Chapter 7 Overview: Limits, Continuity and Differentiability

Derivatives and Integrals are the core practical aspects of Calculus. They were the first things investigated by Archimedes and developed by Liebnitz and Newton. The process involved examining smaller and smaller pieces to get a sense of a progression toward a goal. This process was not formalized algebraically, though, at the time. The theoretical underpinnings of these operations were developed and formalized later by Bolzano, Weierstrauss and others. These core concepts in this area are Limits, Continuity and Differentiability. Derivatives and Integrals are defined in terms of limits. Continuity and Differentiability are important because almost every theorem in Calculus begins with the condition that the function is continuous and differentiable.

The Limit of a function is the function value (y-value) expected by the trend (or sequence) of y-values yielded by a sequence of x-values that approach the *x*-value being investigated. In other words, the Limit is what the *y*-value should be for a given x-value, even if the actual y-value does not exist. The limit was created/defined as an operation that would deal with y-values that were of an indeterminate form.

Indeterminate Form of a Number--Defn: "A number for which further analysis is necessary to determine its value."

Means: the number equals $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞,or other strange things.

The formal definition is rather unwieldy and it will not dealt with it in this course other that to show the Formal Definition and translate it:

Formal Definition of a Limit

$\lim\limits_{x\to a} f(x) = \text{L}$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

Formal Definition of a Limit means:

When x almost equals a, the limit almost equals y.

There are four kinds of Limits:

- Two-sided Limits (most often just referred to as Limits)
- One-sided Limits
- Infinite Limits
- Limits at Infinity

Some Facts to Remember:

$\ln 1 = 0 \qquad \ln e = 1$

$\ln e^u = u \qquad e^{\ln u} = u$

7.1: Two-sided Limits and L'Hopital's Rule

As mentioned in the intro to this chapter and last year, the limit was created/defined as an operation that would deal with y-values that were of an indeterminate form.

$\lim\limits_{x \to a} f(x)$ is read "the limit, as x approaches a, of f of x." What the definition means is, if x is almost equal to a (the difference is smaller than some small number δ), f(x) is almost equal to L (the difference is smaller than some small number ε). In fact, they are so close, one could round off and consider them equal. In practice, usually $\lim\limits_{x \to a} f(x) = f(a)$. In other words, the limit is the y for a given $x = a$--as long as y $\neq$ 0/0. If y=0/0, one is allowed to factor and cancel the terms that gave the zeros. No matter how small the factors get, they cancel to 1 as long as they do not quite equal 0/0.

Ex 1 Find $\lim\limits_{x \to 5} (x+2)$ and $\lim\limits_{x \to -4} \left(x^2+3\right)$

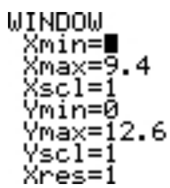
$$\lim_{x \to 5} (x+2) = 5+2 = \ 7$$
$$\lim_{x \to -4} (x^2+3) = (-4)^2 + 3 = \ 19$$

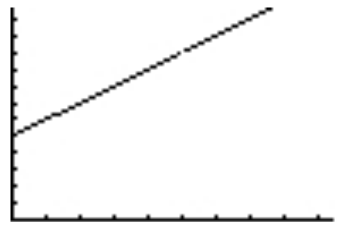
Ex 2 Find $\lim\limits_{x \to 5} \dfrac{x^2-25}{x-5}$

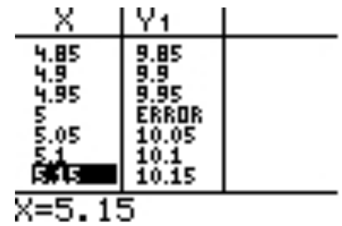
If $x = 5$ here, $\dfrac{x^2-25}{x-5}$, which is $\dfrac{(x-5)(x+5)}{x-5}$, would $= \dfrac{0}{0}$. But with a Limit, x is only *almost* equal to 5, and, therefore, $\dfrac{x-5}{x-5} = 1$. So

$$\begin{aligned}\lim_{x \to 5} \frac{x^2-25}{x-5} &= \lim_{x \to 5} \frac{(x+5)(x-5)}{x-5} \\ &= \lim_{x \to 5} (x+5) \\ &= 5+5 \\ &= 10\end{aligned}$$

Notice that throughout this process, the limit notation is kept in the problem until the limit is actually evaluated (that is, plugged in *a*). Not writing this notation actually makes the problem wrong – it is like getting rid of an operation. The limit is essentially what allows you to do the cancelling and/or plugging in. You must use proper notation when writing these up.







As can be seen from both the graph (in the given window) and the table, while no y-value exists for x=5, the y-values of the points on either side of x=5 show y should be 10.

Basically, the limit allows us to factor and cancel before the number "a" is substitute for x.

OBJECTIVE

Evaluate Limits algebraically.
Evaluate Limits using L'Hopital's Rule.
Recognize and evaluate Limits which are derivatives.
Use the *nDeriv* function on the calculator to find numerical derivatives.

Ex 3 Find $\lim_{x\to 2} \frac{x^2+4x-12}{x^2+3x-10}$

Since if x = 2, $\frac{x^2+4x-12}{x^2+3x-10} = \frac{0}{0}$, the function must be factored and the fraction must be reduced. And, in fact, one of the factors in each must be $(x-2)$, otherwise the fraction would not yield zeros. So,

$$\begin{aligned}\lim_{x\to 2} \frac{x^2+4x-12}{x^2+3x-10} &= \lim_{x\to 2} \frac{(x-2)(x+6)}{(x-2)(x+5)} \\ &= \lim_{x\to 2} \frac{x+6}{x+5} \\ &= \frac{8}{7}\end{aligned}$$

Ex 4 Find $\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6}$

$$\begin{aligned}\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6} &= \lim_{x\to -3} \frac{(x+3)(2x^2-5x+2)}{(x+3)(x-2)} \\ &= \lim_{x\to -3} \frac{2x^2-5x+2}{x-2} \\ &= \frac{35}{-5} \\ &= -7\end{aligned}$$

Ex 5 Find $\lim_{x \to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2}$

Unlike the previous examples, this fraction does not factor. Yet it must simplify, somehow, to eliminate the Indeterminate Number. If conjugates are multiplied to eliminate the radicals from the numerator:

$$\lim_{x \to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2} = \lim_{x \to 2} \frac{\left(\sqrt{4-x}-\sqrt{2}\right)\left(\sqrt{4-x}+\sqrt{2}\right)}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x \to 2} \frac{4-x-2}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x \to 2} \frac{2-x}{(x-2)\left(\sqrt{4-x}+\sqrt{2}\right)}$$

$$= \lim_{x \to 2} \frac{-1}{\sqrt{4-x}+\sqrt{2}}$$

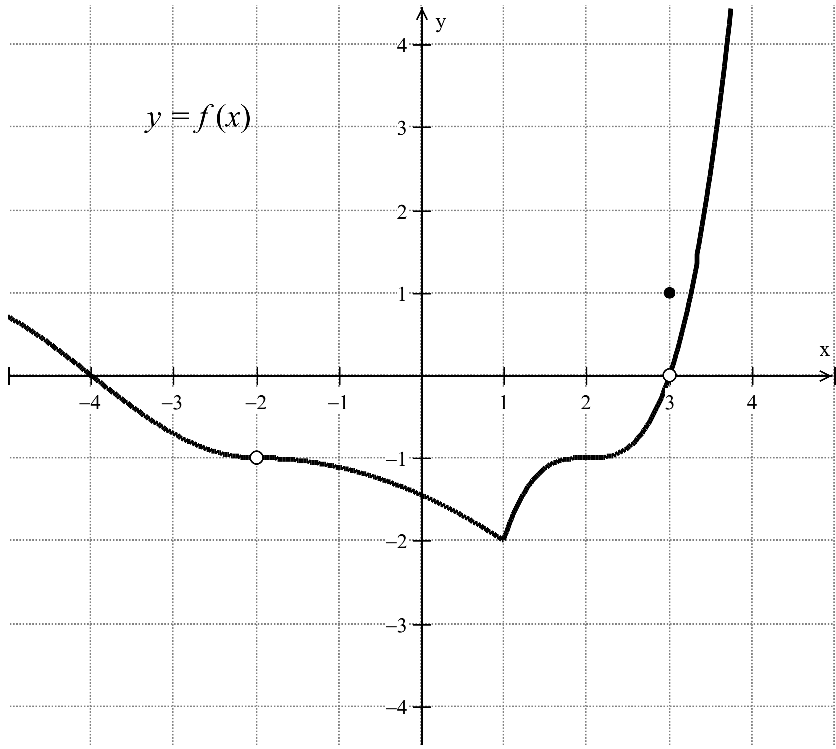
$$= \frac{-1}{\sqrt{2}+\sqrt{2}}$$

$$= \frac{-1}{2\sqrt{2}} \text{ or } \frac{-\sqrt{2}}{4}$$

Again, notation is very important. Essentially, the limit is the operation you are performing. Until you have actually evaluated the expression by plugging in the *a*, you must still write the notation . Just like you would not drop a square root from an equation until you actually do the "square rooting", you don't get rid of your limit notation until you've actually taken your limit. You will get marked wrong in this class, on the AP test, and/or in college if you don't do this.

Fundamentally, a limit is just telling us where a *y*-value **should be** for a particular function. It does not necessarily tell us that the *y*-value does or does not exist, it just tells us where it is supposed to be on a given curve.

Ex 6 For the function illustrated below, find each of the following limits:



a) $\lim_{x\to -2} f(x)$ b) $\lim_{x\to 1} f(x)$ c) $\lim_{x\to 3} f(x)$

Note that the *y*-values are on the function $f(x)$, and it doesn't matter that we don't know what $f(x)$ is as an equation – it is the graph.

a) $\lim_{x\to -2} f(x) = -1$ When $x \to -2$, the *y*-value approaches –1. It doesn't matter that there is a hole there.

b) $\lim_{x\to 1} f(x) = -2$ When $x \to 1$, the *y*-value approaches –2.

c) $\lim_{x\to 3} f(x) = 0$ Even though there is an actual *y*-value ($f(3) = 1$), the curve heads to the hole (at y = 0)

One of the more powerful tools in Calculus for dealing with Indeterminate Forms and limits is called L'Hopital's Rule.

L'Hopital's Rule

If $\dfrac{f(a)}{g(a)}=\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$, then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$.

This rule allows us to evaluate limits of functions that do not factor, such as those that involve transcendental functions.

It is very important to note that this is **not** the Quotient Rule. Since the original problem is a limit and not a derivative, the quotient rule is not being used. L'Hôpital's Rule is a powerful tool to find the limits of quotients.

Derivatives are used in the process of L'Hôpital's Rule, but a quotient is not being differentiated, so the Quotient Rule is not being applied!

Note also that some books spell it "L'Hospital's Rule" – both are acceptable spellings of the French name.

Ex 5 (again) Find $\lim\limits_{x\to 2}\dfrac{\sqrt{4-x}-\sqrt{2}}{x-2}$ using L'Hopital's Rule.

Since, at x = 0, $\lim\limits_{x\to 2}\dfrac{\sqrt{4-x}-\sqrt{2}}{x-2}=\dfrac{\sqrt{4-2}-\sqrt{2}}{2-2}=\dfrac{0}{0}$, the condition for L'Hopital's Rule is satisfied.

$$\lim_{x\to 2}\frac{\sqrt{4-x}-\sqrt{2}}{x-2}=\lim_{x\to 2}\frac{D_x\left(\sqrt{4-x}-\sqrt{2}\right)}{D_x\left(x-2\right)}$$

$$=\lim_{x\to 2}\frac{\frac{1}{2\sqrt{4-x}}(-1)}{1}$$

$$=\frac{-1}{2\sqrt{2}}$$

Obviously, this is a much faster process than the algebraic one.

Usually, when applying L'Hôpital's Rule, "L'H" is written over the equal sign to indicate what was done.

Ex 6 Demonstrate that $\lim_{x\to 0}\frac{\sin x}{x}=1$.

Last year, this limit was used to prove the derivative of sin x. but it was never proven that the limit actually equals 1. This limit is relatively difficult to evaluate without L'Hopital's Rule (and requires a complicated theorem called the Squeeze Theorem), but it is very easy with L'H.

At $x = 0$, $\frac{\sin x}{x}=\frac{0}{0}$, L'H applies.

$$\lim_{x\to 0}\frac{\sin\ x}{x} = \lim_{x\to 0}\frac{\cos\ x}{1}=1$$

Obviously, this is not a proof, because of the circular reasoning. A derivative whose proof involved the limit was used in what was evaluated.

Ex 7 Evaluate $\lim_{x\to 0}\frac{\int_0^{x^2}\sin t^2\,dt}{x^3}$.

At $x = 0$, $\frac{\int_0^{0}\sin t^2\,dt}{0}=\frac{0}{0}$, therefore L'Hôpital's Rule applies.

$$\lim_{x\to 0}\frac{\int_0^{x^2}\sin t^2\,dt}{x^3}\overset{L'H}{=}\lim_{x\to 0}\frac{\frac{d}{dx}\left[\int_0^{x^2}\sin t^2\,dt\right]}{\frac{d}{dx}\left[x^3\right]}$$

According to the First Fundamental Theorem of Calculus,

$$\frac{d}{dx}\left[\int_0^{x^2}\sin t^2\,dt\right]=\sin\left(x^2\right)^2\left(2x\right)=2x\sin x^4\text{, and}$$

$$\lim_{x\to 0}\frac{\frac{d}{dx}\left[\int_0^{x^2}\sin t^2\,dt\right]}{\frac{d}{dx}\left[x^3\right]}=\lim_{x\to 0}\frac{2x\sin x^4}{3x^2}=\lim_{x\to 0}\frac{2\sin x^4}{3x}.$$

Since we still get $\frac{0}{0}$, L'Hôpital's Rule will apply again:

$$\lim_{x\to 0}\frac{\frac{d}{dx}\left[2\sin x^4\right]}{\frac{d}{dx}\left[3x\right]}=\lim_{x\to 0}\frac{\left(2\cos x^4\right)\left(4x^3\right)}{3}=0$$

7.1 Free Response Homework

Evaluate the following Limits.

