AP Calculus AB '23-24

Spring Final Part IIA v1

Calculator Allowed

30 minutes

Name:

Solution Key

1. A particle is moving along the x-axis so that its velocity is given by $E(t) = 5 - 416\left(\frac{t}{5}\right)^4\left(1 - \frac{t}{10}\right)^5$ on $0 \le t \le 10$. The position of the particle at $t = 0$ is $x = -1.6$.

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(2) (a) At what time(s) on $t \in [0, 10]$, if any, does the particle switch directions? At which time does the particle's direction switch from moving left to moving right?

GRAPH $E(t)$

$E(t) = 0 \rightarrow t = 2.292, 6.752$

~~E(t) switch~~

THE PARTICLE SWITCHES FROM LEFT TO RIGHT AT $t = 6.752$

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(1) (b) Find the acceleration at $t = 7.3$.

$a(7.3) = v'(7.3) = 3.536$

(c) What is the position of the particle at $t = 7.3$.

3

$$p(7.3) = -1.6 + \int_0^{7.3} E(t)\,dt$$

$$= -16.408$$

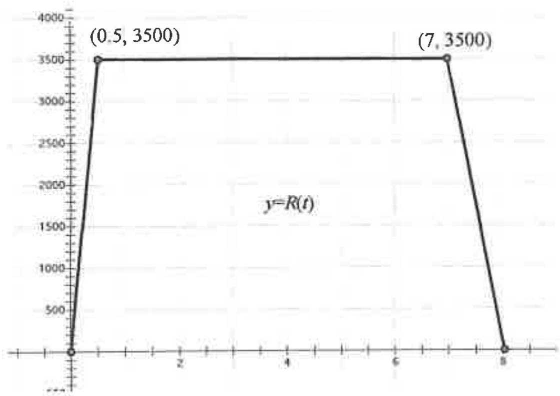
3

(d) What is the total distance traveled by the particle on $t \in [2, 9]$.

$$\text{TOTAL DIST} = \int_2^9 |E(t)|\,dt = 31.831$$

The Groundwater Replenishment System Problem I

2. Disparagingly referred to as Toilet to Tap, Orange County's GWRS (Groundwater Replenishment System) has converted 400 billion gallons of raw sewage into drinkable water over the past fifteen years. That is about 25,000 gallons every eight hours. The process occurs in three phases: microfiltration, reverse osmosis, and ultraviolet disinfection. Let us assume that the rate $R(t)$, in gallons per hour, at which raw sewage enters the process as modeled by the graph below, formed by three linear functions.



Further, let us assume the rate at which the raw sewage is converted to treated sewage, in gallons per hour, in the microfiltration process is modeled by $T(t) = 83e^{0.5t}\sqrt{8t - t^2}$ for $t \in [0, 8]$ hours.

(a) How many gallons of raw sewage have entered the process in these eight hours?

1

$\int R(t)\,dt \approx \frac{1}{2}(.5)(3500) + 3500(6.5) + \frac{1}{2}(3500)$

$= \cancel{24{,}500}$ 25375 GALLONS

3

(b) Find $T'(6.2)$. Using the correct units, explain the meaning of the result in terms of the situation.

$T'(6.2) = 1867.127 \frac{GAL}{HR^2}$

THE RATE AT WHICH TREATED WATER EXITS THE TANKS IS INCREASING BY 1864.127 GALLONS PER HOUR PER HOUR AT $t=6.2$ HOURS

1

(c) Assuming there is no raw sewage left from the previous process, write an expression for the $A(t)$, the total amount of raw sewage present within the process at any time t.

$A(t) = \int_0^t R(x) - T(x)\, dt$

4

(d) Find the maximum amount of raw sewage in process during the micro filtration phase.

