values of the points into $\frac{dy}{dx}=\frac{1-y}{x}$ to determine the slant of the line segments.

$$(-2,2)\rightarrow \frac{dy}{dx}=\frac{1}{2} \qquad (2,2)\rightarrow \frac{dy}{dx}=-\frac{1}{2}$$

$$(-2,0)\rightarrow \frac{dy}{dx}=-\frac{1}{2} \qquad (2,0)\rightarrow \frac{dy}{dx}=\frac{1}{2}$$

$$(-2,-2)\rightarrow \frac{dy}{dx}=-\frac{3}{2} \qquad (2,-2)\rightarrow \frac{dy}{dx}=\frac{3}{2}$$

$$etc.$$

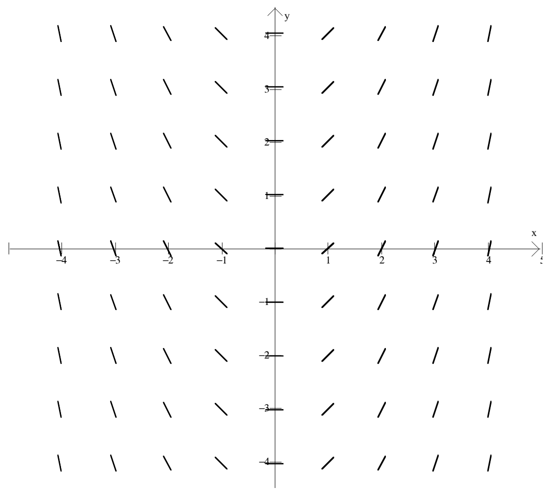



<u>Steps to Sketching a Slope Field:</u>

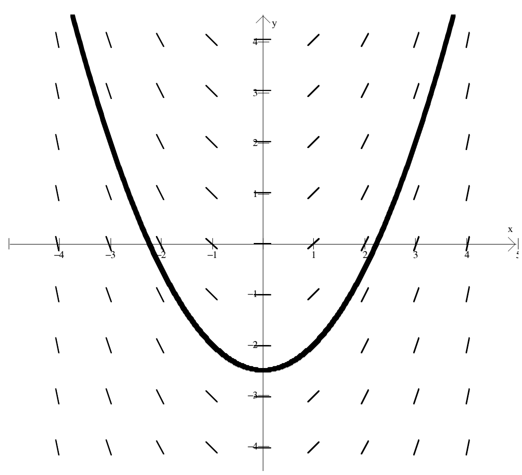
1. Determine the grid of points for which you need to sketch (many times the points are given).
2. Pick your first point. Note its $x$ and $y$ coordinate. Plug these numbers into the differential equation. This is the slope at that point.
3. Find that point on the graph. Make a little line (or dash) at that point whose slope represents the slope that you found in Step 2.
4. Repeat this process for all the points needed.

## 2. Slope Fields Graphically (FRQs):

Ex 2 Given the slope field of $\frac{dy}{dx} = x$ below, sketch the particular solution given the initial condition of $(3, 2)$.



Starting at $(3, 2)$ and following the slope segments, the graph appears thus:



Ex 3 Given the slope field for $x\frac{dy}{dx} = 1$ below, sketch the particular solution given the initial condition of $(1, 3)$.

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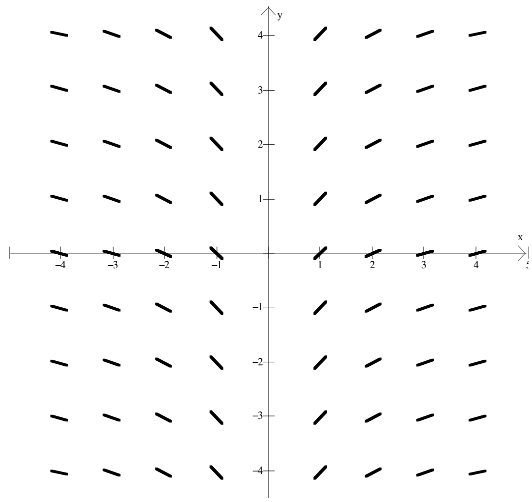
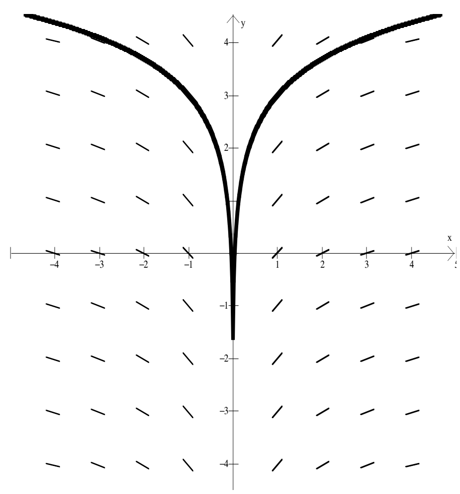


Starting at $(1, 3)$ and following the slope segments, the graph appears thus:



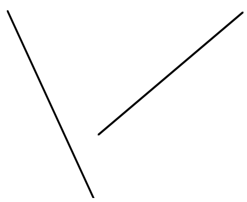
Note that there appears to be a vertical asymptote at $y = 0$.

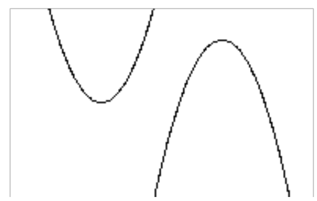
## 3. Slope Fields Graphically (MCQs)

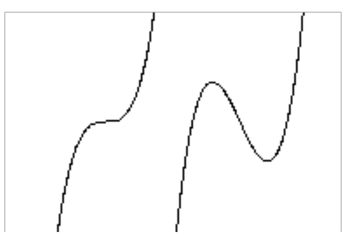
For these problems, one would sketch a solution and decide from among the options based on what was learned about families of functions in PreCalulus.

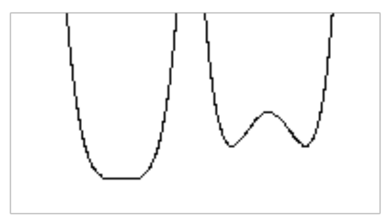
I. *Polynomial*s

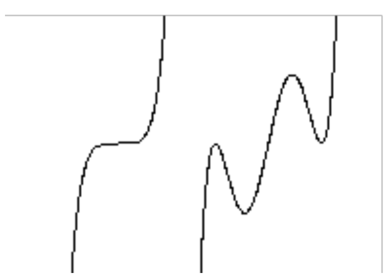
- Defn: "An expression containing no other operations than addition, subtraction, and multiplication performed on the variable."
- Means: any equation of the form $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$, where $n$ is a non-negative integer.
- **Most Important Traits: Zeros (*x*-intercepts) and Extreme Points.**





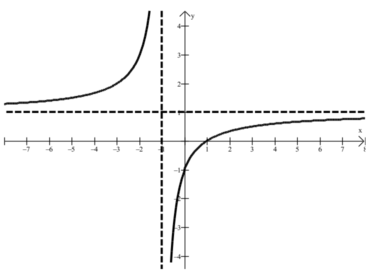
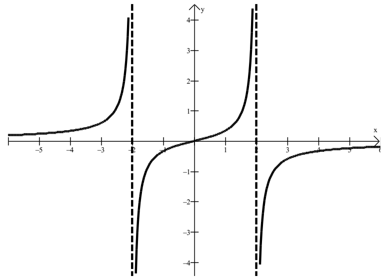
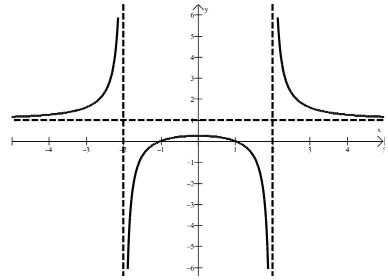
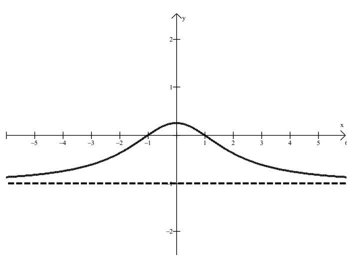
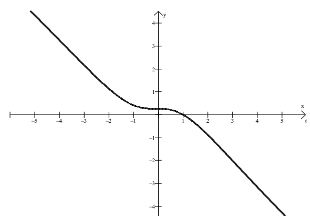
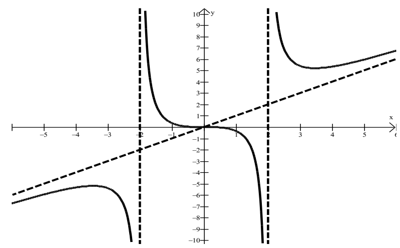






II. *Rationals*

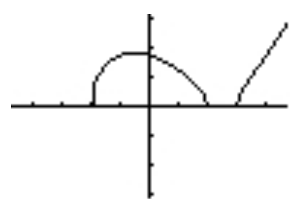
- Defn: "An expression that can be written as the ratio of one polynomial to another."
- Means: an equation with an $x$ in the denominator.
- **Most Important Traits: Zeros vs. VAs vs. POEs and End Behavior.**

III. *Radicals* (*Irrationals*)

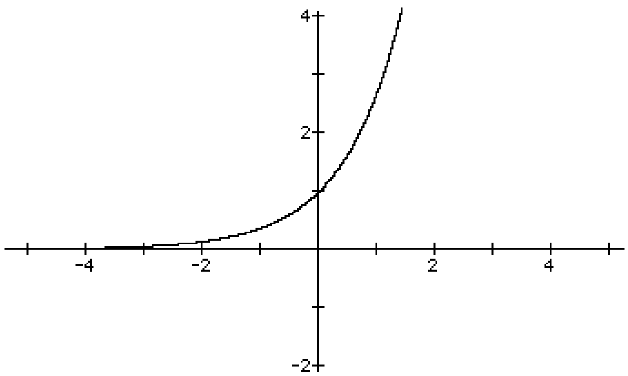
- Defn: "An expression whose general equation contains a root of a variable and possibly addition, subtraction, multiplication, and/or division."
- Means: An equation with an $x$ in a radical.
- **Most Important Traits: Domain and Extreme Points.**