1. $\lim\limits_{x\to -3} \frac{x^2+7x+12}{x^2-9}$

2. $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$

3. $\lim\limits_{x\to 0} \frac{x^2}{\tan x}$

4. $\lim\limits_{x\to 2} \frac{4x-12}{x^2-3x-10}$

5. $\lim\limits_{x\to 0} x(\cot\ x)$

6. $\lim\limits_{x\to 1} \frac{x^2-x}{e^x-1}$

7. $\lim\limits_{x\to 1} \frac{x^3-1}{x^2+3x-4}$

8. $\lim\limits_{x\to 4} \frac{\ln\left(\frac{x}{4}\right)}{\sqrt{x}-2}$

9. $\lim\limits_{x\to 1} \frac{\ln x}{\tan x}$

10. $\lim\limits_{x\to -3} \frac{x^2+4x-1}{x^3+3x}$

11. $\lim\limits_{x\to \sqrt{2}} \frac{x^4+4x^2-12}{x^3+x^2-2x-2}$

12. $\lim\limits_{x\to 9} \frac{3-\sqrt{x}}{x^2-81}$

13. $\lim\limits_{x\to \pi} \frac{\cos^2\ x-1}{2\cos^2\ x-5\cos\ x-7}$

14. $\lim\limits_{x\to 1} \frac{x^2-1}{\ln x}$

15. $\lim\limits_{x\to -1} \frac{1-x^2}{e^{x+1}-1}$

16. $\lim\limits_{x\to \pi} \csc x(1+\sec x)$

17. $\lim\limits_{x\to -1} \frac{x+1}{e^{x+1}}$

18. $\lim\limits_{x\to 5} \frac{1-\sqrt{x-5}}{x-6}$

19. $\lim\limits_{x\to 0} \left(1+\frac{x}{2}\right)^{\cot x}$

20. $\lim\limits_{x\to \frac{\pi}{2}} \frac{1-\sin^2\ x}{3\sin^2\ x-\sin\ x-2}$

21. $\lim\limits_{x\to 5} \csc(x-5)\ln(x-4)$

22. $\lim\limits_{x\to \pi} \frac{1+\cos x}{e^x-1}$

7.1 Multiple Choice Homework

1. If $a \neq 0$, then $\lim_{x \to a} \frac{x^4 - a^4}{x - a}$ is

a) a b) $3a^2$ c) $4a^3$ d) 0 e) nonexistent

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2. $\lim_{x \to 3} \frac{x^4 - 81}{x - 3}$

a) 108 b) 3 c) 81 d) 0 e) DNE

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3. $\lim_{x \to 2} \frac{x^2 + x - 2}{x^2 - 4}$

a) $\frac{5}{4}$ b) 2 c) 0 d) 1 e) DNE

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x	$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$
3	0	0	7	5

4. Given that $f(x)$ is a thrice differentiable, continuous function on the interval $(0, 4)$ with the table values given above. $\lim_{x \to 3} \frac{f(x)}{(x-3)^3} =$

a) 0 b) $\frac{7}{3}$ c) $\frac{5}{3}$ d) $\frac{5}{6}$ e) dne

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5. $\lim_{x\to 1}\frac{\ln x}{e^x - 1} =$

a) 0 b) e c) $\frac{1}{e}$ d) -e e) Undefined

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6. $\lim_{x\to 1}\frac{x}{\ln x}$ is

a) 0 b) $\frac{1}{e}$ c) 1 d) e e) nonexistent

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7. $\lim_{x\to 1}\frac{\ln x}{x^2 - 1}$ is

a) -1 b) 0 c) $\frac{1}{2}$ d) 2 e) undefined

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8. $\lim_{x\to 3}\frac{\ln\left(\frac{x-1}{2}\right)}{3-x}$

a) -1 b) $-\frac{1}{2}$ c) 0 d) $\frac{1}{2}$ e) 1

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9. $\lim_{x\to 2}\frac{x^2 - 4}{\int_2^x \cos(\pi t)\,dt} =$

a) 0 b) 1 c) 2 d) 4 e) DNE

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10. $\lim_{x\to\pi}\frac{\int_{\pi}^{x}\left(\cos^2 t\right)dt}{\sin 2x} =$

a) -1 b) $-\frac{1}{2}$ c) 0 d) $\frac{1}{2}$ e) 1

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11. $\lim_{x\to 2}\frac{\int_{-2}^{x} t^3 dt}{x^2-4}$

a) 0 b) 2 c) 4 d) 6 e) DNE

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12. $\lim_{x\to 1}\frac{\int_{1}^{x} e^{t^2}\,dt}{x^2-1}$

a) 0 b) 1 c) $\frac{e}{2}$ d) e e) DNE

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13. $\lim_{x\to 0}\frac{\int_{0}^{x^3}\cos t^2\,dt}{x^3} =$

a) 0 b) 1 c) $\frac{1}{3}$ d) 3 e) DNE

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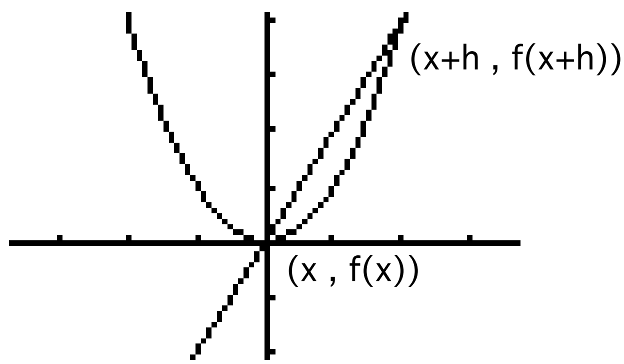
14. If $h(x)=\int_{9}^{x^2}\cos^3\left(\pi t\right)dt$, then $\lim_{x\to 3}\frac{h(x)}{3x-9} =$

a) -2 b) -1 c) $-\frac{1}{3}$ d) $-\frac{2}{3}$ e) DNE

7.2: The Limit Definitions of a Derivative

As one might recall from last year, the derivative was initially created as a function that would yield the slope of the tangent line. The slope formula for any line though two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$. When considering a tangent line, though, that equation has too many variables. This can be simplified some by realizing $y = f(x)$. Considering *h* to represent the horizontal distance between the points and realize that $y = f(x)$, then the two points that form the secant line would be $(x, f(x))$ and $(x+h, f(x+h))$.



Then the slope formula becomes $m = \frac{f(x+h) - f(x)}{x+h-x}$ or $m = \frac{f(x+h) - f(x)}{h}$.
When h = 0, the points would merge and the secant line becomes the tangent line. h = 0 gives $m = \frac{0}{0}$, therefore, $\lim_{h \to 0}$ can be used and $m_{\text{tangent line}} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ will represent the slope of the tangent line.

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ is the derivative.}$$

Since the derivative rules are already known from the last chapter, this will not be used too often. What one might see instead is the Numerical Derivative, which yields the slope of the tangent line at a specific point.

There are two versions of the Numerical Derivative formula:

The Numerical Derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \text{ or } f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Often, there are questions on the AP test that look like limit questions, but which are really questions about recognition of these formulas.

You might notice that the second one is simply the same as the general form for the limit definition of the derivative where $x = a$, but the first one looks a little different.

The first Numerical Derivative is just the slope formula through the points .

Ex 1 Evaluate $\lim_{h \to 0} \frac{(2+h)^3 - (2)^3}{h}$.

FOIL the numerator out (using Pascal's Triangle for ease) and solve this limit algebraically:

$$\begin{aligned} \lim_{h \to 0} \frac{(2+h)^3 - (2)^3}{h} &= \lim_{h \to 0} \frac{8 + 12h + 3h^2 + h^3 - 8}{h} \\ &= \lim_{h \to 0} \frac{12h + 3h^2 + h^3}{h} \\ &= \lim_{h \to 0} \left(12 + 2h + h^2\right) \\ &= 12 \end{aligned}$$

or apply L'Hopital's Rule:

$$\lim_{h\to 0}\frac{(2+h)^3-(2)^3}{h}=\lim_{h\to 0}\frac{3(2+h)^2}{1}$$
$$=3(2)^2$$
$$=12$$

But the quickest way is to recognize that $\lim_{h\to 0}\frac{(2+h)^3-(2)^3}{h}$ is really $f\ '(2)$, where $f(x)=x^3$. Of course, $f\ '(x)=3x^2$ and $f\ '(2)=3(2)^2=12$.

Ex 2 Evaluate $\lim_{h\to 0}\frac{\ln(e+h)-1}{h}$

$$\lim_{h\to 0}\frac{\ln(e+h)-1}{h}=\left.\frac{d}{dx}\right|_{x=e}(\ln x)=\left.\frac{1}{x}\right|_{x=e}=\frac{1}{e}$$

Ex 3 Evaluate $\lim_{x\to 2\pi}\frac{\cos x-1}{x-2\pi}$

$$\lim_{x\to 2\pi}\frac{\cos x-1}{x-2\pi}=\left.\frac{d}{dx}\right|_{x=2\pi}(\cos x)$$
$$=-\sin(2\pi)$$
$$=0$$

7.2 Free Response Homework

Evaluate the following Limits.

1. $\lim\limits_{h\to 0}\frac{(3+h)^3-27}{h}$

2. $\lim\limits_{x\to \pi}\frac{\sin x}{x-\pi}$

3. $\lim\limits_{h\to 0}\frac{e^{3+h}-e^3}{h}$

4. $\lim\limits_{x\to \frac{\pi}{2}}\frac{1+\sin x}{x+\frac{\pi}{2}}$

5. $\lim\limits_{h\to 0}\frac{\cos(\pi+h)+1}{h}$

6. $\lim\limits_{x\to 2}\frac{\ln x-\ln 2}{x-2}$

8. $\lim\limits_{x\to 4}\frac{\sin x-\sin 4}{x-3}$

8. $\lim\limits_{x\to a}\frac{x^3-a^3}{x-a}$

9. $\lim\limits_{x\to a}\frac{x^3-a^3}{x^2-a^2}$

10. $\lim\limits_{h\to 0}\frac{2(3+h)^4-162}{h}$

7.2 Multiple Choice Homework

1. If *f* is a differentiable function such that $f(3)=8$ and $f'(3)=5$, which of the following statements must be false?

a) $\lim\limits_{x\to 3}f(x)=8$ b) $\lim\limits_{x\to 3^+}f(x)=\lim\limits_{x\to 3^-}f(x)$ c) $\lim\limits_{x\to 3}\frac{f(x)-8}{x-3}=5$

d) $\lim\limits_{h\to 0}\frac{f(3+x)-5}{h}=8$ e) $\lim\limits_{x\to 3}f'(x)=5$

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2. $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}=$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}$ c) 0 d) $-\frac{\pi}{4}$ e) Does not exist

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3. If $f(x)=\sqrt{x^2-1}$, which of the following is equal to $f'(3)$?

a) $\lim_{x\to 3}\frac{\sqrt{(x+h)^2-1}-\sqrt{8}}{x-3}$ b) $\lim_{h\to 0}\frac{\sqrt{(x+h)^2-1}-\sqrt{x^2-1}}{h}$

c) $\lim_{h\to 0}\frac{\sqrt{(x+h)^2-1}-\sqrt{8}}{h}$ d) $\lim_{x\to 3}\frac{\sqrt{x^2-1}-\sqrt{8}}{x-3}$

e) $\lim_{h\to 0}\frac{\sqrt{x^2-1}-\sqrt{8}}{x-3}$

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4. $\lim_{h\to 0}\frac{\sin^2\left(\frac{\pi}{3}+h\right)-\frac{3}{4}}{h}=$

a) $\sqrt{3}$ b) $\frac{1}{4}$ c) $\frac{\sqrt{3}}{2}$ d) $\frac{1}{2}$ e) DNE

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5. $\lim_{h\to 0}\frac{2(x+h)^5-5(x+h)^3-2x^5+5x^3}{h}=$

a) 0 b) $10x^3-15x$ c) $10x^4+15x^2$

d) $10x^4-15x^2$ e) $-10x^4+15x^2$

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6. If $f(x)=\sqrt{x^2-1}$, which of the following is equal to $f'(3)$?

I. $\lim\limits_{x\to 3}\dfrac{\sqrt{x^2-1}-\sqrt{8}}{x-3}$

II. $\lim\limits_{h\to 0}\dfrac{\sqrt{(x+h)^2-1}-\sqrt{8}}{h}$

III. $\lim\limits_{h\to 0}\dfrac{\sqrt{(3+h)^2-1}-\sqrt{3^2-1}}{h}$

a) I only b) II only c) III only d) I and II e) II and III only

e) I and III only ac) I, II, and III ad) None of these

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7. $\lim\limits_{h\to 0}\dfrac{\tan\left(\frac{1}{4}\pi+h\right)-\tan\frac{1}{4}\pi}{h}=$

a) -1 b) 2 c) $\dfrac{\pi}{4}$ d) π e) 1

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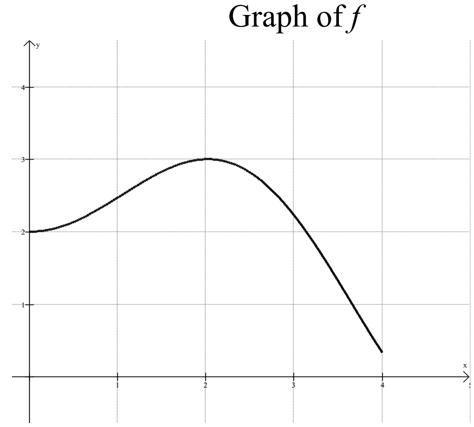
8. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim\limits_{x\to 2}\dfrac{f(x)-3}{x-2}$ does not exist.