$A' = R(t) - T(t) = 0 \rightarrow t = 4.746$

t	A
0	0
4.746	9707.291
8	-17.6

9707.291 GALLONS

EC. ALL THE RAW SEWAGE WAS CONVERTED BECAUSE $A(8) < 0$

End of

AP Calculus AB '23-24

Spring Final Part IIA v1

AP Calculus AB '23-24

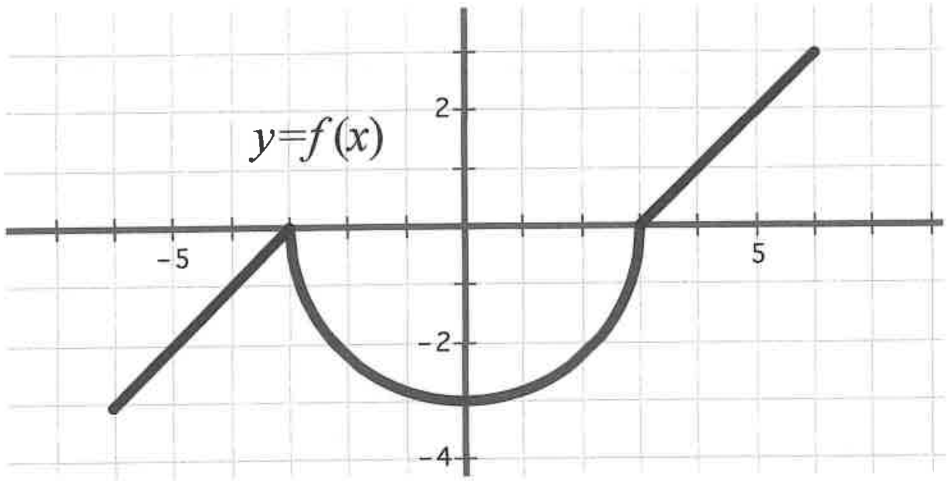
Spring Final Part IIB v1

NO Calculator Allowed

60 minutes

Name:

Solution Key



3. The graph above, $f(x)$ on $-6 \le x \le 6$, is comprised of two line segments and a semi-circle. Let $g(x) = -1 + \int_{-3}^{x} f(t)\, dt$.

(a) Find $g(3)$, $g'(3)$, and $g''(3)$.

3

$g(3) = -1 - \frac{9\pi}{2}$

$g'(3) = 0$

$g''(3) = \text{DNE}$

(b) At what x-value(s) on $-6 \le x \le 6$ does $g(x)$ have a relative minimum. Explainn your reasoning.

2

MIN @ $x=3$ BECAUSE $g' = f$ AND f SWITCHES FROM $-$ TO $+$

(c) Find $\lim\limits_{x \to -3} \dfrac{g(x)+1}{x^2+2x-3}$. Justify your answer.

~~$x^2+2x-15$ $\overset{LH}{=} \lim\limits_{x \to 5} \dfrac{g'(x)}{2x+2} = \dfrac{f(5)}{2(5)+2} = \dfrac{2}{12} = \dfrac{1}{6}$~~

2 $\overset{LH}{=} \lim\limits_{x \to -3} \dfrac{g'(x)}{2x-2} = \dfrac{f(-3)}{} = \dfrac{0}{-8} = 0$

(d) On what interval(s) is $g(x)$ both decreasing and concave up? Explain why.

2 f IS NEGATIVE AND INCREASING

$x \in (0,3) \cup (-6,-3)$

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x	0	2	4	8
$f(x)$	2	8	0	4
$f'(x)$	1	0	-3	2
$g(x)$	8	4	2	0
$g'(x)$	1	-2	0	-3

4. The functions f and g are twice differentiable. The table shown above gives values of the functions and their derivatives at selected values of x.

(a) Let h be a differentiable function defined by $h(x) = g(f(x))$. Find $h'(0)$. Show the works that leads to your answer.

3

$h'(0) = g'(f(0)) \cdot f'(0)$

$= g'(2) \cdot f'(0)$

$= (-2)(1) = -2$

(b) Let k be a differentiable function such that $k'(x) = [g(x)]^2 \cdot f(x)$. Is the graph of k concave up or concave down at the point where $x = 2$? Give a reason for your answer.

3

$k'' = [g(2)]^2 f'(2) + f(2)\, 2(g(2)) \cdot g'(2)$

$= (-2)^2(0) + 8(2)(4)(-2)$

$= -128$

CONCAVE DOWN BECAUSE $k''(2) < 0$

(c) Let k be a differentiable function such that $k'(x) = [g(x)]^2 \cdot f(x)$. Is the graph of k concave up or concave down at the point where $x = 2$? Give a reason for your answer.