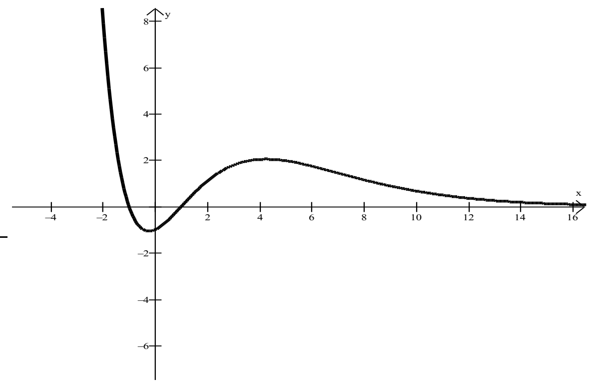




IV. *Exponentials*

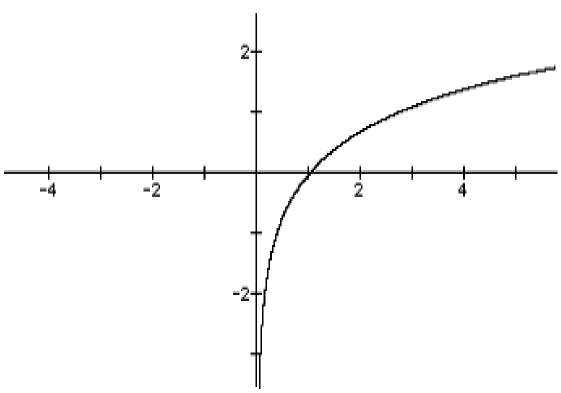
- Defn: "A function whose general equation is of the form $y = a \cdot b^x$."
- Means: there is an *x* in the exponent.
- **Most Important Traits: Extreme Points and End Behavior.**

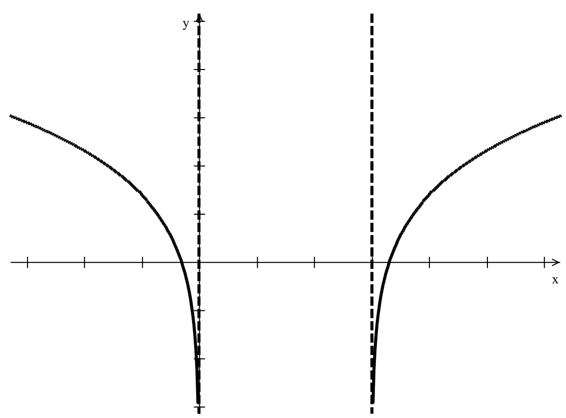



V. *Logarithmic Functions*

- Defn: "The inverse of an exponential function."
- Means: there is a Log or Ln in the equation.
- Most Important Traits: Domain, VAs, and Extreme Points.



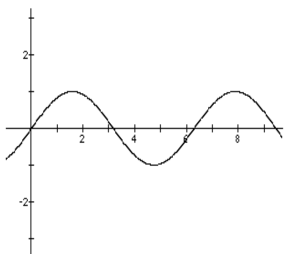



VI. *Trigonometric Functions*

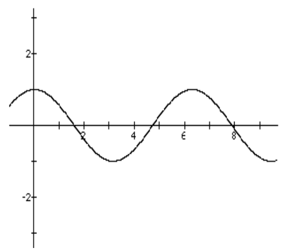
Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."

Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

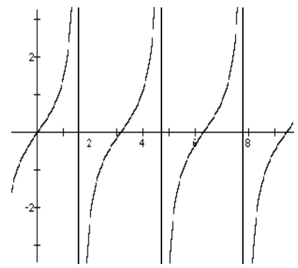
- **Most Important Traits: VAs, Axis Points, and Extreme Points.**



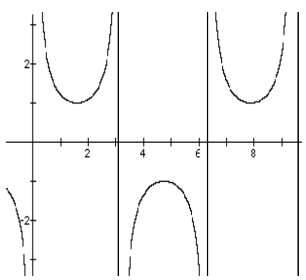
$y = \sin x$



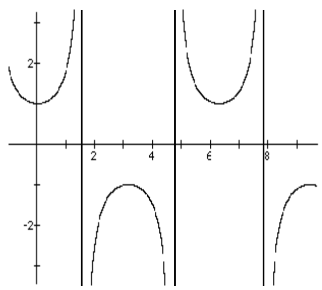
$y = \cos x$



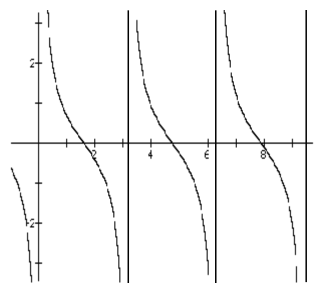
$y = \tan x$



$y = \csc x$



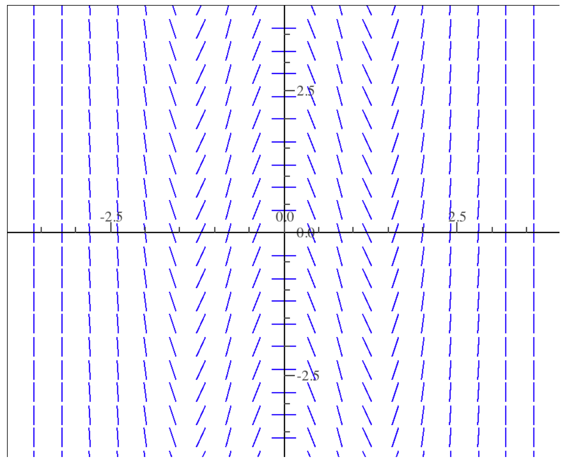
$y = \sec x$



$y = \cot x$

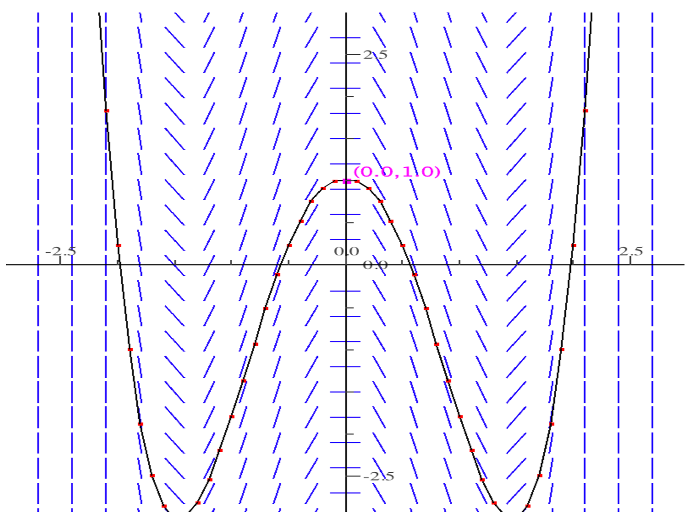
Finding the solution equation entails looking at the pattern of the slopes and matching it against the graphs we know.

Ex 4 Which of the following equations might be the solution to the slope field shown in the figure below?



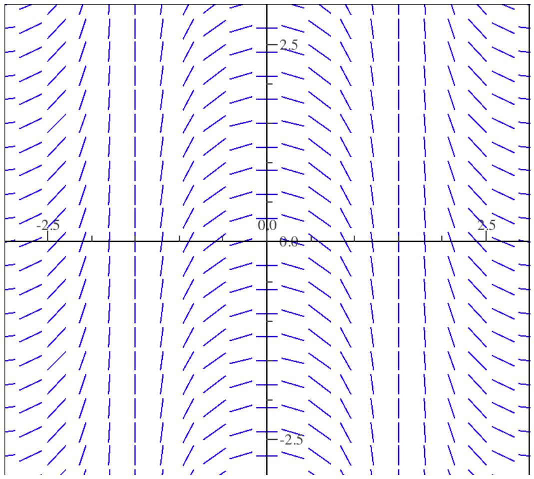
a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:



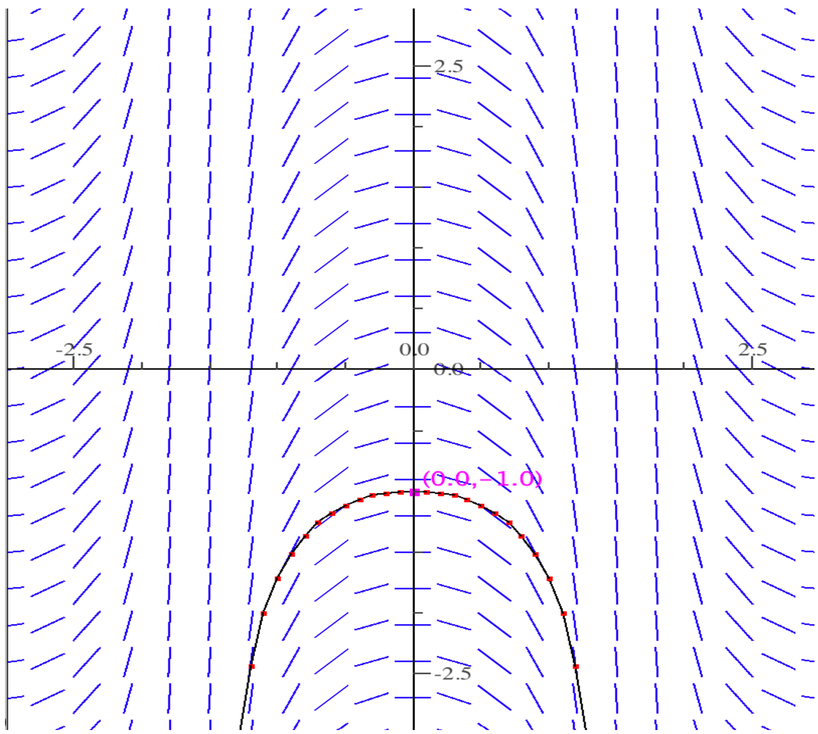

This is a quartic function, so the answer is a) $y = x^4 - 4x^2$

Ex 5   Which of the following equations might be the solution to the slope field shown in the figure below?