II. $f(2)=3$

III. $\lim\limits_{x\to 2} f(x)=3$

Graph of f



a) I only b) I and II only c) II and III only

d) I, II, and III e) III only

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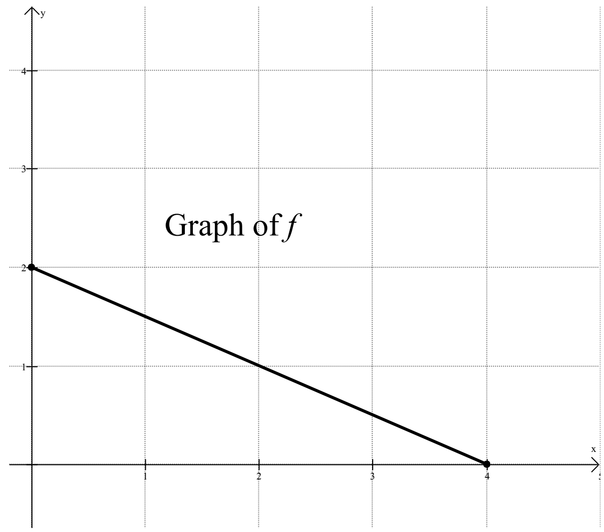
9. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim_{h\to 0}\frac{f(2+h)-1}{h}=\frac{-1}{2}$

II. $f(2)=1$

III. $\lim_{x\to 2} f(x)=1$



a) I only b) I and II only c) II and III only

d) I, II, and III e) III only

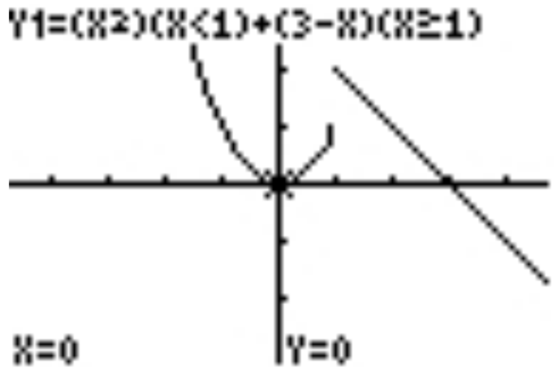
---

10. If the average rate of change of a function f over the interval from $x=2$ to $x=2+h$ is given by $4e^h-4\sin h$, then $f'(2)=$

a) 0 b) 1 c) 2 d) 3 e) 4

7.3: One-sided Limits and Infinite Limits

Previously, it was said that $\lim_{x \to a} f(x)$ is basically what the y-value should be when $x = a$, even if a is not in the domain. For all of the families of functions studied last year, this was enough. But what if that y-value might be two different numbers? In a graph of a piece-wise defined function like this,



It is not clear whether the y-value for x=1 is 1 or 2. As can be seen from the table of values, x-values less than 1 have y-values that approach 1 while x-values greater than 1 have y-values that approach 2:

X	Y1
.85	.7225
.9	.81
.95	.9025
1	2
1.05	1.95
1.1	1.9
1.15	1.85

X=1

Basically, different y-values are achieved from the left and the right of $x = 1$ from the left or the right.

The algebraic ways to describe these differences are one-sided limits. The symbols used are:

$$\lim_{x \to a^-} f(x) \text{ or } \lim_{x \to a^+} f(x)$$

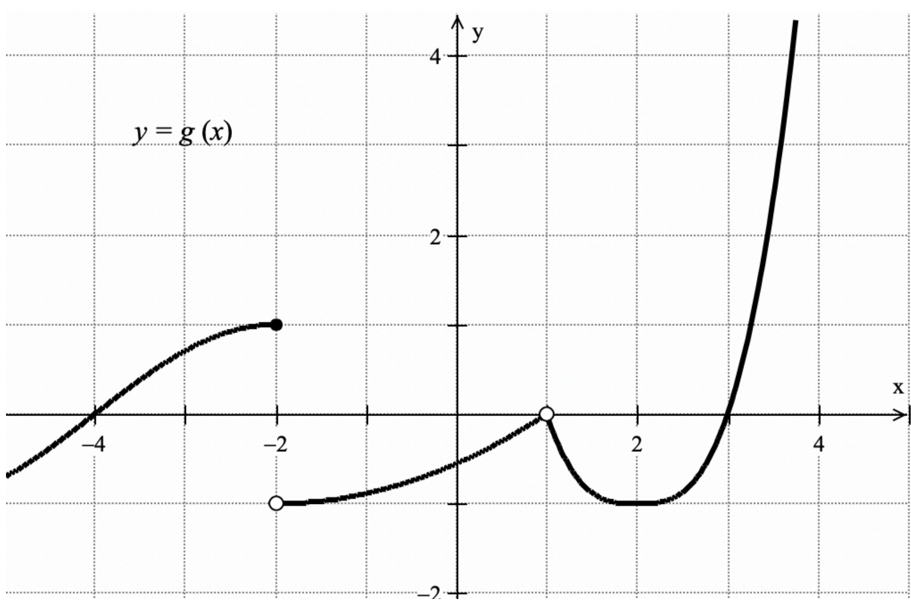
$\lim_{x\to a^-} f(x)$ reads "the limit, as x approaches a from the left, of $f(x)$" while $\lim_{x\to a^+} f(x)$ reads "the limit, as x approaches a from the right, of $f(x)$."

In this example, $\lim_{x\to 1^-} f(x)=1$ and $\lim_{x\to 1^+} f(x)=2$. The $\lim_{x\to 1} f(x)$ does not exist, because the one-sided limits do not equal each other. The "$\lim_{x\to 1} f(x)$ does not exist" means there is not <u>one</u> REAL number that the limit equals.

OBJECTIVE
Evaluate one-sided limits graphically, numerically, and algebraically. Evaluate two-sided limits in terms of one-sided limits. Prove continuity or discontinuity of a given function. Interpret Vertical Asymptotes in terms of one-sided limits.

This concept may be easier to understand visually.

Ex 1 Evaluate each of the limits given the function pictured below.



a) $\lim_{x\to -2^-} g(x)$ b) $\lim_{x\to -2^+} g(x)$ c) $\lim_{x\to -2} g(x)$ d) $g(-2)$

e) $\lim_{x\to 1^-} g(x)$ f) $\lim_{x\to 1^+} g(x)$ g) $\lim_{x\to 1} g(x)$ h) $g(1)$

i) $\lim\limits_{x \to 2^-} g(x)$ j) $\lim\limits_{x \to 2^+} g(x)$ k) $\lim\limits_{x \to 2} g(x)$ l) $g(2)$

a) 1 b) -1 c) DNE d) 1

The left and right limits are different because the curve approaches different values from each side, therefore the does not exist. The actual value of the function () is 1 because that is where the function actually has a value.

e) 0 f) 0 g) 0 h) DNE

The left and right limits are the same because the curve approaches the same value from each side, therefore the is 0 (the *y*-value the curve approaches). The actual value of the function () does not exist because that is where the function has no value (there is a hole in the graph).

i) -1 j) -1 k) -1 l) -1

The left and right limits are the same because the curve approaches the same value from each side, therefore the is -1. The actual value of the function () is -1 because that is where the function actually has a value.

A limit for a function only exists if the left and right limits are equal. Since this was not much of a concern for the functions in Precalculus, this fact was pretty much ignored, but in Calculus (both here and in college) teachers love to deal with this fact.

$$\lim_{x \to a} f(x) \text{ exists if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

Ex 2 Does $\lim\limits_{x \to -1} f(x)$ exist for $f(x) = \begin{cases} x^2+1, & \text{if } x \geq -1 \\ 3-x, & \text{if } x < -1 \end{cases}$

For $\lim\limits_{x \to -1} f(x)$ to exist, $\lim\limits_{x \to -1^-} f(x)$ must equal $\lim\limits_{x \to -1^+} f(x)$. The domain states that any number less than -1 goes into 3 - x. Therefore,

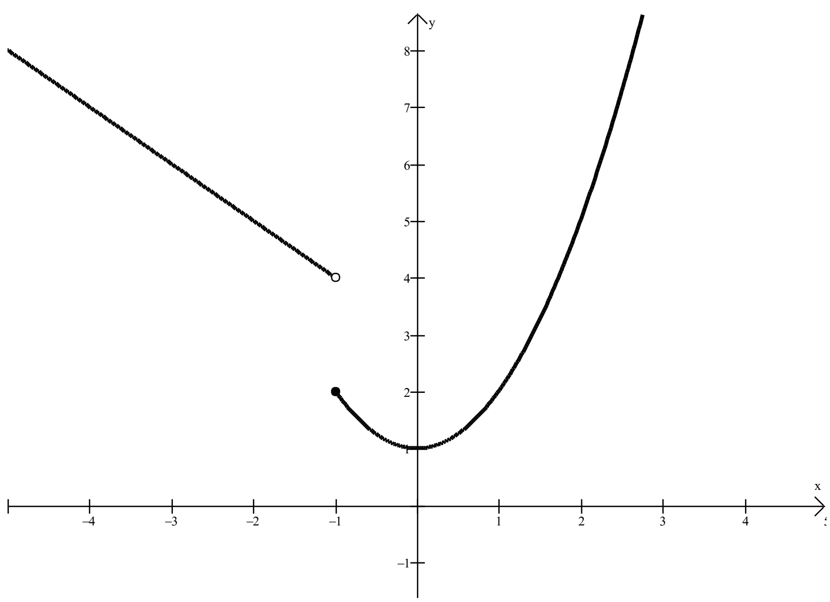
$$
\begin{aligned}
\lim_{x \to -1^-} f(x) &= \lim_{x \to -1}(3 - x) \\
&= 3 - (-1) \\
&= 4
\end{aligned}
$$

Similarly, numbers greater than -1 go into $x^2 + 1$, and

$$
\begin{aligned}
\lim_{x \to -1^+} f(x) &= \lim_{x \to -1}\left(x^2 + 1\right) \\
&= (-1)^2 + 1 \\
&= 2
\end{aligned}
$$

You can see that the two one-sided limits are not equal. Therefore, $\lim_{x \to -1} f(x)$ does not exist.

As see in the graph, the two parts do not come together:



Another situation where one-sided limits come into play is at vertical asymptotes. Here, the y-value goes to infinity (or negative infinity), which is why these limits are also called **Infinite Limits**.

EX 4 Evaluate a) $\lim_{x \to 2^-} \frac{2}{x-2}$ and b) $\lim_{x \to 2^+} \frac{1-x}{2-x}$

In both these cases, there is a vertical asymptote to consider. The limits, being *y* values, would be either positive or negative infinity, depending on if the curve went up or down on that side of the asymptote. Looking at the limit algebraically (as opposed to graphically) though:

a) $$\lim_{x \to 2^-} \frac{2}{x-2} = \frac{2}{0^-} = -\infty$$

Note that the 0^- is not "from the left" because the 2 is from the left, but rather that $x-2$ is negative for any *x* values less than 2.

b) $$\lim_{x \to 2^+} \frac{1-x}{2-x} = \frac{-1}{0^-} = +\infty$$

Note in this case that the numerator's sign affects the outcome (two negatives make a positive).

Note: It is debatable whether the infinite limits exist or not. It depends on whether "exist" is defined as equal to a real number or not. Some books would say that $\lim_{x \to 2^-} \frac{2}{x-2}$ ***exists because there is one answer. It just happens to be a transfinite number. Other books would say*** $\lim_{x \to 2^-} \frac{2}{x-2}$ ***does not exist (DNE) because the answer is not a real number.***

There are certain infinite limits that must be known:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x \to 0^+} (\ln x) = -\infty$$

Summary of Infinite Limit Process

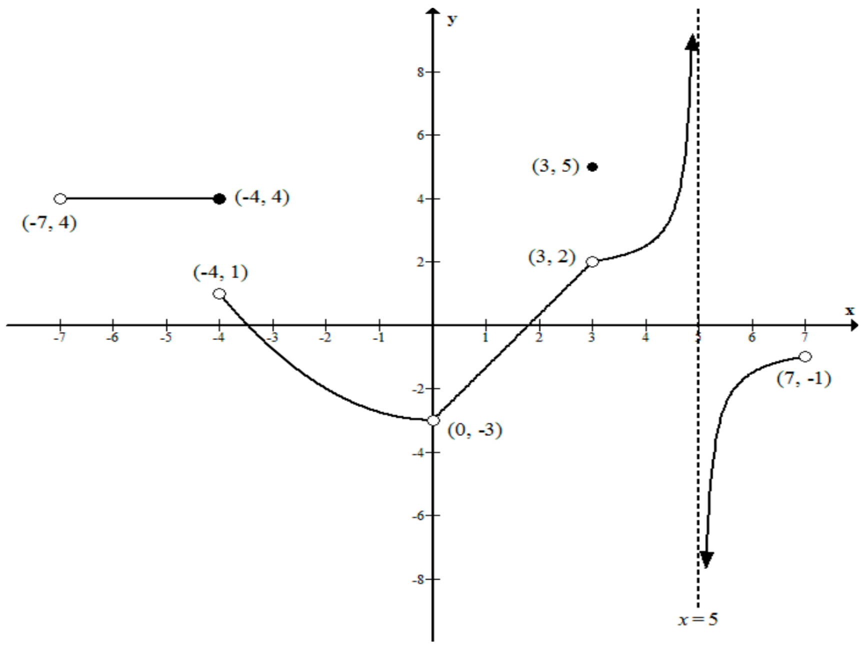
1. Determine that $\lim_{x \to a} f(x) = \frac{non-zero}{0}$

2. Determine whether the 0 is reached through positive numbers or negative numbers (i.e., “is the 0 positive or negative).