2

CON. $m(4) = 3(4)^2 + \int_0^4 g'(t)\,dt$

$= 48 + g(4) - g(0)$

$= 48 + 2 - 8$

$= 42$

(d) Let k be a differentiable function such that $k'(x) = [g(x)]^2 \cdot f(x)$. Is the graph of k concave up or concave down at the point where $x = 2$? Give a reason for your answer.

2

$m'(8) = 6(8) + g'(8)$

$= 48 + (-3)$

$= 45$

~~IS M INC, DEC OR~~ ~~NEITHER~~ AT X=8 INCR

~~NEITHER~~ INCREASING $m' > 0$

~~CONCAVE DOWN UP BECAUSE $m'(8) > 0$~~

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5. Consider the curve given by $xy + y^3 = 4x$.

(a) Show that $\frac{dy}{dx} = \frac{4-y}{3y^2+x}$.

2

$$\frac{d}{dx}(xy + y^3 = 4x)$$

$$x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 4$$

$$(3y^2+x)\frac{dy}{dx} = 4-y$$

$$\frac{dy}{dx} = \frac{4-y}{3y^2+x}$$

2 (b) Show there are no points where the tangent line is horizontal.

$\frac{dy}{dx} = 0 \Rightarrow y = 4$ $\qquad$ $4x + (4)^3 = 4x$

2

$4^3 = 0$

No Sol

(c) Show there are no points where the tangent line is vertical.

use $\frac{dy}{dx}$ DNE $\Rightarrow -3y^2 = x \Rightarrow$ x

$(-3y^2)y + y^3 = 4(-3y^2)$

3

$-3y^3 + y^3 = -12y^2$

$-2y^3 + 12y^2 = 0$ $\quad \cancel{4y^3} - 12y^2 = 0$

$-2y^2(y-6) = 0$ $\quad 4\cancel{y^2(y-3)}$ $\quad y = 0, 3$

$y = 0, 6$

$(0, 0)$, $(-108, 6)$

8

(d) A particle is moving along the curve. At the instant when the particle is at the point $\left(\frac{1}{3}, 1\right)$, its horizontal position is changing at a rate $\frac{dx}{dt} = 3$ units per second. What is the value of $\frac{dy}{dt}$, the rate of change of the particle's vertical position, at that instant?

2

$\frac{d}{dt}\left[xy + y^3 = 4x\right]$

$x\frac{dy}{dt} + y\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 4\frac{dx}{dt}$

$\left(\frac{1}{3}\right)\frac{dy}{dt} + 1(3) + 3\left(\frac{dy}{dt}\right) = 4(3)$

$\frac{10}{3}\frac{dy}{dx} = 9$

$\frac{dy}{dt} = \frac{27}{10}$

The Boiled Potato Problem.

6. At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius (°C) at time $t = 0$, and the internal temperature of the potato is greater than 27°C for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where H(t) is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

2

$$\left.\frac{dH}{dt}\right|_{H=91} = -16$$

$$y - 91 = -16t$$

$$H(3) \approx y(3) = 91 - 16(3) = 43°C$$

1

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

$$\frac{d^2H}{dt^2} = -\frac{1}{4}\frac{dH}{dt} = \frac{1}{16}(H-27)$$

$\frac{d^2H}{dt^2} > 0$ SO THE ESTIMATE IS AN UNDERESTIMATE

(c) For $0 \le t \le 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation

6

$\frac{dG}{dt} = -(G-27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$.
Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

$$\cancel{4} \quad \int (G-27)^{-2/3}\, dG = \int -\, dt$$

$$\frac{(G-27)^{1/3}}{1/3} = -t + C$$

$$3(G-27)^{1/3} = \overset{12}{\cancel{192}} - t \qquad (0, 91) \to \overset{12}{\cancel{192}}$$

$$(G-27)^{1/3} = \cancel{64} \; 4 - \tfrac{1}{3}t$$

$$G = 27 + (\cancel{2}4 - \tfrac{1}{3}t)^3 = 54^\circ C$$