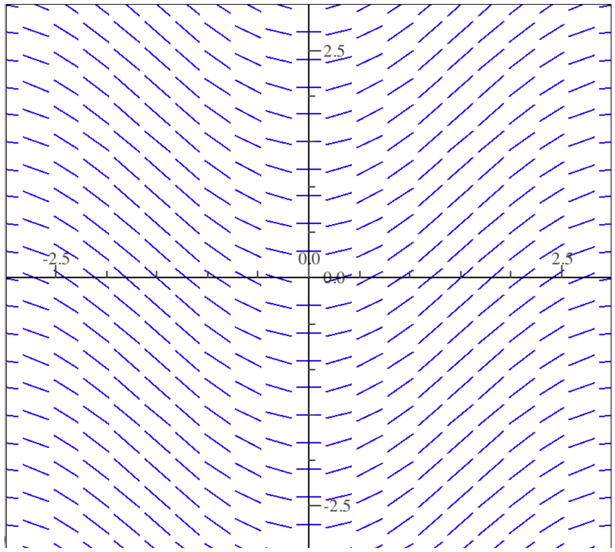

a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:



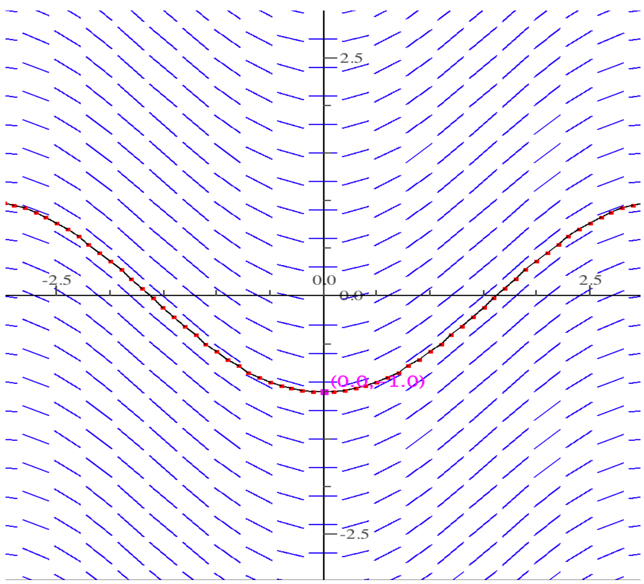

This is a secant curve, so the answer is d) $y = -\sec x$

Ex 6 Which of the following equations might be the solution to the slope field shown in the figure below?




a) $y = x^4 - 4x^2$ b) $y = 4x - x^3$ c) $y = -\cos x$ d) $y = -\sec x$

Tracing along the slope segments we see one solution curve is:




This is a negative cosine wave, so the answer is c) $y = -\cos x$

## 4. Slope Fields Numerically (MCQs)

Let's summarize what we know about slopes of lines in terms of numbers:

1. Horizontal lines have $\frac{dy}{dx} = 0$

2. Vertical lines have $\frac{dy}{dx} = dne$

3. Lines with positive slopes go up from left to right

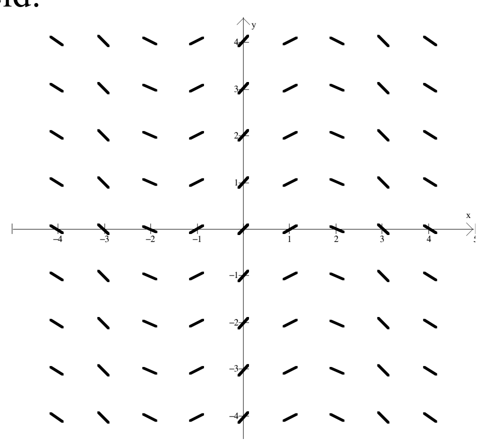
4. Lines with negative slopes go down from left to right

Two other facts are apparent from viewing a slope field and its differential equation:

5. If all Dashes in each column are // to each other, then $\frac{dy}{dx}$ has no *y*.

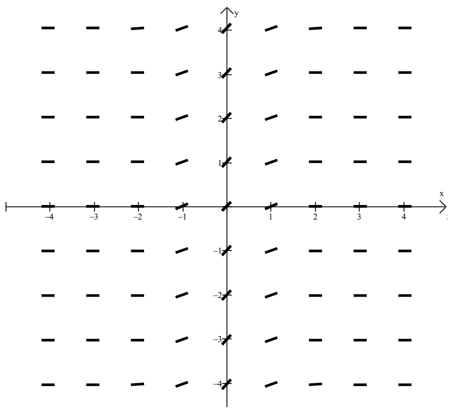
6. If all Dashes in each row are // to each other, then $\frac{dy}{dx}$ has no *x*.

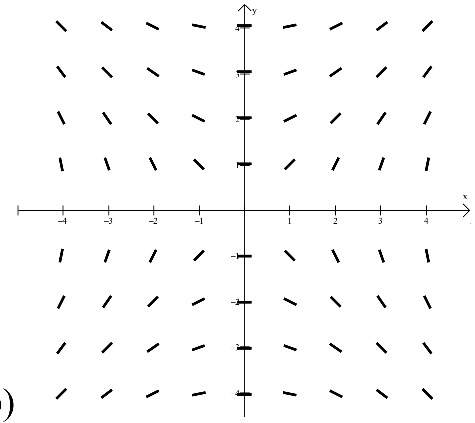
So, consider this slope field:

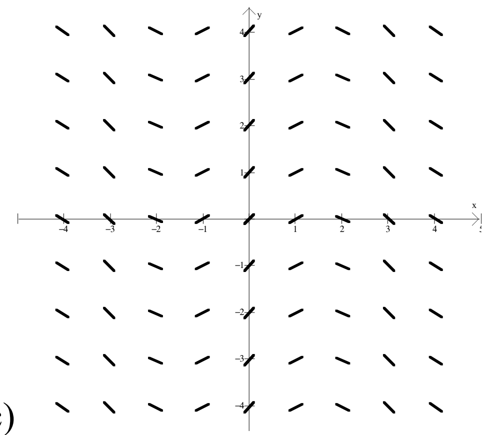


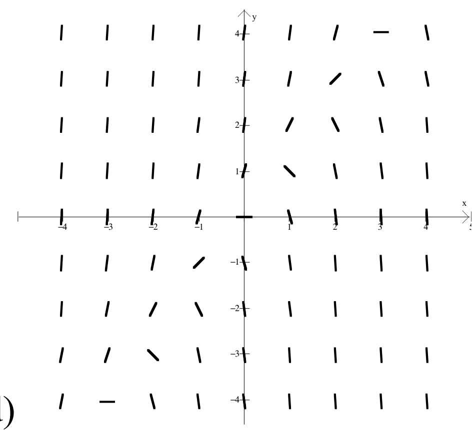
The differential equations that yields this would not have a *y* in the equation because of the segments being parallel in. each column.

Ex 7 Which of the following slope fields matches $\frac{dy}{dx} = 3y - 4x$.

a) 

b) 

c) 

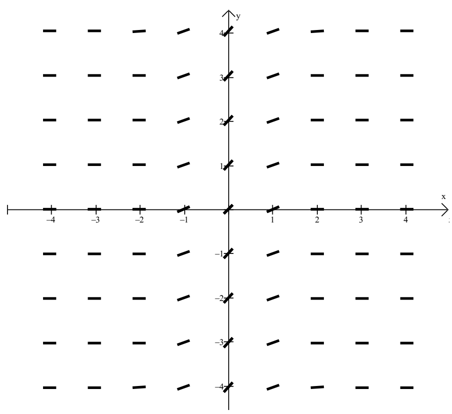
d) 

a) and c) have all slopes in each column parallel to one another, therefore, there is no $y$ in those equations. Neither can be the $\frac{dy}{dx} = 3y - 4x$.

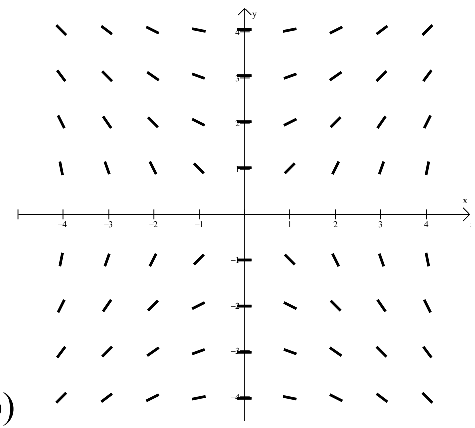
b) appears to have horizontal slopes at $x = 0$, but $\frac{dy}{dx} = 3y - 4x$ does not always equal 0 at $x = 0$. b) cannot be the slope field for $\frac{dy}{dx} = 3y - 4x$.

Therefore, by process of elimination, the correct answer is d).

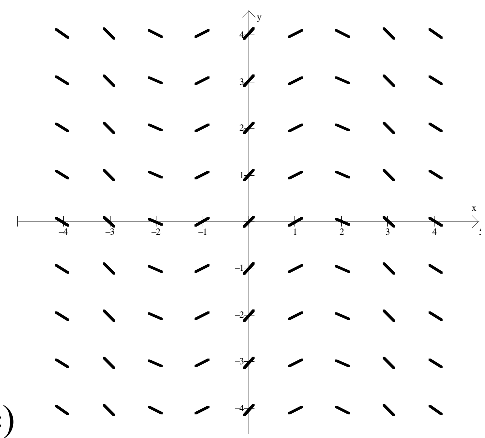
Ex 8   Which of the following slope fields matches $\frac{dy}{dx} = e^{-x^2}$.



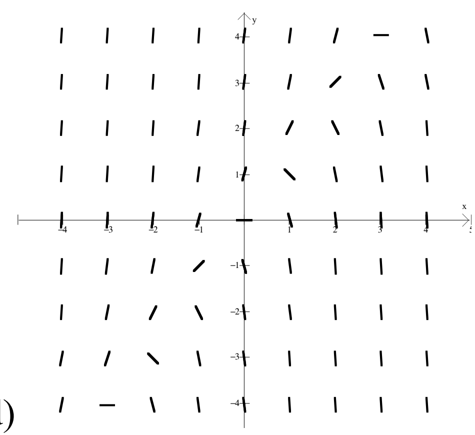
a)



b)



c)



d)

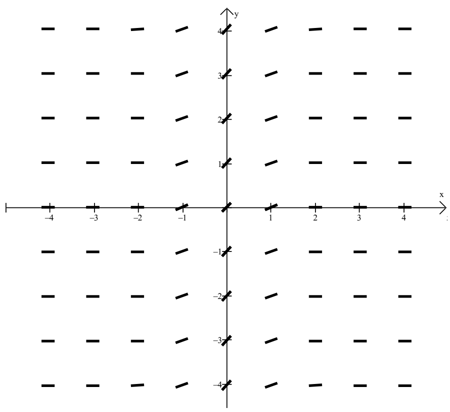
$\frac{dy}{dx} = e^{-x^2}$ has no $y$ in the equation, therefore, the segments in each column must be parallel to each other.  The answer must be either a) or c).

In c), the slopes at $x = 4$ are negative, but $\frac{dy}{dx} = e^{-4^2}$ is positive.

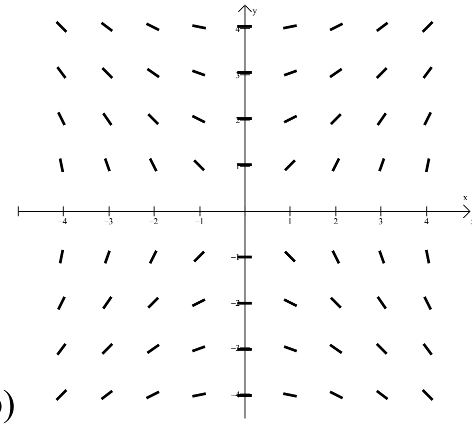
Again, by process of elimination, the correct answer is a).

Ex 9 Which of the following slope fields matches $\frac{dy}{dx}=\frac{x}{y}$?