3. Count the number of negative to determine if the ∞ is positive or negative.

7.3 Free Response Homework

1. Evaluate each of the following for the graph of $f(x)$, shown below.



a) $\lim_{x\to -4^-} f(x) =$

b) $\lim_{x\to -4^+} f(x) =$

c) $\lim_{x\to 3} f(x) =$

d) $\lim_{x\to -4} f(x) =$

e) $\lim_{x\to 5^+} f(x) =$

f) $\lim_{x\to 5^-} f(x) =$

g) $f(-4) =$

h) $\lim_{x\to 0} f(x) =$

i) $f(0) =$

j) $f(3) =$

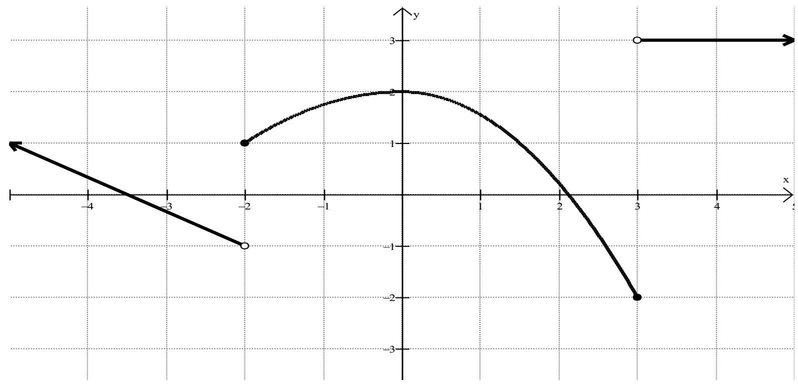
k) $\lim_{x\to 0^+} f(x) =$

l) $\lim_{x\to 0^-} f(x) =$

m) $\lim_{x\to 3} f(x) =$

n) $f(5) =$

2. Evaluate each of the following for the graph of $f(x)$, shown below.



a. $\lim_{x \to -2^-} f(x) =$

b. $\lim_{x \to -2^+} f(x) =$

c. $\lim_{x \to -2} f(x) =$

d. $f(-2) =$

e. $\lim_{x \to 0^+} f(x) =$

f. $\lim_{x \to 0^-} f(x) =$

g $\lim_{x \to 0} f(x) =$

h. $f(0) =$

i. $\lim_{x \to 3^+} f(x) =$

j. $\lim_{x \to 3^-} f(x) =$

k. $\lim_{x \to 3} f(x) =$

l. $f(3) =$

m. $\lim_{x \to 4^+} f(x) =$

n. $\lim_{x \to 4^-} f(x) =$

Evaluate the following limits.

3. $\lim_{x \to 1^-} \frac{2x}{x^2 - 1}$

4. $\lim_{x \to -3^+} \frac{(x+4)(x-3)}{(x-2)(x+3)(x-1)}$

5. $\lim_{x \to 2^-} \frac{1-2x}{x^2 - 4x + 4}$

6. $\lim_{x \to 2^+} \frac{7x}{x^2 - 4}$

7. $\lim_{x \to 0^+} \frac{\ln x}{\sin x}$

8. $\lim_{x \to 2^-} \frac{7x}{x^2 - 4}$

9. $\lim_{x \to 0^-} \frac{\cot(x^2 - 5x)}{\ln x^2}$

10. Given the function $f(x)=\dfrac{-2x+1}{x^2-5x-6}$ find each of the following:

a. $\lim_{x\to\infty} f(x)$ b. $\lim_{x\to-\infty} f(x)$ c. $\lim_{x\to6^+} f(x)$

d. $\lim_{x\to6^-} f(x)$ e. $\lim_{x\to-1^+} f(x)$

7.3 Multiple Choice Homework

1. A function $f(x)$ has a vertical asymptote at $x=2$. The derivative of $f(x)$ is negative for all $x<-2$ and positive for all $-2<x$. . Which of the following statements are **false**?

I. $\lim_{x\to-2} f(x)=-\infty$ II. $\lim_{x\to-2^-} f(x)=-\infty$ III. $\lim_{x\to-2^+} f(x)=+\infty$

a) I only b) II only c) III only

d) I and II only e) II and III only

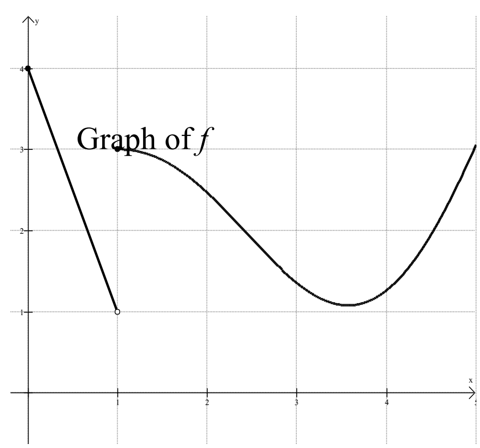
---

2. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?

I. $\lim_{x\to1} f(x)$ does not exist.

II. $\lim_{x\to1^+} f(x)=1$

III. $\lim_{x\to1^-} f(x)=3$

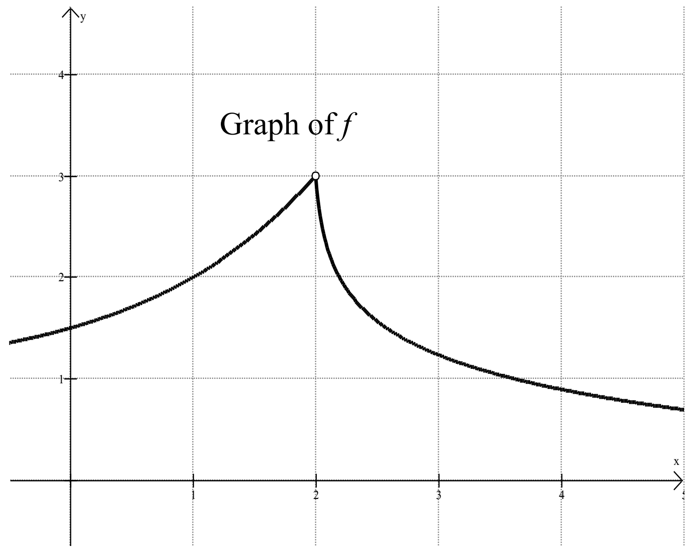


a) I only b) I and II only c) II and III only

d) I, II, and III e) III only

---

3. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?



I. $\lim_{x \to 2} f(x)$ does not exist.

II. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

III. $\lim_{x \to 2} f(x) = 3$

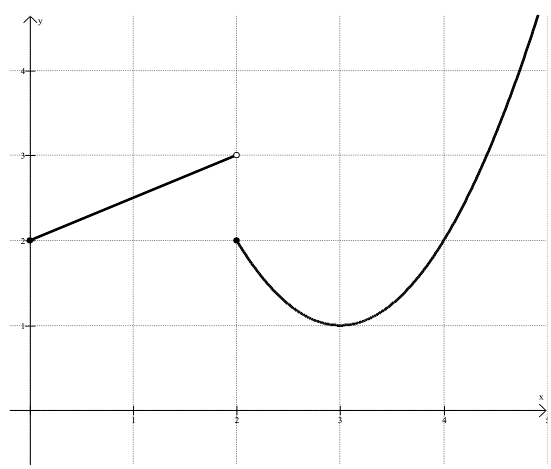
a) I only b) I and II only c) II and III only

d) I, II, and III e) III only

---

4. The graph of a function, *f*, is shown below. Which of the following statements are **TRUE**?

Graph of *f*



I. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

II. $f(2) = 2$

III. $\lim_{x \to 2} f(x)$ does not exist

a) I only b) I and II only c) II and III only

d) I, II, and III e) III only

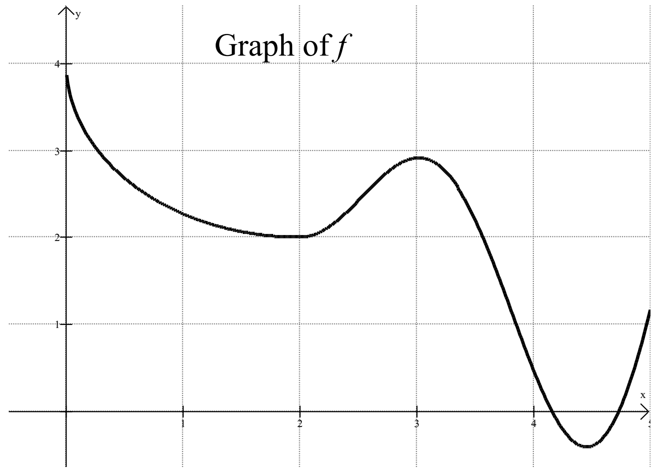
---

5. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?

I. $f'(2) > \lim_{x \to 2} f(x)$

II. $f'(1) < \lim_{x \to 2} f(x)$

III. $\lim_{x \to 2} f(x)$ does not exist



a) I only b) II only c) I, II, and III

d) I and III only e) II and III only

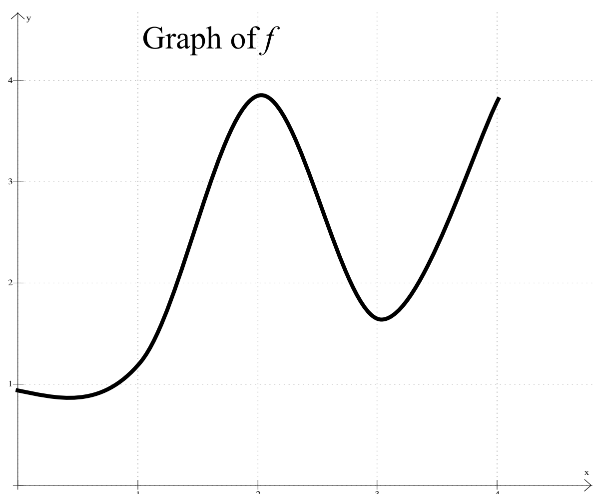
---

6. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?

I. $\lim_{x \to 2} f(x) > \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

II. $\lim_{x \to 2} f(x) = \lim_{x \to 3} f(x)$

III. $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}$

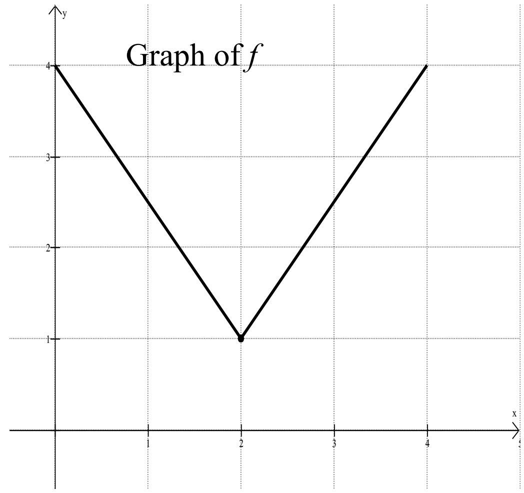


a) I only b) II only c) I, II, and III

d) I and III only e) II and III only

---

7. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. $\lim_{x\to 2} f(x) = f(2)$

II. $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

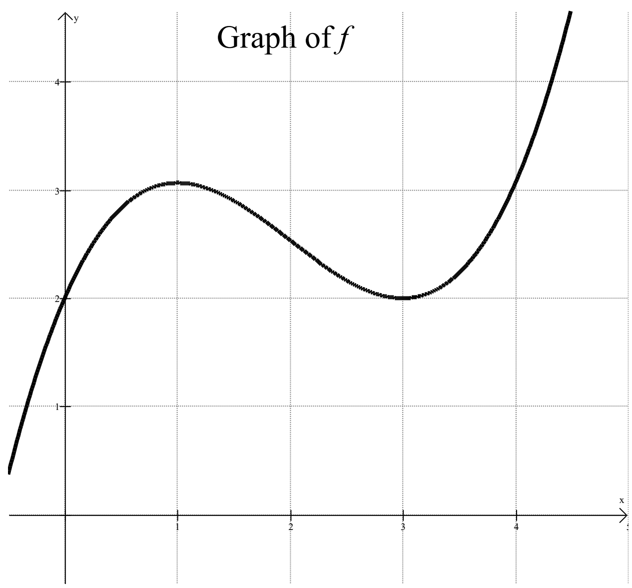
III. $\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ does not exist

a) I only b) II only c) I, II, and III

d) I and III only e) II and III only

---

8. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. $\lim_{x\to 1} f(x) = f(1)$

II. $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x)$

III. $\lim_{x\to 2} \dfrac{f(x)-f(2)}{x-2} < \lim_{x\to 4} \dfrac{f(x)-f(4)}{x-4}$

a) I only b) I and III only c) I, II, and III

d) II only e) II and III only

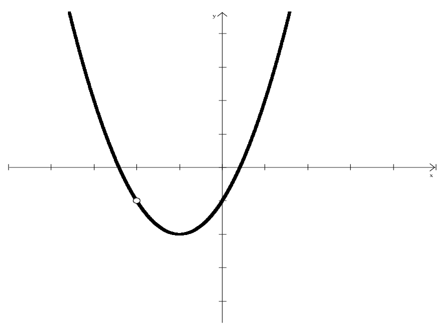
7.4: Continuity

Continuity basically means a function's graph has no breaks in it. In general, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

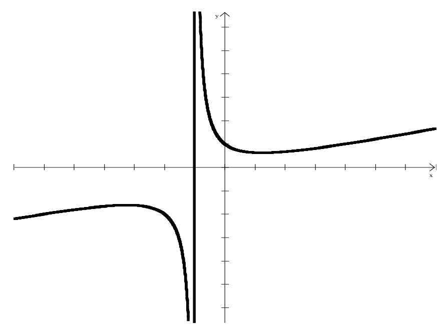
Removable Discontinuity ($\lim_{x \to a} f(x)$ does exist)

Essential Discontinuity ($\lim_{x \to a} f(x)$ does not exist)

f(a) does not exist

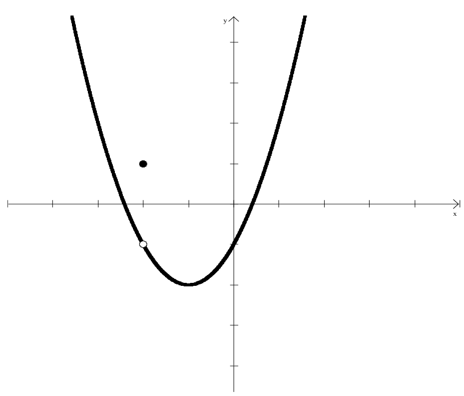


Point of Exclusion (POE)

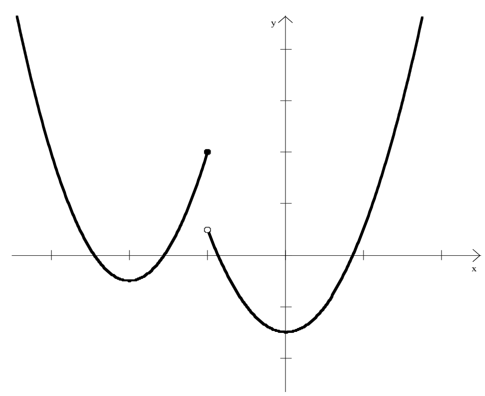


Vertical asymptotes

f(a) exists



Point of Displacement (POD)



Jump Discontinuity

NB. It will be a great advantage to know the shapes of the graphs of various equations. Please review Chapter Appendix A.1.

The formal definition involved limits.

Continuous--Defn: "A function f(x) is continuous at x = a if and only if:

i. *f(a)* exists*,

ii. $\lim_{x\to a} f(x)$ exists*,

and iii. $\lim_{x\to a} f(x) = f(a)$."

**By "exists," it means "equals a real number."*

i) "*f(a)* exists" means *a* must be in the domain.

ii) "$\lim_{x\to a} f(x)$ exists" means $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

iii) "$\lim_{x\to a} f(x) = f(a)$" should be self-explanatory.

NB. All the families of functions which were explored in PreCalculus are continuous in their domain.

Ex 3 $g(x) = \begin{cases} x^2 - 2x + 1, & if\ x > -1 \\ 2\ , & if\ x = -1 \\ 3 - x, & if\ x < -1 \end{cases}$ Is $g(x)$ continuous at x = -1?

To answer this question, one must check each part of the definition.

i) Does *g*(-1) exist? Yes, the middle line says that -1 is in the domain and it tells us that *y* = 2 if *x* = -1.

ii) Does the $\lim_{x\to -1} g(x)$ exist? Yes.

Since both one-sided limits equal 4, then

iii) Does $\lim_{x\to -1} g(x) = g(-1)$? No. $\lim_{x\to -1} g(x)$=4, while $g(-1) = 2$

So *g(x)* is not continuous at $x = -1$ because the limit does not equal the function.

Ex 4 $F(x)=\begin{cases} x^2-5, & \text{if } x>0 \\ x+2, & \text{if } x<0 \end{cases}$ Is *F(x)* continuous at $x = 0$? Why not?

F(x) is not continuous at $x = 0$ because 0 is not in the domain. Notice neither inequality includes an equals sign.

Ex 5 If $G(x)=\begin{cases} x+2, & \text{if } x>-1 \\ 3, & \text{if } x=-1 \\ x^2-5, & \text{if } x<-1 \end{cases}$, is *G(x)* continuous at $x = 1$? Why not?

i) Does $G(1)$ exist? Yes, the second line says that 1 is in the domain (and it tells us that $y = 3$, if $x = 1$).

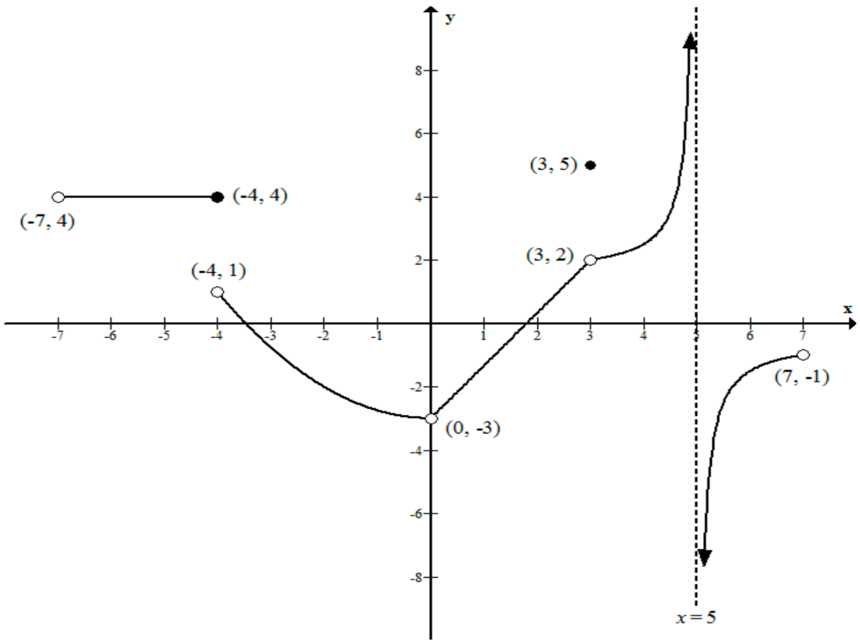
ii) Does the $\lim_{x\to -1} g(x)$ exist? No. The two-sided limit only exists if the two one-sided limits are equal. But,

The two one-sided limits are not equal. Therefore,

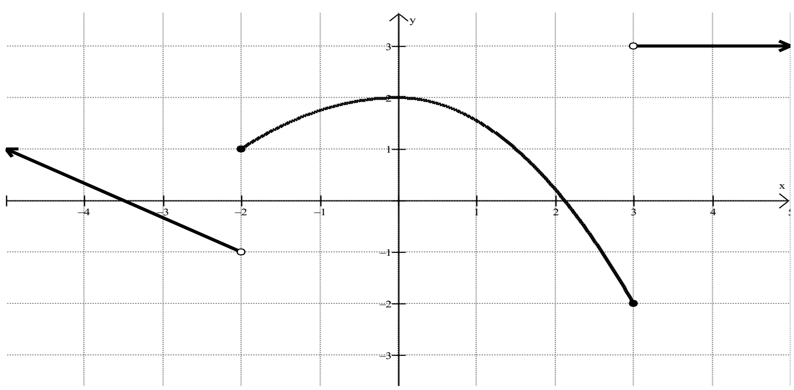
G(x) is not continuous at $x = 1$, because $\lim_{x\to 1} G(x)$ does not exist.

7.4 Free Response Homework

1. The graph of $f(x)$ is shown below. Where is $f(x)$ discontinuous and why?



2. The graph of $g(x)$ is shown below. Where is $g(x)$ discontinuous and why?



Determine if each of the following functions is continuous at $x = a$ and use the definition to prove it.

3. $f(x)=\begin{cases} x^2-1, & \text{if } x>-1 \\ 0, & \text{if } x=-1 \\ 4-x, & \text{if } x<-1 \end{cases}$; $a = -1$

4. $g(x)=\dfrac{x^2-4x-5}{x^2-1};\ a=-1$

5. $h(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & if\ x\neq 2\\ 4, & if\ x=2\end{cases};\ a=2$

6. $k(x)=\begin{cases}2x+1, & if\ x\leq -1\\ 3-x^2, & if\ x>-1\end{cases};\ a=3$

7. $k(x)=\begin{cases}2x+1, & if\ x\leq -1\\ 3-x^2, & if\ x>-1\end{cases};\ a=-1$

8. $F(x)=\begin{cases}2x+1, & if\ x\leq -1\\ -x^2, & if\ x>-1\end{cases};\ a=-1$

9. $H(x)=\begin{cases}2x+1, & if\ x>-1\\ 0, & if\ x=-1\\ -x^2, & if\ x<-1\end{cases};\ a=-1$

10. $G(x)=\begin{cases}\cos x, & if\ x\leq \pi\\ x^2-\pi x-1, & if\ x>\pi\end{cases};\ a=\pi$

11. $f(x)=\begin{cases}\tan^{-1}(x-3) & \text{if } x\leq 3\\ 1-\cos(x-3) & \text{if } x>3\end{cases};\ a=3$

12. $f(x)=\begin{cases}\ln(1+x) & \text{if } x<0\\ x^2+5x & \text{if } x>0\end{cases};\ a=0$

7.4 Multiple Choice Homework

1. The function f is continuous at $x = b$. Which of the following statements **must** be false?

a) $\lim_{x \to b} f(x)$ DNE b) $\lim_{x \to b} f(x) = f(b)$ c) $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$

d) $f(b)$ is defined e) $f'(b)$ exists

---

2. The function f is not continuous at $x = b$. Which of the following statements **must** be false?

a) $\lim_{x \to b} f(x)$ dne b) $\lim_{x \to b} f(x) \neq f(b)$ c) $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$

d) $\lim_{x \to b^-} f'(x) = \lim_{x \to b^+} f'(x)$ e) None of these

---

3. Which of the following is true about the function f if $f(x) = \dfrac{(x-1)^2}{2x^2 - 5x + 3}$?

I. f is continuous at $x = 1$
II. The graph of f has a vertical asymptote at $x = 1$
III. The graph of f has a horizontal asymptote at $y = \dfrac{1}{2}$

a) I only b) II only c) III only

d) II and III only e) I, II, and III

---

4. Which function is **not** continuous everywhere?

a) $y=|x|$ b) $y=x^{2/3}$ c) $y=\sqrt{x^2+1}$

d) $y=\dfrac{x}{x^2+1}$ e) $y=\dfrac{4x}{(x+1)^2}$

---

5. If f is continuous at $x=1$, and if $f(x)=\begin{cases}\dfrac{\sqrt{x+3}-\sqrt{3x+1}}{x-1} & \text{for } x\neq 1\\ k & \text{for } x=1\end{cases}$,

then $k=$

a) 0 b) 1 c) $\dfrac{1}{2}$ d) $-\dfrac{1}{2}$ e) None of these

---

6. Let f be defined by $f(x)=\begin{cases}x^2+kx & \text{for } x<4\\ 4\cos\left(\dfrac{\pi}{2}x\right) & \text{for } x\geq 4\end{cases}$. Determine the value of k

for which is continuous for all real x.

a) –5 b) –4 c) –3 d) –2 e) None of these

---

7. A function $f(x)$ has a vertical asymptote at $x = 2$. The derivative of $f(x)$ is negative for all $x < -2$ and positive for all $-2 < x$. . Which of the following statements are **false**?

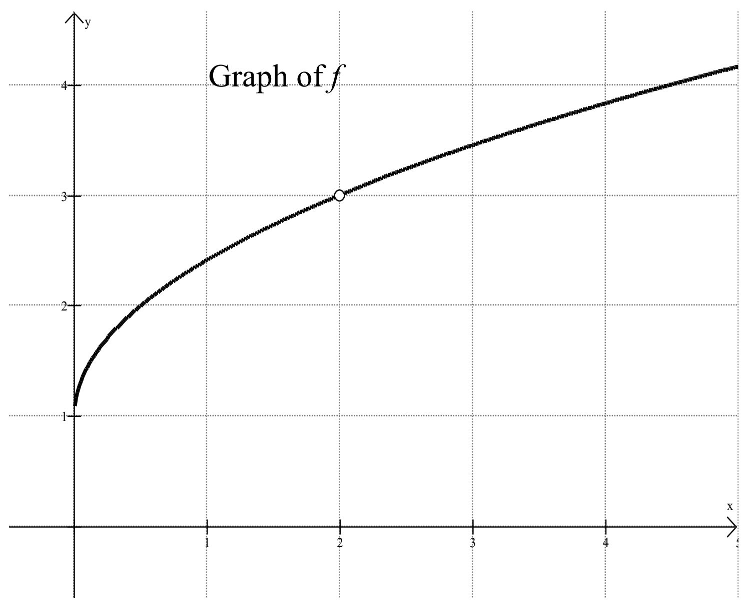
I. $\lim_{x \to -2} f(x) = -\infty$ II. $\lim_{x \to -2^-} f(x) = -\infty$ III. $\lim_{x \to -2^+} f(x) = +\infty$

a) I only b) II only c) III only

d) I and II only e) II and III only

---

12. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. f is continuous at $x = 3$

II. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

III. $\lim_{x \to 2} f(x)$ does not exist

a) I only b) II only c) I, II, and III

d) I and II only e) II and III only

7.5: Differentiability and Smoothness

The second (after Continuity) important underlying concept to Calculus is differentiability. Almost all theorems in Calculus begin with "If a function is continuous and differentiable…" It is actually a very simple idea.

Differentiability—Defn: the derivative exists. *F(x)* is differentiable at *a* if and only if *F'(a)* exists and is differentiable on *[a, b]* if and only *F(x)* is differentiable at every point in the interval.