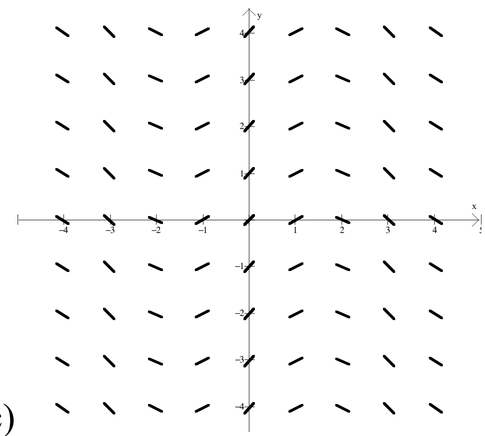
a)



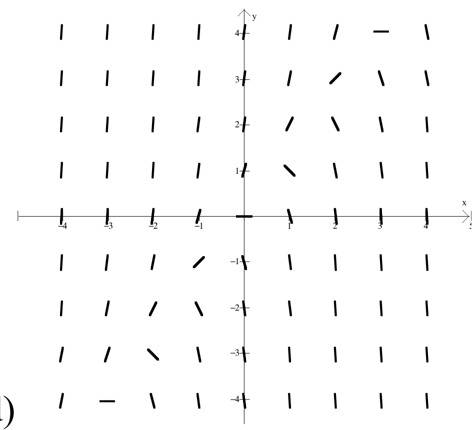
b)



c)



d)



$\frac{dy}{dx}=\frac{x}{y}$ has both $x$ and $y$ in it, therefore there cannot be parallel slopes in columns or rows. A) and c) must be wrong.

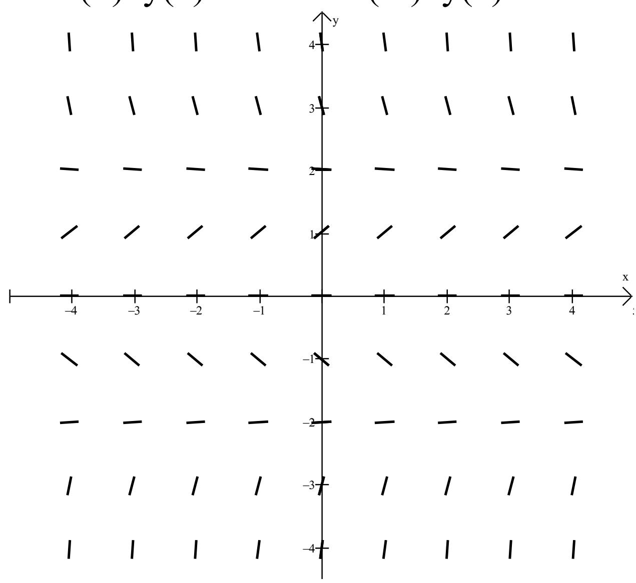
$\frac{dy}{dx}=\frac{x}{y}=0$ when $x=0$, so the answer must be B

## 2.7 Free Response Homework

1. A slope field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$ is shown.

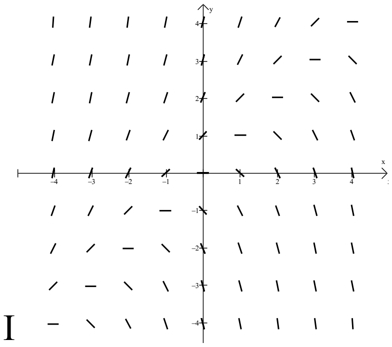
(a) Sketch the graphs of the solutions that satisfy the given initial conditions.

(i) y(0) = 1 (ii) y(0) = -1 (iii) y(0) = -3 (iv) y(0) = 3

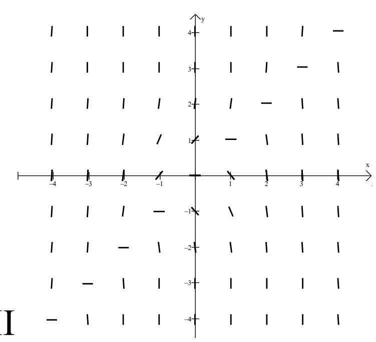


Match the differential equation with its slope field (labeled I-IV). Give reasons for your answer.

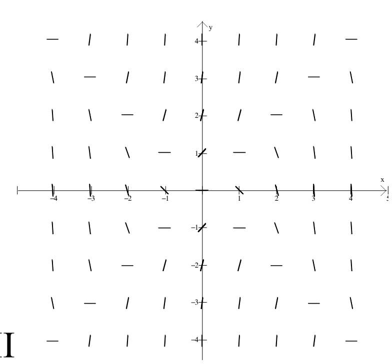
2. $\frac{dy}{dx} = y - 1$ 3. $\frac{dy}{dx} = y - x$ 4. $\frac{dy}{dx} = y^2 - x^2$ 5. $\frac{dy}{dx} = y^3 - x^3$



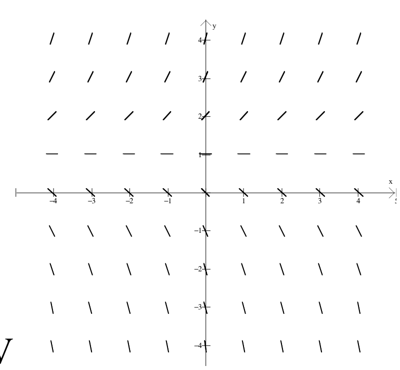
I



II



III



IV

6. Use the slope field labeled I (for exercises 2-5) to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) y(0) = 1 (b) y(0) = 0 (c) y(0) = -1

7. Sketch a slope field for the differential equation. Then use it to sketch three solution curves.

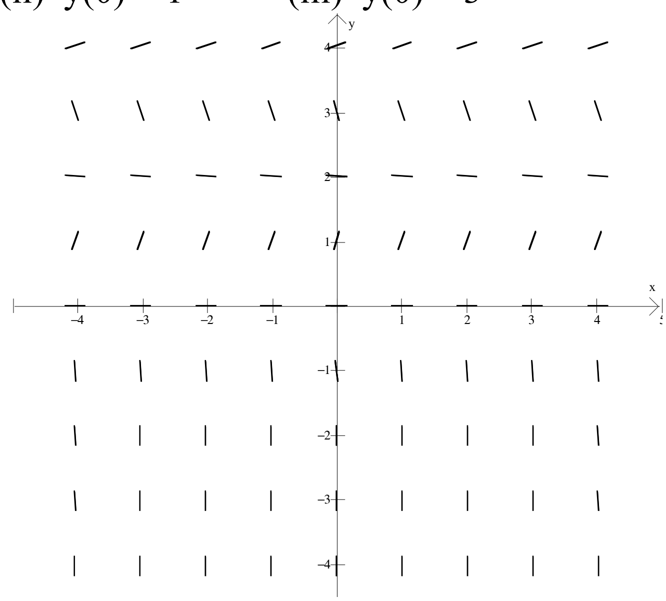
$y' = 1 + y$

8. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.

$y' = y - 2x$; (1,0)

9(a). A slope field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.

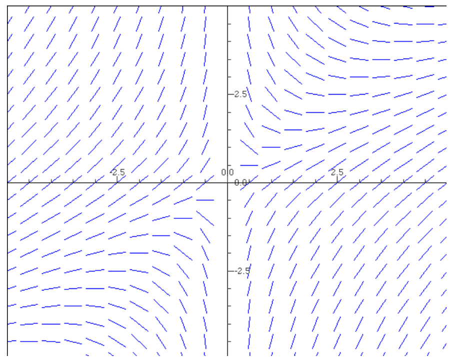
(i) y(0) = -0.3 (ii) y(0) = 1 (iii) y(0) = 3 (iv) y(0) = 4.3




9(b). If the initial condition is y(0) = c, for what values of c is $\lim_{t \to \infty} y(t)$ finite?

**2.7 Multiple Choice Homework**

1. Which of the following differential equations corresponds to the slope field shown in the figure below?

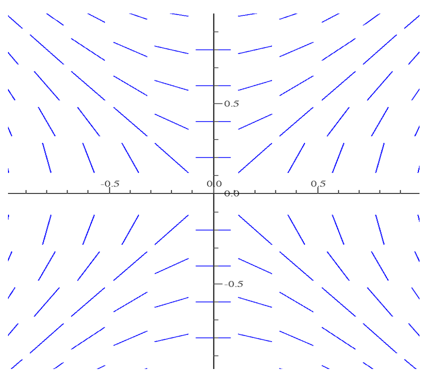


a) $\frac{dy}{dx} = -\frac{y^2}{x}$ b) $\frac{dy}{dx} = 1 - \frac{y}{x}$ c) $\frac{dy}{dx} = -y^3$

d) $\frac{dy}{dx} = x - \frac{1}{2}x^3$ e) $\frac{dy}{dx} = x + y$

---

2. Which of the following differential equations corresponds to the slope field shown in the figure below?

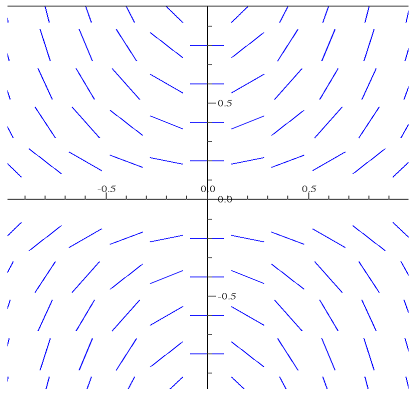


a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $\frac{dy}{dx} = 5xy$ c) $\frac{dy}{dx} = \frac{1}{10}xy$

d) $\frac{dy}{dx} = \frac{y}{x}$ e) $\frac{dy}{dx} = \frac{x}{y}$

---

3. Which of the following differential equations corresponds to the slope field shown in the figure below?

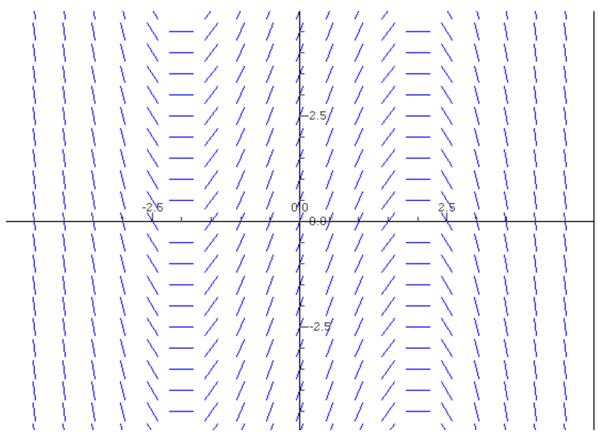



a) $\frac{dy}{dx} = -\frac{y}{x}$ b) $\frac{dy}{dx} = 5xy$ c) $\frac{dy}{dx} = \frac{1}{10}xy$