> *F(x)* is differentiable at $x=a$ if and only if
>
> i. *F(x)* is continuous at $x=a$,
>
> and ii. $\lim_{x\to a^-} F'(x) = \lim_{x\to a^+} F'(x)$ as a real number.

NB. All the families of functions studied in PreCaculus are continuous and differentiable in their domains.

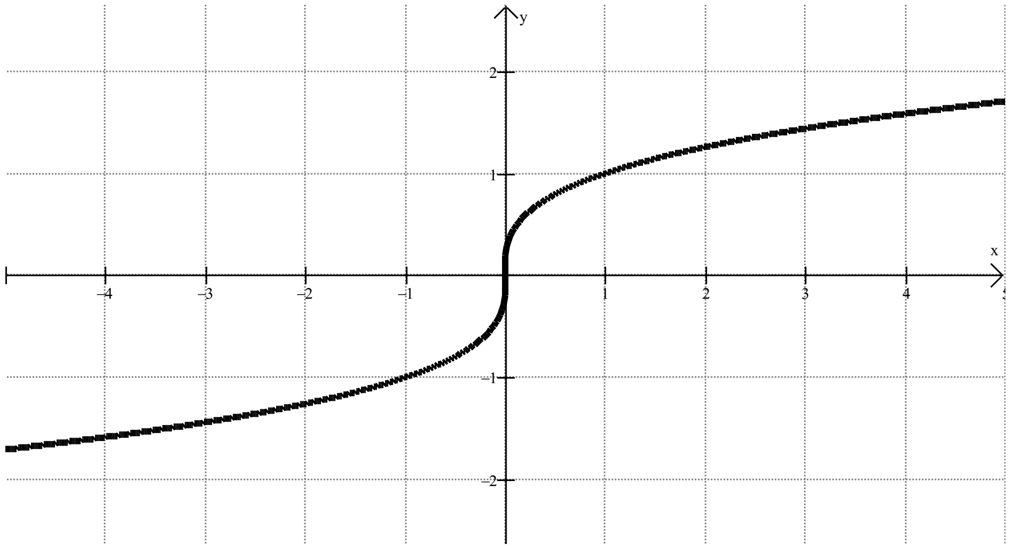
> OBJECTIVE
>
> Determine if a function is differentiable or not.
> Demonstrate understanding of the connections and differences between differentiability and continuity.

There are two ways that a function would not be differentiable:

I. The tangent line could be vertical, causing the slope to be infinite.

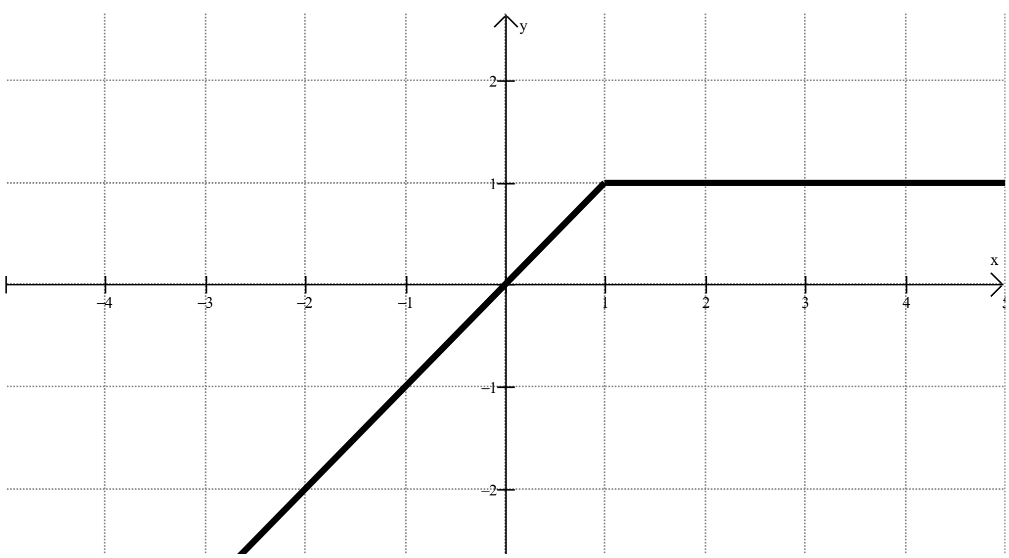
Ex 1 Is $y = \sqrt[3]{x}$ differentiable at x = 0?

If $y = \sqrt[3]{x}$, then $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$. At $x = 0$, $\frac{dy}{dx} = \frac{1}{0} = dne$.



II. Just as the Limit does not exist if the two one-sided limits are not equal, a derivative would not exist if the two one-sided derivatives are not equal. That is, the slopes of the tangent lines to the left of a point are not equal to the tangent slopes to the right. A curve like this is called non-smooth.

Ex 2 Is the function represented by this curve differentiable at $x=1$?



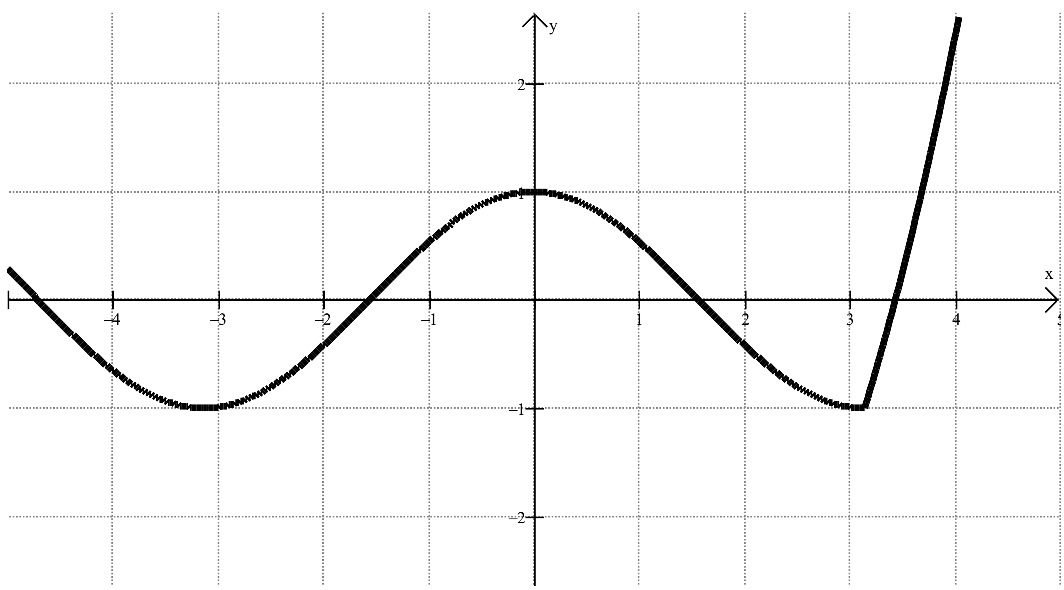
One can see that the slope to the left of x=1 is 1, while the slope to the right of x=1 is 0. So this function is not differentiable at x=1. This is an example of a curve that is not smooth.

Ex 3 Is $G(x)=\begin{cases} \cos x, & if\ x \le \pi \\ x^2-\pi x-1, & if\ x>\pi \end{cases}$ differentiable at $x=\pi$?

$$G'(x)=\begin{cases} -\sin x, & if\ x \le \pi \\ 2x-\pi, & if\ x>\pi \end{cases}$$

$$\lim_{x\to\pi^-} G'(x)=\lim_{x\to\pi}\left(-Sin\ x\right)=0$$
$$\lim_{x\to\pi^+} G'(x)=\lim_{x\to\pi}\left(2x-\pi x\right)=2\pi-\pi=\pi$$

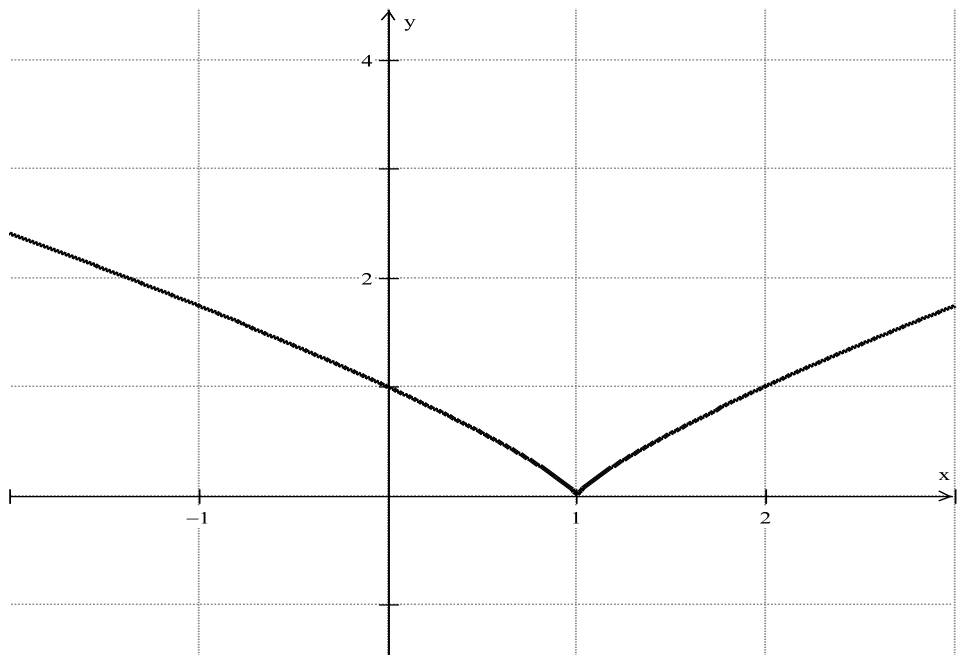
Clearly, these two one-sided derivatives are not equal. Therefore, *G(x)* is not differentiable at $x=\pi$.



Ex 4 Is $h(x)=(x-1)^{4/5}$ differentiable at $x=1$?

$$h'(x)=\frac{4}{5}(x-1)^{-1/5}$$
$$h'(x)=\frac{4}{5\sqrt[5]{x-1}}$$
$$h'(1)=\frac{4}{5\sqrt[5]{1-1}}=\frac{4}{0}=\text{D.N.E.}$$

Therefore, *h*(*x*) is not differentiable at *x* = 1. The graph below illustrates how the slopes from the left and right aren't the same.



You can always identify if a function is differentiable visually (these are the same two rules as before):

I. If the curve has a vertical tangent, it is not differentiable at that point.

II. If the curve has a corner (also called a *cusp*), it is not differentiable at that point.

If the curve is not continuous, it cannot be differentiable (there is no point at which to find a slope).

All four of these examples show functions that are continuous, but not differentiable. Continuity does not ensure differentiability. But the converse is true.

> If *f(x)* is differentiable, it MUST BE continuous
>
> and
>
> If *f(x)* is not continuous, it CANNOT BE differentiable

Ex 5 Given that $f(x)=\begin{cases} k\sqrt{x+1}, \ if \ x\le 3 \\ 2+mx, \ if \ x>3 \end{cases}$ is differentiable at *x*=3, find m and k.

$$f'(x)=\begin{cases} \dfrac{k}{2\sqrt{x+1}}, \ if \ x\le 3 \\ m, \ if \ x>3 \end{cases}$$

If *f(x)* is differentiable at *x*=3, then $\dfrac{k}{2\sqrt{3+1}}=m \to \dfrac{1}{4}k=m$.

If *f(x)* is differentiable at *x*=3, it is also continuous.

Therefore, at *x*=3, $k\sqrt{x+1}=2+mx$

$$k\sqrt{3+1}=2+m3$$

$$2k=3m+2$$

Since both $\dfrac{1}{4}k=m$ and $2k=3m+2$ must be true, one can use linear combination or substitution to solve for k and m.