d) $\frac{dy}{dx} = \frac{y}{x}$ e) $\frac{dy}{dx} = \frac{x}{y}$

---

4. Which of the following equations might be the solution to the slope field shown in the figure below?

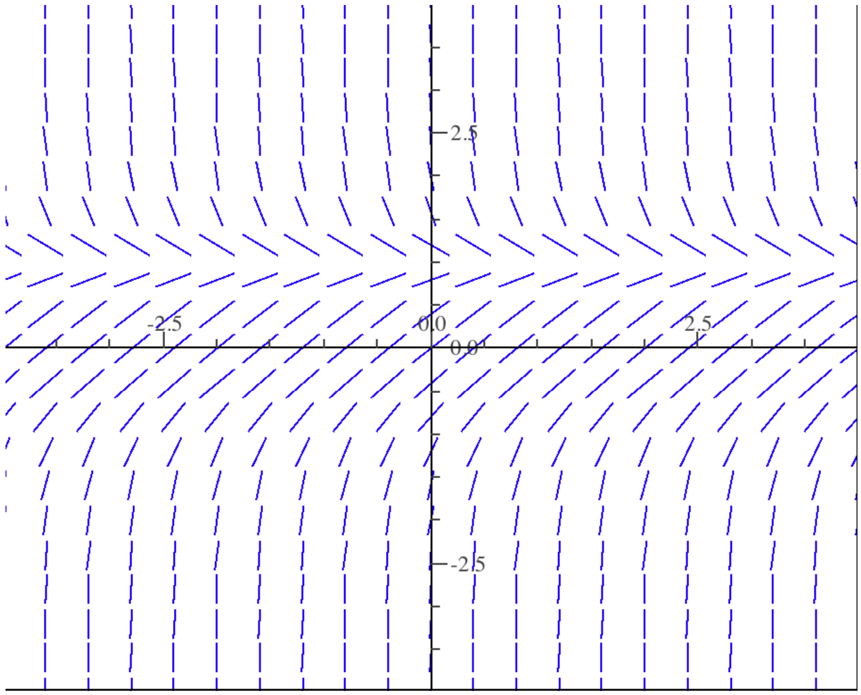



a) $y = 12x - x^3$ b) $y = -\cos x$ c) $y = \sec x$

d) $x = -y^2$ e) $x = -y^3$

---

5. Which of the following differential equations corresponds to the slope field shown in the figure below?

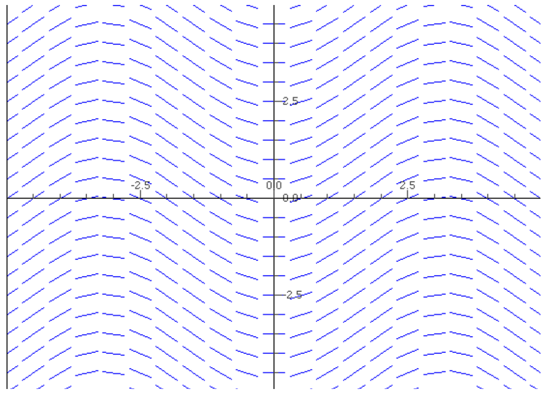



a) $\dfrac{dy}{dx} = 1 - y^3$ b) $\dfrac{dy}{dx} = y^2 - 1$ c) $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$

d) $\dfrac{dy}{dx} = x^2 y$ e) $\dfrac{dy}{dx} = x + y$

---

6. Which of the following equations might be the solution to the slope field shown in the figure below?

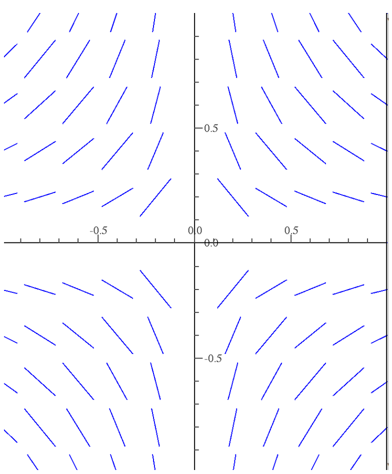


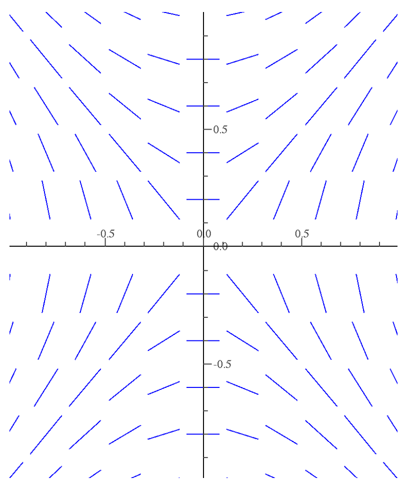
a) $y = 4x - x^3$  b) $y = -\cos x$  c) $y = \sec x$

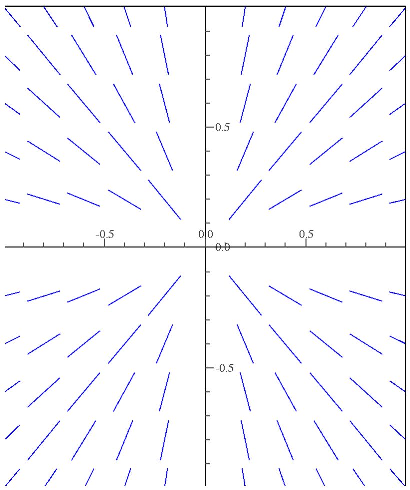
d) $x = -y^2$  e) $x = -y^3$

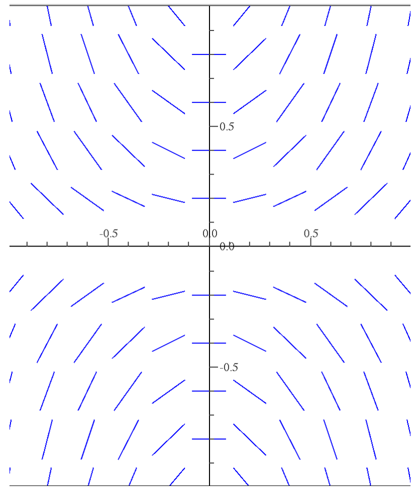
---

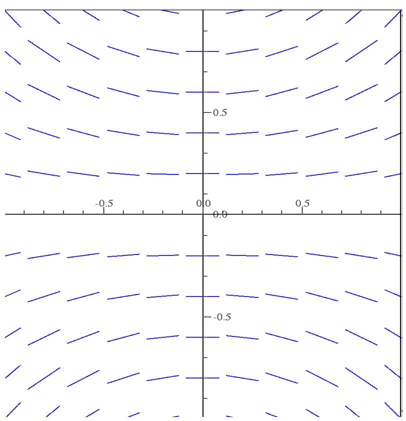
7. Which of the slope field shown below corresponds to $\dfrac{dy}{dx} = -\dfrac{y}{x}$?

a) 

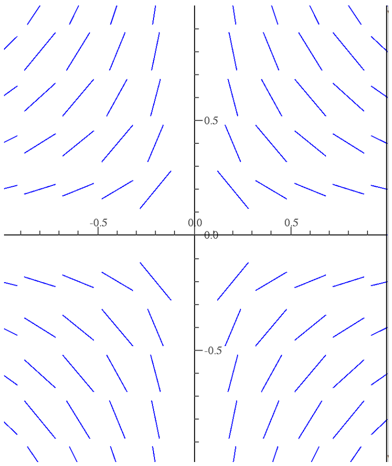
b) 

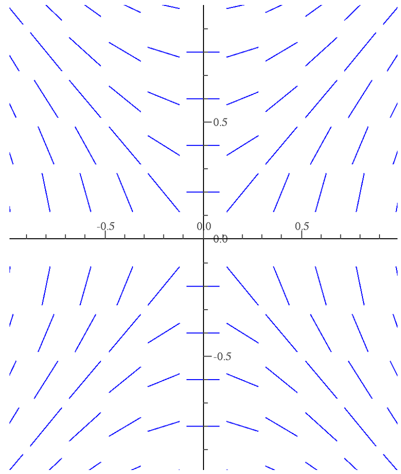
c) 

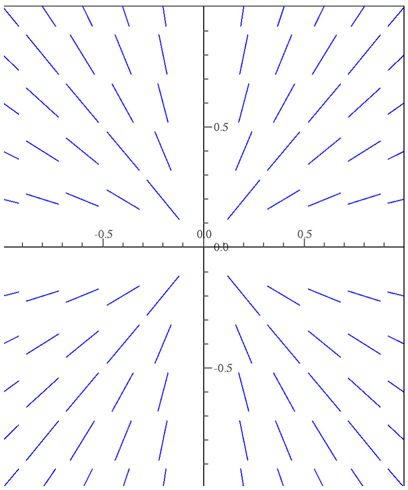

d) 

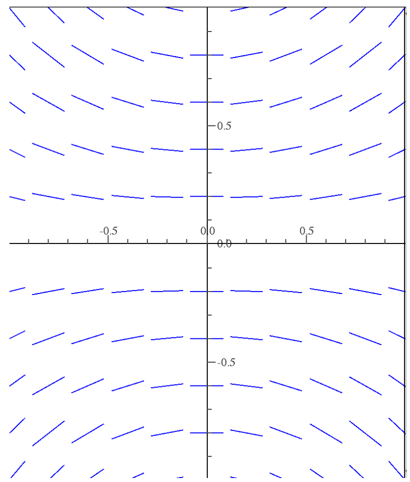

e) 

8. Which of the slope field shown below corresponds to $\dfrac{dy}{dx} = yx$?

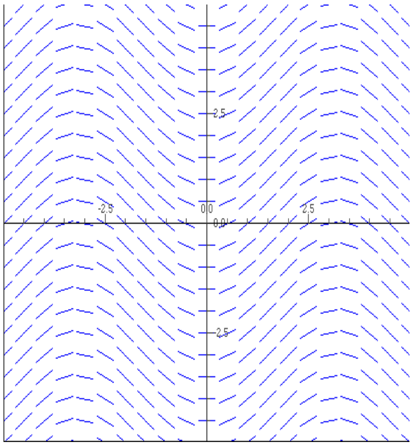
a) 

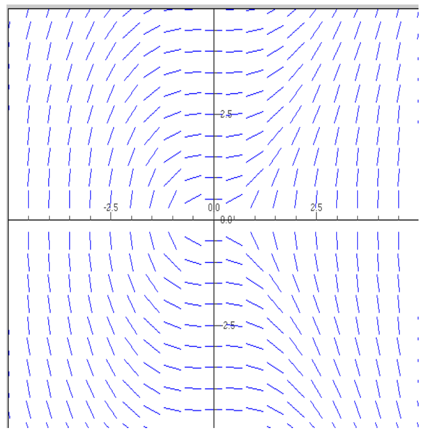

b) 

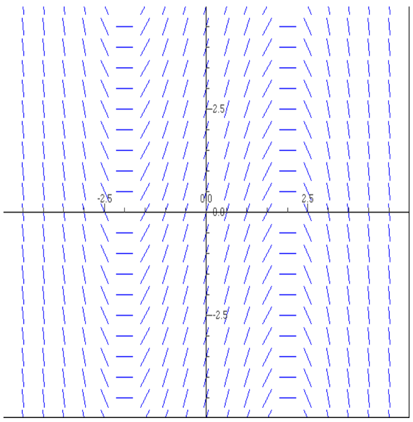

c) 

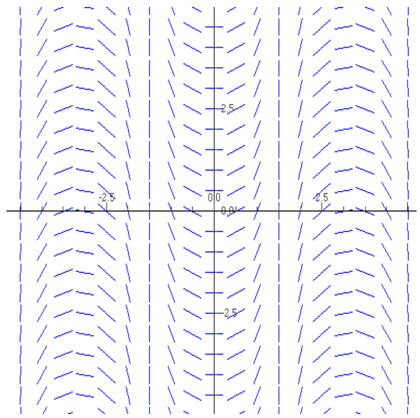

d) 

9. Which of the slope field shown below corresponds to $|y| = e^{x^3}$?

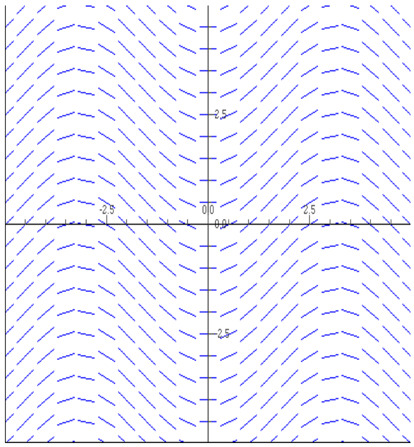
a) 

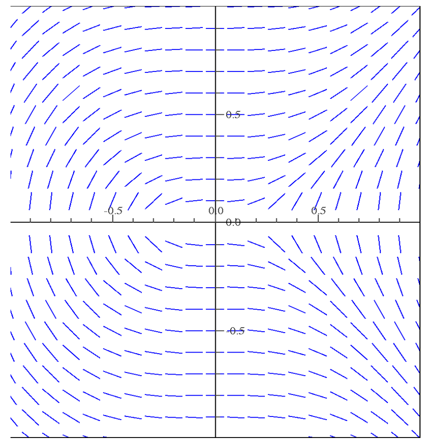

b) 

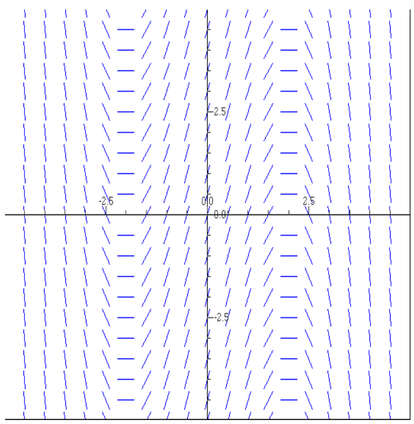

c) 

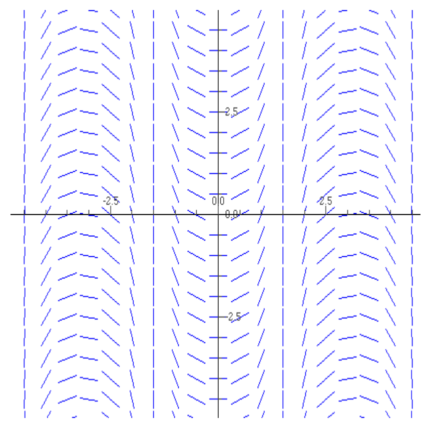

d) 

10. Which of the slope field shown below corresponds to $y = \sec x$?

a)


b)


c)


d)


**Anti-Derivative Chapter Test**

1. Which of the following statements are true?

I. $\int \left( \left(x^3 + x\right) \sqrt[4]{x^4 + 2x^2 - 5} \right) dx = \frac{1}{5}\left(x^4 + 2x^2 - 5\right)^{5/4} + c$

II. $\int \left( x^5 \sin x^6 \right) dx = -\frac{1}{6} \cos x^6 + c$

III. $\int \csc x \, dx = \ln\left|\csc x + \cot x\right| + c$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

2. $\int \frac{x-2}{x-1} dx =$

a) $-\ln|x-1| + c$ b) $x + \ln|x-1| + c$ c) $x - \ln|x-1| + c$

d) $x - \sqrt{x-1} + c$ e) $x + \sqrt{x-1} + c$

---

3. If $\frac{dy}{dx} = \sin x \cos^3 x$ and if $y = 1$ when $x = \pi$, what is the value of $y$ when $x =$ 0?

a) –3 b) –2 c) 1 d) 2 e) 3

---

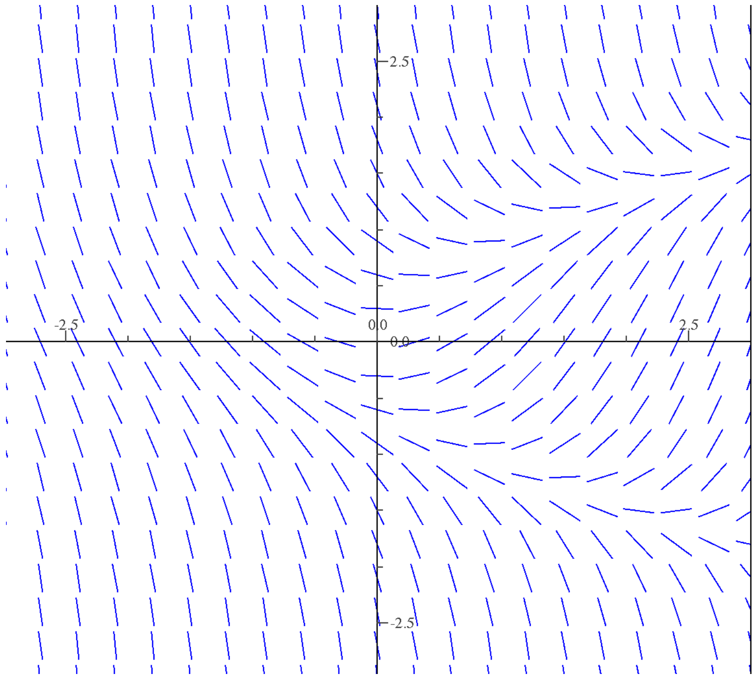
4. $\int x\sqrt{1-x^2}\,dx$

a) $\dfrac{\left(1-x^2\right)^{3/2}}{3}+c$  b) $-\left(1-x^2\right)^{3/2}+c$  c) $\dfrac{x^2\left(1-x^2\right)^{3/2}}{3}+c$

d) $\dfrac{-x^2\left(1-x^2\right)^{3/2}}{3}+c$  e) $\dfrac{-\left(1-x^2\right)^{3/2}}{3}+c$

---

5. Which of the following differential equations corresponds to the slope field shown in the figure below?

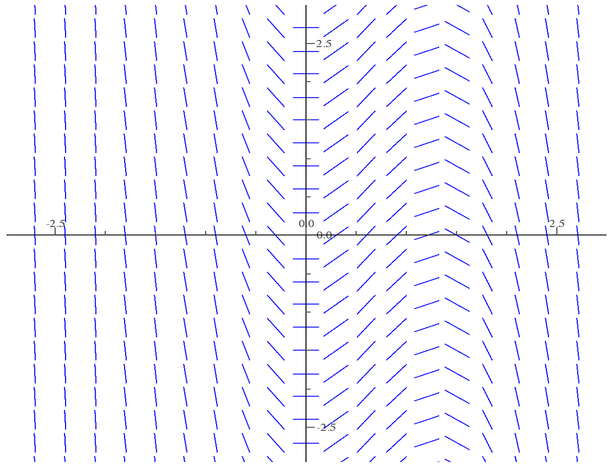



a) $\dfrac{dy}{dx} = x - y^2$  b) $\dfrac{dy}{dx} = 1 - \dfrac{y}{x}$  c) $\dfrac{dy}{dx} = -y^3$

d) $\dfrac{dy}{dx} = x - \dfrac{1}{2}x^3$  e) $\dfrac{dy}{dx} = x + y$

---

6. Which of the following equations might be the solution to the slope field shown in the figure below?



a) $y = 4x - x^3$  b) $y = x^3 - 4x$  c) $y = 4x^4 - x^6$

d) $y = x^3 - 15x^5$  e) $y = \sec x$

---

7. Identify is the first mistake (if any) in this process:

$$\frac{dy}{dx} = xy + x$$

Step 1: $\frac{1}{y+1} dy = x dx$

Step 2: $\ln|y+1| = x^2 + c$

Step 3: $|y+1| = e^{x^2} + c$

Step 4: $y = e^{x^2} + c$

a) Step 1  b) Step 2  c) Step 3

d) Step 4  e) There is no mistake.

---

8. $\int \left( \frac{t^3 - 4t - 3}{5t^{2/3}} \right) dt$

9. $\int \frac{x^2}{\left(x^3 - 1\right)^{3/2}}\, dx$

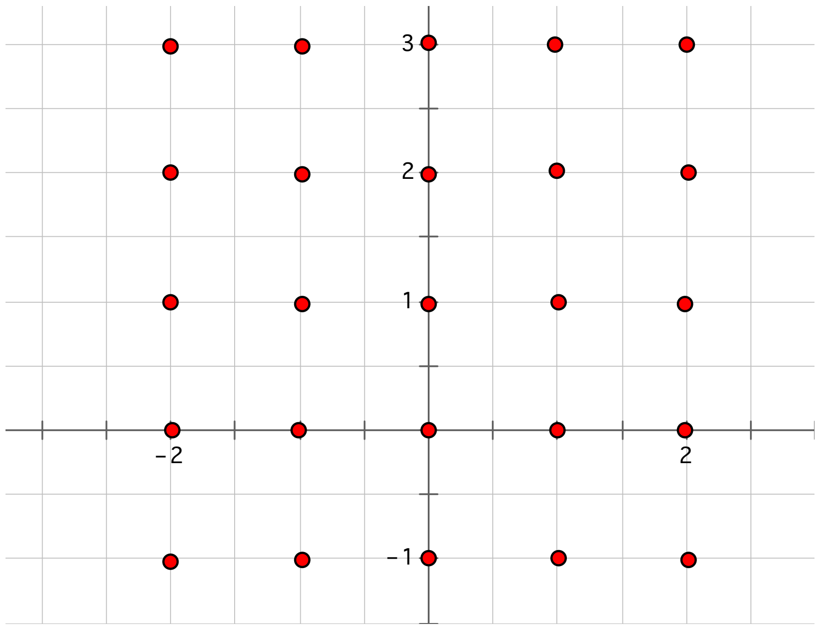
10. $\int\left(3x^5+\frac{\csc^2 x}{e^{\cot x}}-x^3\csc\left(x^4\right)\right)dx$

11. $\int\left(x\sqrt{-3x^2+17}\right)dx$

12. The acceleration of a particle is described by $a(t)=48t^2-18t+6$. Find the distance equation for *x(t)* if *v*(1) = 1 and *x*(1) = 3.