$$2k = 3\left(\frac{1}{4}k\right) + 2$$

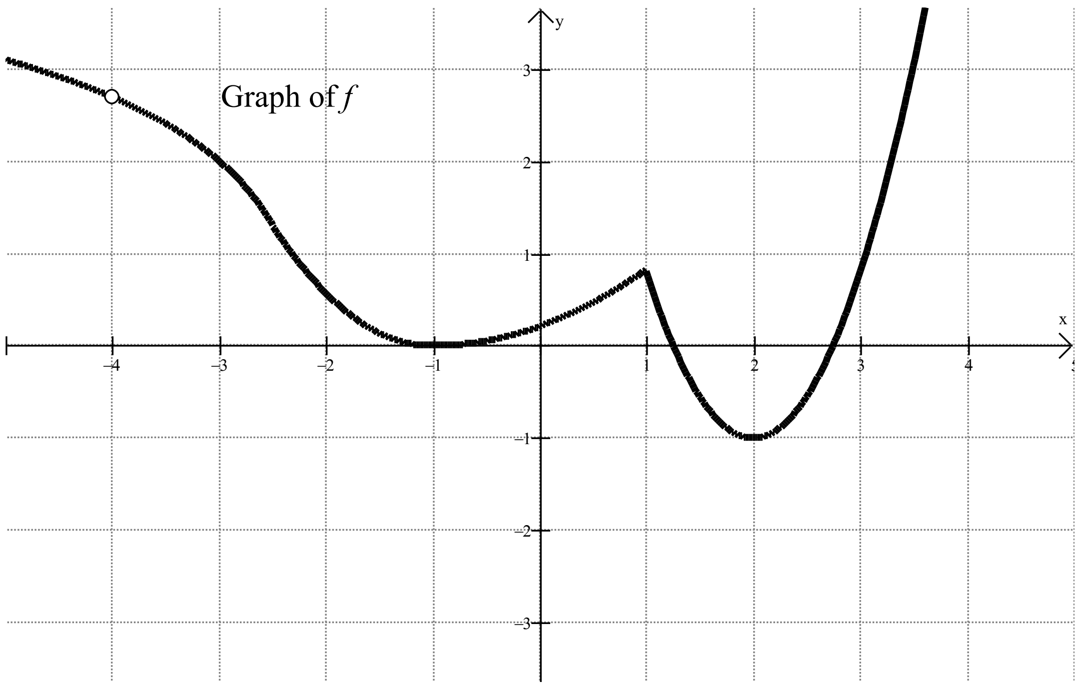
$$2k = \frac{3}{4}k + 2$$

$$\frac{5}{4}k = 2$$

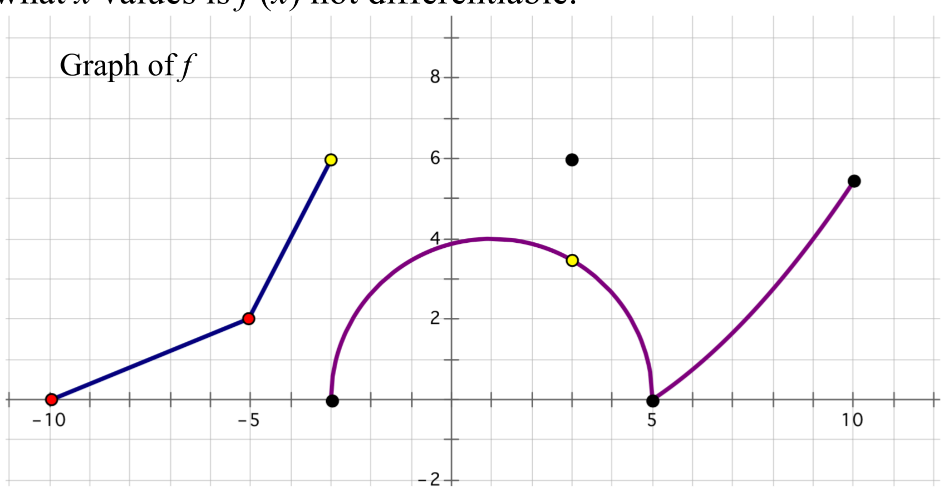
$$k = \frac{8}{5} \rightarrow m = \frac{2}{5}$$

7.5 Free Response Homework

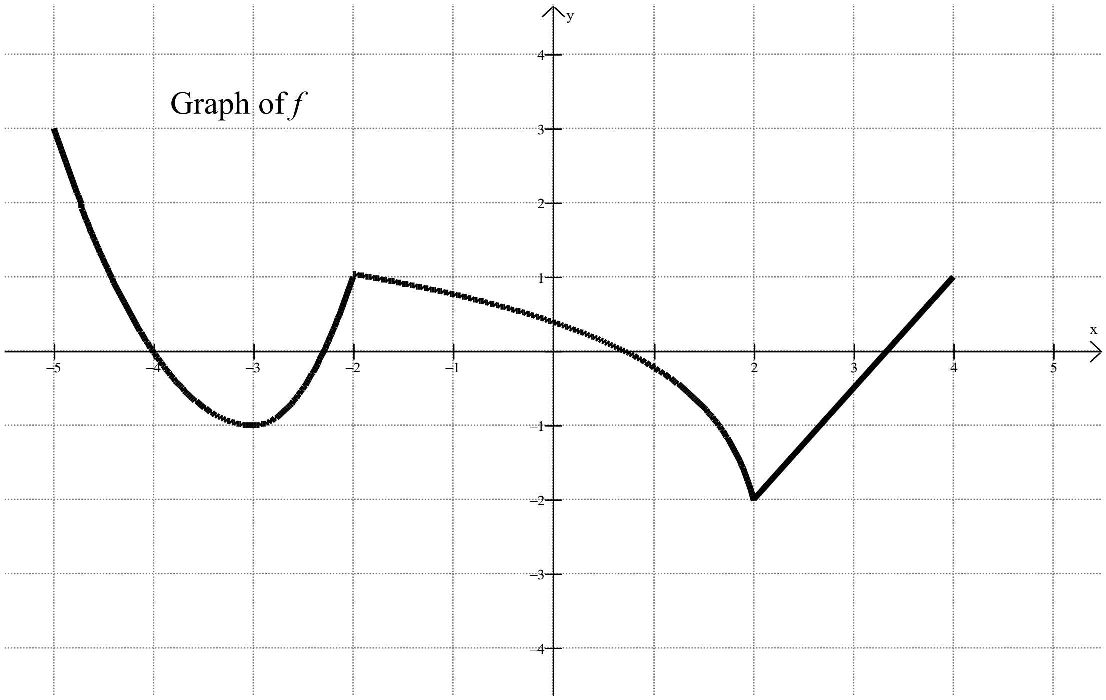
1. At what x values is $f(x)$ not differentiable.



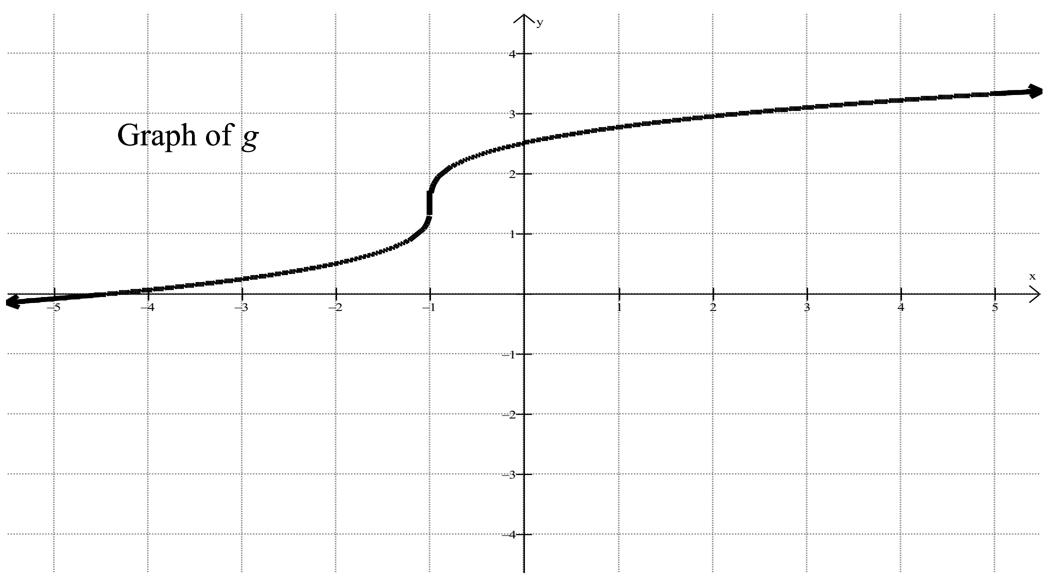
2. At what x values is $f(x)$ not differentiable.



3. At what x values is $f(x)$ not differentiable.



4. At what x value is function g not differentiable.



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5. Determine if $f(x)=\sqrt[3]{x^2}$ is differentiable or not.

6. Determine if $f(x)=\begin{cases}\sin^{-1}x, \text{ if } -1\le x<1\\ \ln x, \text{ if } x\ge 1\end{cases}$ is differentiable or not.

7. Determine if $f(x)=\begin{cases}x^2-2x+1, \text{ if } x\le 1\\ \ln x, \text{ if } x>1\end{cases}$ is differentiable or not.

8. Determine if $f(x)=\begin{cases}x^2+2x-5, \text{ if } x\le 1\\ x^3+x-4, \text{ if } x>1\end{cases}$ is differentiable or not.

9. Given that $f(x)=\begin{cases}mx+2, \text{ if } x\le 1\\ k\ln x, \text{ if } x>1\end{cases}$ is differentiable at $x = 1$, find m and k.

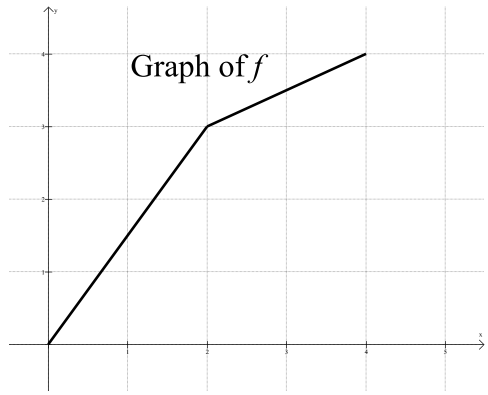
10. Given that $f(x)=\begin{cases}mx-5, \text{ if } x\le -2\\ kx^2+1, \text{ if } x>-2\end{cases}$ is differentiable at $x = -2$, find m and k.

11. Given that $f(x)=\begin{cases}ke^{2x}, \text{ if } x\le 0\\ 3-mx, \text{ if } x>0\end{cases}$ is differentiable at $x = 0$, find m and k.

12. Given that $f(x)=\begin{cases}mx-2, \text{ if } x\le 2\\ k\sqrt{x^2-3}, \text{ if } x>2\end{cases}$ is differentiable at $x = 2$, find m and k.

7.5 Multiple Choice Homework

1. Use the graph of *f* below to select the correct answer from the choices below.



a) f has no extremes

b) f is continuous at $x = 2$

c) f is differentiable for $x \in (0,4)$

d) f has a relative maximum at 2

e) f is concave up for $x \in (0,4)$

---

2. The function *f* is not differentiable at $x = b$. Which of the following statements must be false?

a) $\lim_{x \to b} f(x)$ dne

b) $\lim_{x \to b} f(x) \neq f(b)$

c) $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$

d) $\lim_{x \to b^-} f'(x) = \lim_{x \to b^+} f'(x)$

e) None of these

---

3. The function *f* is differentiable at $x = b$. Which of the following statements could be false?

a) $\lim_{x \to b} f(x)$ exists

b) $\lim_{x \to b} f(x) = f(b)$

c) $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$

d) $\lim_{x \to b^-} f'(x) = \lim_{x \to b^+} f'(x)$

e) None of these

---

4. Let m and b be real numbers and let the function f be defined by

$$f(x)=\begin{cases} 3x^2-mx+5 & \text{for } x\le 1 \\ mx+b & \text{for } x>1 \end{cases}$$

If f is both continuous and differentiable at $x=1$, then

a) $m=3,\ b=2$
b) $m=3,\ b=-2$
c) $m=-3,\ b=2$
d) $m=-3,\ b=-2$
e) None of these

---

5. The function f defined on all the Reals such that

$f(x)=\begin{cases} x^2+kx-3 & \text{for } x\le 1 \\ 3x+b & \text{for } x>1 \end{cases}$. For which of the following values of k and b will the function f be both continuous and differentiable on its entire domain?

a) $k=1,\ b=-3$
b) $k=1,\ b=3$
c) $k=-1,\ b=-3$
d) $k=-1,\ b=3$
e) $k=-1,\ b=6$

---

6. Let $f(x)=\begin{cases} -x+5, & \text{if } x<-2 \\ x^2+3, & \text{if } -2\le x\le 1 \\ 2x^3, & \text{if } 1<x \end{cases}$. Which of the following statements is **true** about f?

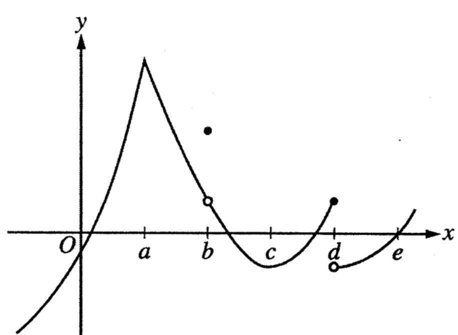
I. f is continuous at $x = 1$.
II. f is differentiable at $x = 1$.
III. f has a local minimum at $x = 0$.

a) I only b) II only c) III only

d) I and III only e) II and III only

---

7. The graph of the function f is shown below. At which value of x is f continuous, but not differentiable?

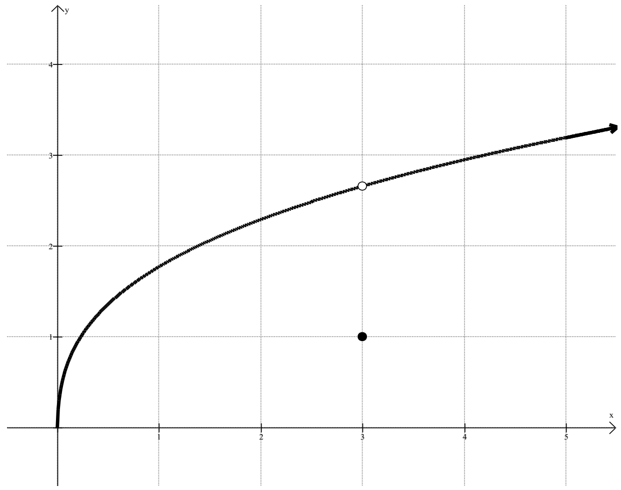


Graph of f

a) a b) b c) c d) d e) e

---

8. Use the graph of f below to select the correct answer from the choices below.



a) $\lim_{x\to 3} f(x) = f(3)$

b) f is not continuous at $x = 3$

c) f is differentiable at $x = 3$

d) $f'(1) < f'(4)$

e) $\lim_{x\to 3} f(x)$ does not exist

---

9. Let f be the function given below. Which of the following statements are true about f?

$$f(x) = \begin{cases} x+2 & \text{if } x \le 3 \\ 4x-7 & \text{if } x > 3 \end{cases}$$

I. $\lim_{x\to 3} f(x)$ exists
II. f is continuous at $x = 3$
III. f is differentiable at $x = 3$

a) None b) I only c) II only d) I and II only e) I, II, and III

---

10. Let f be the function defined below, where c and d are constants. If f is differentiable at $x = 2$, what is the value of $c + d$?

$$f(x) = \begin{cases} cx + d & \text{if } x \le 2 \\ x^2 - cx & \text{if } x > 2 \end{cases}$$

a) –4 b) –2 c) 0 d) 2 e) 4

---

11. Let $P(x)$ and $Q(x)$ be polynomials. Find $\lim_{x \to \infty} \frac{P(x)}{Q(x)}$ if the degree of $P(x)$ is 5 and the degree of $Q(x)$ is 9.

a) $\frac{5}{9}$ b) $\frac{9}{5}$ c) 0 d) DNE

e) There is not enough information to answer the question

---

12. Use the graph of f below to select the correct answer from the choices below.

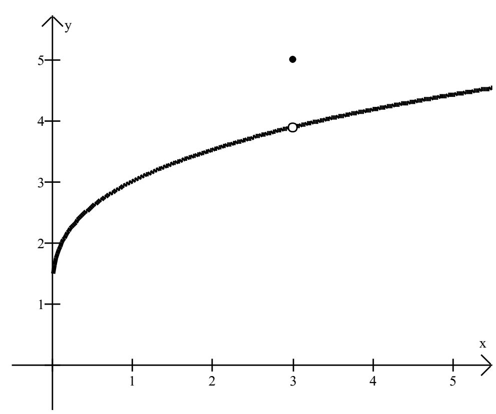
a) f has a relative maximum at $x = 3$

b) f is continuous at $x = 3$

c) f is differentiable at $x = 3$

d) $f'(3) < f(3)$

e) None of the above



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13. If $\lim_{x \to 2} \frac{f(x)}{x-2} = f'(2) = 0$, which of the following must be true?