13. Given the differential equation, $\frac{dy}{dx} = \frac{y-2}{x+1}$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.




b. If the solution curve passes through the point (0, 0), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve of $\frac{dy}{dx} = (y-2)(x+1)$ given that $y(0) = 5$

# Chapter 2 Answer Key

## 2.1 Free Response Answer Key

1. $2x^3 - x^2 + 3x + c$

2. $\frac{1}{4}x^4 + x^3 - x^2 + 4x + c$

3. $\frac{3}{5}t^{5/3} + \frac{4}{5}t^{3/2} + c$

4. $\frac{8}{5}x^5 - x^4 + 3x^3 + x^2 + x + c$

5. $-x^{-2} - 3\ln|x| + 6x^{2/3} + c$

6. $\frac{4}{7}x^7 - \frac{2}{3}x^3 - 7\ln|x| + \frac{7}{11}x^{11/7} - \frac{2}{3}x^{-3/2} + c$

7. $x^7 - x^3 - 8\ln|x| + \frac{5}{9}x^{9/5} - \frac{2}{5}x^{-5/2} + c$

8. $\frac{5}{4}x^4 - x^2 - 6|x| + \frac{6}{13}x^{13/6} - \frac{2}{7}x^{-7/2} + c$

9. $\frac{2}{3}x^{3/2} + \frac{6}{5}x^{5/2} - 12x^{1/2} + c$

10. $\frac{2}{3}x^{3/2} - 12x^{1/2} + c$

11. $-\frac{3}{5}x^5 + \frac{7^x}{\ln 7} + 15x^{-1/5} - \frac{1}{12}\ln|x| + c$

12. $= \frac{1}{8}x^8 - \frac{32}{15}x^{15/8} + \frac{7^x}{\ln 7} - \frac{7}{3}x^{3/7} + \frac{1}{5}\ln|x| + c$

13. $\frac{1}{3}x^3 - \frac{1}{x} - 5e^x + 3x + c$

14. $\frac{1}{5}x^5 - \frac{49}{8}x^{16/7} + \frac{8^x}{\ln 8} + \frac{4}{3}x^{-4/3} + \frac{1}{8}\ln|x| + c$

15. $\frac{1}{7}x^7 - \frac{5^x}{\ln 5} + \frac{3}{2}x^{-2/3} + \frac{1}{2}\ln|x| + c$

16. $\frac{1}{3}x^3 + x^2 + x + c$

17. $\frac{2}{3}x^6 + \frac{5}{4}x^4 + C$

18. $4x^3 + \frac{29}{2}x^2 - 8x + c$

19. $\frac{2}{5}x^{5/2} + \frac{8}{3}x^{3/2} + 6x^{1/2} + c$

20. $\frac{1}{2}x^2 + \frac{2}{3}x^{3/2} + 3\ln|x| + c$

21. $2t^6 + \frac{3}{4}t^4 + \frac{28}{3}t^3 + 7t + c$

22. $\frac{16}{3}x^3 - 12x^2 + 9x + c$

23. $2x^2 - 2x^{-1/2} - \frac{3}{2}x^{-1} + c$

24. $\frac{1}{2}\mathrm{x}^2 - 4\mathrm{x} + 7\ln|\mathrm{x}| + \mathrm{c}$

25. $\frac{1}{10}\mathrm{x}^5 - \frac{7}{6}\mathrm{x}^3 + \mathrm{x} - \frac{9}{2}\ln|\mathrm{x}| + \mathrm{c}$

26. $\frac{1}{3}x^3 + x^2 + x + c$

27. $\frac{1}{5}y^5 + \frac{10}{3}y^3 + 5y + c$

28. $\frac{1}{4}x^4 + x^3 + \frac{3}{2}x + x + c$

## 2.1 Multiple Choice Answer Key

1. D 2. A 3. A 4. D 5. A 6. D

## 2.2 Free Response Answers

1. $y = 3x^3 - 6x^2 + 5x + 10$
2. $y = 4x^3 - 18x^2 + 5x + 10$
3. $y = -x^2 + 8x + 33$
4. $y = 12x^3 - 6x^2 + 8x - 14$
5. $f(x) = x^3 - 3x^2 + 3x + 2$

6. $y=-\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2+3x+\frac{37}{12}$

7. $y=\frac{3}{2}x^2-\frac{10}{3}x^{3/2}-2x+\frac{35}{2}$

8. $y=-\frac{1}{4}x^4+2x^3-6x^2+8x+\frac{321}{4}$

## 2.2 Multiple Choice Answers

1. D 2. E 3. B 4. A 5. E 6. B

## 2.3 Free Response Answers

1a. $t\in(-\infty,\ -2)\cup(6,\ \infty)$
b. $a(3)=2$
c. $x(t)=\frac{1}{3}t^3-2t^2-12t+\frac{41}{3}$

2a. $t\in(-4,\ 0)\cup(4,\ \infty)$
b. $a(3)=-27$
c. $x(t)=\frac{1}{6}t^6-4t^4+2$

3a. $t\in(-2,\ 2)\cup(9,\ \infty)$
b. $a(3)=-31$
c. $x(t)=\frac{1}{4}t^4-3t^3-2t^2+36t+4$

4a. $t\in(6,\ \infty)$
b. $a(3)=18$
c. $x(t)=\frac{1}{4}t^4-2t^3+\frac{9}{2}t^2-54t+\frac{217}{4}$

5a. $t \in (-2,\ 2) \cup (2.5,\ \infty)$
b. $a(3) = 16$
c. $x(t) = \frac{1}{2}t^4 - \frac{5}{2}t^3 + 4t^2 + 20t - 4$

6a. $t \in (-\infty,\ -3) \cup \left(3/2,\ 3\right)$
b. $a(3) = -18$
c. $x(t) = -\frac{1}{4}t^4 - t^3 + 9t^2 + 27t - 35.45$

7a. $t \in (-4,\ 0) \cup (4,\ \infty)$
b. $a(3) = -27$
c. $x(t) = \frac{1}{6}t^6 - 4t^4 + 2$

8a. $t \in (-\infty,\ -3) \cup (3,\ \infty)$
b. $a(3) = 78$
c. $x(t) = \frac{1}{5}t^5 - \frac{5}{3}t^3 - 36t + \frac{2209}{15}$

9. $x(t) = 3t^4 - 2t^3 + 4t^2 - 13t + 11$

10. $x(t) = \frac{1}{6}t^4 - \frac{1}{3}t^3 + 2t^2 + 2t + 4$

11. $x(t) = \frac{1}{30}t^6 - \frac{1}{6}t^4 + \frac{1}{2}t^2 + 2t + 4$

12. $x(t) = = \frac{1}{12}t^4 - \frac{4}{3}t^{3/2} + \frac{20}{3}t - \frac{80}{3}$

## 2.3 Multiple Choice Answers

1. B 2. B 3. D 4. D 5. C

## 2.4 Free Response Answer Key

1. $\frac{1}{20}(5x+3)^4+c$

2. $\frac{1}{100}\left(x^4+5\right)^{25}+c$

3. $\frac{1}{7}x^7+\frac{1}{2}x^4+x+c$

4. $-\frac{3}{5}(2-x)^{5/3}+c$

5. $\frac{1}{6}\left(2x^2+3\right)^{3/2}+c$

6. $\frac{1}{-10(5x+2)^2}+c$

7. $\frac{1}{2}\sqrt{1+x^4}+c$

8. $\frac{3}{4}\left(x^2+2x+3\right)^{2/3}+c$

9. $-\frac{1}{3}\ln\left|5-v^3\right|+c$

10. $\frac{5}{12}\left(2x^2+6x+1\right)^{6/5}+c$

11. $\frac{1}{7}\left(x^2+5x+6\right)^7+c$

12. $\frac{1}{6}\left(t^3+1\right)^6+c$

13. $\frac{20}{3}\left(m^2+3m+1\right)^{3/4}+c$

14. $-\frac{1}{4}\left(1+x^3\right)^{-4}+c$

15. $\frac{1}{24}(4s+1)^6+c$

16. $\frac{5}{2}\ln\left(t^2+1\right)+c$

17. $\ln\left|m^3+8\right|+c$

18. $\frac{1}{1086}(181x+1)^6+c$

19. $\frac{1}{3}e^{x^3}-3x+c$

20. $2e^{\sqrt{x}}+c$

21. $e^{2x}-\frac{1}{3\ln 5}\left(5^{3x}\right)+c$

22. $\frac{1}{3}e^{x^3}+\frac{1}{6\ln 5}5^{3x^2}+c$

## 2.4 Multiple Choice Answer Key

1. C 2. E 3. B 4. A 5. B 6. B

7. D 8. A

## 2.5 Free Response Answer Key

1. $\frac{1}{5}\sin x^5 + c$

2. $-\frac{1}{7}\cos(7x+1)+c$

3. $\frac{1}{3}\tan(3x-1)+c$

4. $-2\cos\sqrt{x}+c$

5. $\frac{1}{5}\tan^5 x + c$

6. $-\frac{2}{3}\cot^{3/2} x + c$

7. $\frac{1}{6}e^{6x}+c$

8. $-\frac{1}{4}\csc^2 2x + c$

9. $\frac{1}{2}\ln\ |\sec 2x + \tan 2x| + c$

10. $\cot e^{-x} + c$

11. $2e^{\sqrt{x}}+c$

12. $\frac{1}{2}Ln^2 x + c$

13. $\frac{1}{2}\tan^{-1}x^2 + c$

14. $x + c$

15. $\frac{1}{3}\sec(\ln x)+c$

16. $-\csc e^x + c$

17. $\ln|1+\sin x| + c$

18. $\frac{1}{12}\tan^{3/2}(4t^2)+c$

19. $\frac{1}{4}Ln^2\left(x^2+1\right)+c$

20. $-\frac{1}{2}\sin^2\frac{1}{x}+c$

21. $\frac{1}{3}\tan\left(x^3\right)+\frac{1}{4}\ln^4 x+c$

22. $\frac{1}{3}\tan x^3+e^{x^2}+c$

23. $\ln\left|x^2+5\right|-\frac{1}{3}\tan 3x+\frac{1}{2}e^{x^2}-\pi x+c$

24. $\frac{1}{6}x^6+\frac{1}{3}\cos(3x)+\frac{1}{2}e^{x^2}+c$

25. $\frac{1}{4}x^4-\frac{1}{6}\ln\left|\sec\left(2x^3\right)+\tan\left(2x^3\right)\right|+\frac{1}{4}e^{x^4}+c$

26. $\frac{3}{2}\sqrt{x^3}-\frac{1}{2}\ln|sec(2x)|-\frac{1}{6}e^{-3x^2}+c$

27. $2\sqrt{x^5}-\frac{1}{3}\ln|\sec(3x)+\tan(3x)|-\frac{1}{2}e^{-4x^2}+c$

28. $\frac{6}{5}\sqrt{x^5}-\frac{1}{2}\ln\left|\sec(2x)+\tan(2x)\right|-\frac{1}{8}e^{-4x^2}+c$

29. $\frac{1}{5}sec(5x)+\frac{1}{3}\tan(3x)+\frac{1}{7}\ln|sec(7x)+\tan(7x)|+c$

30. $\frac{1}{3}\sin(3x)+\frac{1}{2}x+\frac{1}{20}\sin(10x)-\frac{2}{21}\cos^{3/2}(7x)+\mathrm{c}$

31. $\frac{-1}{(9+x)}+\frac{1}{3}\tan^{-1}\frac{x}{3}+\frac{1}{2}\ln\left(9+x^2\right)+c$

32. $\sin^{-1}\frac{x}{\sqrt{3}}-\sqrt{3-x^2}-\ln|3-x|+c$

33. $\frac{1}{2}sec^{-1}\frac{x}{2}+\sqrt{x^2-4}+\frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|+c$

34. $\frac{1}{(1-x)}-\frac{1}{2}\ln\left|1-x^2\right|+\tan^{-1}x+c$

35. $v(t)=2\cos 2t-2,\ x(t)=\sin 2t-2t-3$

36. $v(t)=\frac{1}{3}\sin 3t-1;\ \ x(t)=-\frac{1}{9}\cos(3t)-t+\frac{19}{9}$

37. $v(t)=e^t+\frac{1}{2}\sin 2t-1;\ x(t)=e^t-\frac{1}{4}\cos 2t-t+\frac{5}{4}$

38. $v(t)=\frac{1}{2}e^{2t}-\sin t-\frac{1}{2};\ x(t)=\frac{1}{4}e^{2t}+\cos t-\frac{1}{2}t+\frac{7}{4}$

39. $v(t)=\frac{1}{3}e^{3t}-t^3+\frac{1}{3};\ \ x(t)=\frac{1}{4}e^{2t}-\frac{1}{4}t^4+\frac{1}{3}t+\frac{8}{9}$

**2.5 Multiple Choice Answer Key**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | A | 2. | D | 3. | C | 4. | A | 5. | D | 6. | C |
| 7. | D | 8. | E | 9. | C | 10. | A | 11. | D | 12. | A |
| 13. | E | 14. | C | 15. | B | 16. | C | 17. | A | 18. | B |
| 19. | D | | | | | | | | | | |

**2.6 Free Response Answer Key**

1. $y=kx$

2. $y=\frac{2}{-x^2+C}$

3. $y=k\sqrt{x^2+1}$

4. $y=\frac{1}{C-3\tan^{-1}x}$

5. $y=\left(\frac{2}{3}\left(x^3-3\right)^{3/2}+C\right)^{1/3}$

6. $y = \pm\frac{1}{5}\sqrt{\ln|5\tan x + C|}$

7. $y = \left(\frac{1}{2}x^4 - 2\right)^{1/3}$

7. $y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

8. $y = \sec^{-1}\left(\frac{x^3}{3} + x + C\right)$

9. $y = \frac{1}{\pm\sqrt{c - 4x^2}}$

10. $y = \frac{1}{c - \sin x}$

11. $v = 1 + ke^{2t + \frac{1}{2}t^2}$

12. $y = \pm\sqrt{\left(\frac{3}{2}t^2 + c\right)^{2/3} - 1}$

13. $\theta = \left(\frac{3}{2}r + r^{3/2} + c\right)^{2/3}$

14. $y = \frac{-2}{5x^2 + 1}$

15. $y = \sqrt{2x^2 + 1}$

16. $y = \left(\frac{1}{2}x^4 - 2\right)^{1/3}$

17. $y = 2 - e^{\frac{\pi}{4}}e^{-\tan^{-1}x}$

18. $y = \tan(x - 1)$

19. $y = -\sqrt{2e^{x^2} - 1}$

20. $u = -\sqrt{t^2 + \tan t + 25}$

21. $y = 5e^{\frac{1}{2}x^2 + \cos x}$

22. $y = \cos^{-1}(\cos x - 1)$

23. $y = \ln\left(\frac{1}{\sin x + C}\right)$

24. $y = 7e^{x^4}$

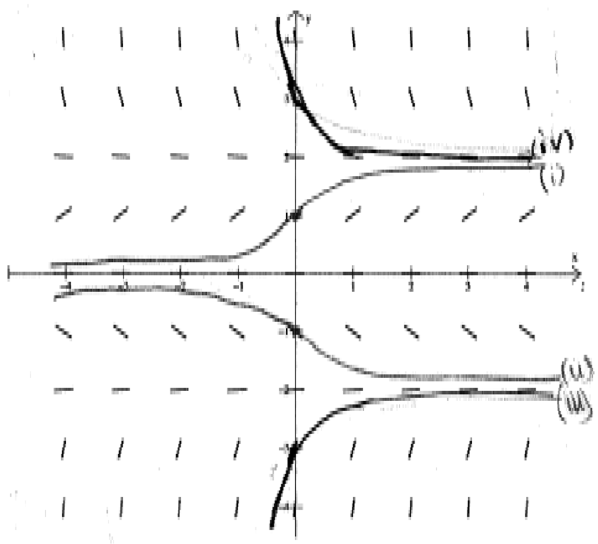
**2.6 Multiple Choice Answer Key**

1. D 2. A 3. C 4. B 5. E

6. C 7. A

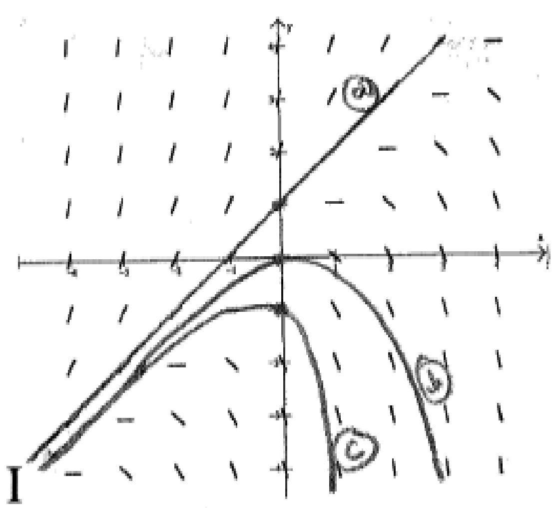
## 2.7 Free Response Answer Key

1.

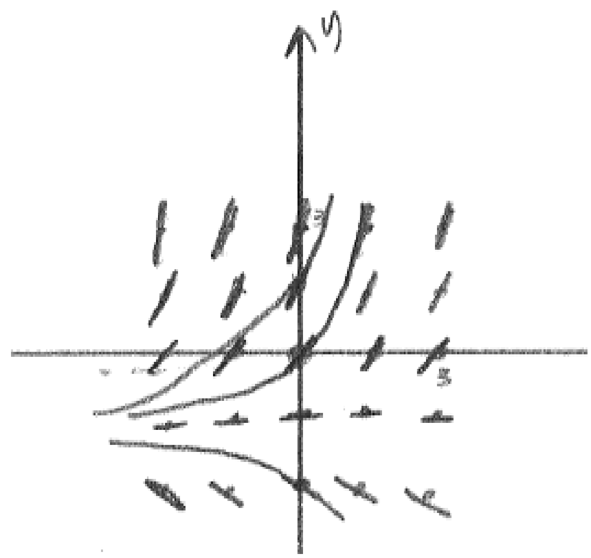



2. IV 3. I 4. III 5. II

6.

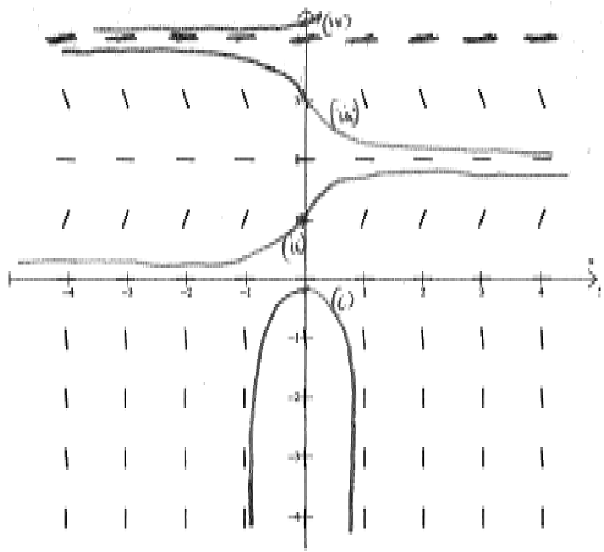



7.




8.

9.



## 2.7 Multiple Choice Answer Key

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | B | 2. | E | 3. | B | 4. | A | 5. | A | 6. | A |
| 7. | A | 8. | D | 9. | B | 10. | D | | | | |

## Chapter 2 Practice Test Key

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | D | 2. | B | 3. | C | 4. | E | 5. | A | 6. | A |
| 7. | B | | | | | | | | | | |

8. $\frac{3}{50}t^{10/3}-\frac{12}{25}t^{5/3}-\frac{9}{5}t^{1/3}+c$

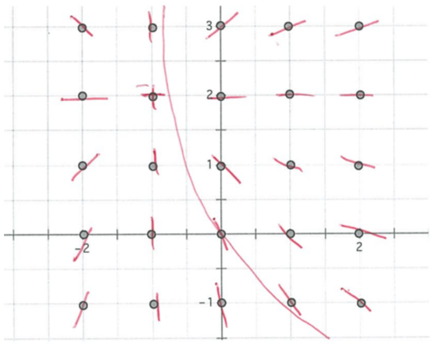
9. $=-\frac{2}{3}\left(x^3-1\right)^{-1/2}+c$

10. $\frac{1}{6}x^6+e^{-\cot x}-\frac{1}{4}\ln\left|\csc x^4-\cot x^4\right|+c$

11. $-\frac{1}{9}\left(-3x^2+17\right)^{3/2}+c$

12. $x(t) = 4t^4 - 3t^3 + 3t^2 - 12t + 8$

13a & b.




13c. $y = 2 + 3e^{\frac{1}{2}x^2 + x}$