I. $f(2) = 0$
II. $f(x)$ is continuous at $x = 2$
III. $f(x)$ has a horizontal tangent line at $x = 2$

a) I only b) II only c) I and II only d) II and III only e) I, II, and III

7.6: Limits at Infinity and End Behavior

Vocabulary

Limit at Infinity—Defn: the y-value when x approaches infinity or negative infinity

Infinite Limit— Defn: a limit where y approaches infinity

End Behavior—Defn: The graphical interpretation of a limit at infinity

OBJECTIVE
Evaluate Limits at infinity. Interpret Limits at infinity in terms of end behavior of the graph.

Evaluating limits at infinity for algebraic functions relies on one fact:

$$\lim_{x\to\pm\infty}\frac{1}{x}=0$$

This fact is what led to our general rule about horizontal asymptotes.

Ex 1 Evaluate $\lim_{x\to-\infty}\frac{1-3x^2}{4x^2+3}$.

It is already known, from last year, that this function, $y=\frac{1-3x^2}{4x^2+3}$, has a horizontal asymptote at $y=-\frac{3}{4}$. Why is this true? Because

$$\lim_{x\to-\infty}\frac{1-3x^2}{4x^2+3}=\lim_{x\to-\infty}\frac{\frac{1-3x^2}{x^2}}{\frac{4x^2+3}{x^2}}=\lim_{x\to-\infty}\frac{\frac{1}{x^2}-3}{4+\frac{3}{x^2}}$$

$$=\frac{\lim_{x\to-\infty}\frac{1}{x^2}-3}{4+\underset{x\to-\infty}{Lim}\frac{3}{x^2}}$$

$$=\frac{0-3}{4+0}$$

$$=-\frac{3}{4}$$

Honestly, one never really needs to do this algebraic process because of the end behavior rules from last year. These kinds of limits are generally more intuitive than analytical – while one could do this analytically (as on the previous page), it is much simpler to understand what is going on and use what is known from last year.

Ex 2 Find $\lim_{x\to\infty} e^x$ and $\lim_{x\to-\infty} e^x$ and interpret these limits in terms of the end behavior of $y=e^x$.

$\lim_{x\to\infty} e^x = e^{\infty} = \infty$. This means that the right end (+ ∞) goes up.

$\lim_{x\to-\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$. This means that, on the left (- ∞), $y=e^x$ has a horizontal asymptote at $y = 0$.

****NB. The syntax used here—treating ∞ as if it were a real number and "plugging" it in—is incorrect. Though it works as a practical approach, writing this on the AP test will lose you points for process.** In particular, this will come up with Improper Integrals. This will be discussed again in Chapter 7.

There are certain Limits at Infinity (which come from our study of end behavior) that must be known.

$$\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2} \quad \text{and} \quad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$
$$\lim_{x\to-\infty} \cot^{-1} x = \pi \quad \text{and} \quad \lim_{x\to\infty} \cot^{-1} x = 0$$

$$\lim_{x\to\infty} e^{x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{x} = \infty$$
$$\lim_{x\to-\infty} e^{-x} = \infty \quad \text{and} \quad \lim_{x\to\infty} e^{-x} = 0$$

For $y' = ky(A-y)$ and its solution $y = \dfrac{A}{1 + Be^{-kt}}$, $\lim_{x\to\infty} f(x) = A$

These last two equations are the Logistic Growth equations. These will discussed further in a later chapter.

Ex 3 Evaluate $\lim_{x\to\infty} \tan^{-1}\left(x - x^3\right)$

$$\begin{aligned}\lim_{x\to\infty} \tan^{-1}\left(x-x^3\right) &= \tan^{-1}\left(\lim_{x\to\infty}\left(x-x^3\right)\right)\\ &= \tan^{-1}(-\infty)\\ &= -\frac{\pi}{2}\end{aligned}$$

While we could go through all the algebraic machinations above, it is easier to just look at the functions as if we are "plugging in" infinity, and evaluating each expression as we would in algebra.

In the above case, we would have to recognize that the polynomial is governed by the highest term (), and when we plug in infinity, we get . Then we see that we actually have , which is .

Ex 4 Evaluate

When infinity is "plug in", the result is the indeterminate form, , which means L'Hôpital's Rule applies.

But the result is still , so L'Hôpital's Rule can be reapplied.

This is still , so L'Hôpital's Rule needs to again be reapplied. But what one may have already noticed is that the numerator's degree keeps decreasing, while the denominator stays the same (because it is an exponential function). This means that eventually, if enough derivatives are taken, the numerator will become a constant, while the denominator stays an exponential function. This means eventually the limit will become a constant over infinity, which gives us 0.

Therefore, .

Limits at Infinity often involve ratios and L'Hospital's Rule can be applied. But some math teachers consider this to be "using a cannon to kill a fly." It is easier to just know the relative growth rates of functions—i.e., which families of functions grow faster than which. The End Behavior studied in PreCalculus gives a good sense of order.

The Hierarchy of Functions:

0. Domain trumps all!!!
1. Exponential functions grow faster than the others. (In BC Calculus, it will be seen that the factorial function, $y = n!$, grows the fastest, but factorials are not continuous functions.)
2. Polynomial, rational and radical functions grow faster than logarithms, and the degree of the EBM determines which algebraic function grows fastest. For example, $y = x^{1/2}$ grows more slowly than $y = x^2$.
3. The trigonometric inverses fall in between the algebraic functions at the value of their respective horizontal asymptotes.
4. Logarithms grow the slowest.
5. Period functions, by nature, do not have the same kind of end behavior as other families of functions. (See the next chapter.)

The fastest growing function in the combination determines the size end behavior, just as the highest degree term did so among the algebraic functions.

Using the Hierarchy of Functions:

Basically, the fastest growing function dominates the problem and that is the function that determines the limit. With fractions, it's easiest to extend the rules from the end behavior model of last year:

Where *a* is whatever the functions cancel to. By comparison to a fast function, the slow function basically acts like a constant. So if you look at it like that, we have the same rules from the previous two sections,

$$x \in [-3,1] \cup [2,\infty)$$

The last rule is essentially the same as the End Behavior Model from last year.

Ex 5 Evaluate (a) $\lim_{x\to\infty} \frac{x-x^3}{e^x-95}$ and (b) $\lim_{x\to\infty} \frac{\sqrt{x-x^3}}{x^2-\ln x}$

(a) Repeated iterations of L'Hospital's Rule will give the same result, but, because exponential grow faster than polynomials, $\lim_{x\to\infty} \frac{x-x^3}{e^x-95} = 0$

(b) Since $\sqrt{x-x^3}$ essentially has a degree of $\frac{3}{2}$, the denominator has a higher degree and $\lim_{x\to\infty} \frac{\sqrt{x-x^3}}{x^2-\ln x} = 0$.

EX 6 Use the concept of the Hierarchy of Functions to find the end behavior of: (a) $y = 2xe^x$, (b) $y = x \ln x$, and (c)

(a) For $y = 2xe^x$, the exponential $y = e^x$ determines the end behavior. So the right end goes up and the left end has a horizontal asymptote at $y = 0$.

(b) We still must consider the domain, first. There is no end behavior on left, because the domain is $y \in (0, \infty)$. For $y = x \ln x$, $y = x$ dominates, so, on the right, the curve goes up.

(c) For , the exponential determines the end behavior, but the polynomial might have an influence. The right end goes up, and the left end goes down. This is because goes up on both ends, but when negative infinity is "plug in" to the , is multiplied by a negative, so the function goes down.

Ex 7 Evaluate a) $\lim\limits_{x \to -\infty} \dfrac{x - x^3}{e^x - 95}$ and b) $\lim\limits_{x \to 0^+} x^2 \ln x$

a) At first glance, this appears to be a "hierarchy of functions" issue, but as we "plug in" negative infinity, only the x^3 matters in the numerator, and e^x goes to zero as x approaches negative infinity. This gives us $\dfrac{\infty}{-95}$ which gives us a result of $\lim\limits_{x \to -\infty} \dfrac{x - x^3}{e^x - 95} = -\infty$

b) Zero times negative infinity. **This is not necessarily zero!** It is an indeterminate form of a number. Either convert it to a fraction to look at it with L'Hôpital's Rule or go the more intuitive approach with the Hierarchy of Functions.

i) L'Hôpital's Rule:

$$\lim_{x \to 0^+} x^2 \ln x$$

$$= \lim_{x \to 0^+} \frac{\ln x}{x^{-2}} \qquad \text{(this gives negative infinity over infinity)}$$

$$\overset{L'H}{=} \lim_{x \to 0^+} \frac{\frac{1}{x}}{-2x^{-3}}$$

$$= \lim_{x \to 0^+} \frac{-1}{2} x^2 = 0$$

ii) Hierarchy of Functions:
Since x^2 is a "faster" function than $\ln x$, the x^2 is dominating the entire expression. It essentially gets to 0 "faster" than the

natural logarithm reaches negative infinity. Therefore, $\lim_{x\to 0^+} x^2 \ln x = 0$

Ex 8 Evaluate $\lim_{x\to\infty} \frac{e^x + x}{2 - 5e^x}$

Again, evaluating this by using L'Hôpital's Rule (because it yields $\frac{\infty}{-\infty}$) or the Hierarchy could be used. With the Heirarchy of Functions,

$\lim_{x\to\infty} \frac{e^x + x}{2 - 5e^x}$	Ignore the *x* and the 2 because they are insignificant compared with e^x.
$\lim_{x\to\infty} \frac{e^x + x}{2 - 5e^x} = -\frac{1}{5}$	Simply cancel the e^x leaving us with the final value of $-\frac{1}{5}$

It would have taken two applications of L'Hôpital's Rule to get the analytical result reached more easily with an intuitive understanding of how these functions actually behave.

7.6 Free Response Homework

Evaluate the Limit.

1. $\lim_{x\to\infty}\frac{1-15x+12x^2}{16x^2-1}$

2. $\lim_{x\to-\infty} xe^x$

3. $\lim_{x\to-\infty}\left(2+e^x\right)\tan^{-1}\left(x^2+1\right)$

4. $\lim_{x\to\infty}\ln\left(\tan^{-1}\left(\frac{x}{e^x}+1\right)\right)$

5. $\lim_{x\to\infty}\frac{\cos x}{18-4x+x^2}$

6. $\lim_{x\to\infty}\frac{2^x+1}{x^{48}-48x}$

7, $\lim_{x\to-\infty}\frac{\sin x}{\ln(-x)}$

8. $\lim_{x\to-\infty}\left(\tan^{-1}\left(\frac{x^2+5x+1}{x-2}\right)\right)$

9. $\lim_{x\to-\infty}\left(\tan^{-1}\left(\frac{x^3+7x^2+10x}{x+5}\right)\right)$

10. $\lim_{x\to-\infty}\frac{\tan^{-1}\left(\frac{x^3}{x^3-3x}\right)}{\left(e^x+1\right)}$

11. $\lim_{x\to\infty}\frac{\cot^{-1}x}{e^{-x}+1}$

Use limits to determine the end behavior of the following functions.

12. $f(x)=\left(e^x+1\right)\left(\tan^{-1}x^3\right)$

13. $f(x)=\frac{\cos x}{x^2}$

14. $f(x)=\frac{\ln x}{\sqrt{x-4}}$

15. $f(x)=\frac{\ln x^2}{\sqrt{4-x}}$

7.6 Multiple Choice Homework

1. $\lim_{x\to\infty} \dfrac{4x^4+2x^3+x^2+1}{3x^5-9x^4+4x^3+15} =$

a) 0 b) $\dfrac{3}{4}$ c) $\dfrac{4}{3}$ d) 3 e) DNE

---

2. If a and b are positive constants, then $\lim_{x\to\infty} \dfrac{\ln(bx+1)}{\ln(ax^2+3)} =$

a) 0 b) $\dfrac{1}{2}$ c) $\dfrac{ab}{2}$ d) 2 e) ∞

---

3. $\lim_{x\to\infty} \dfrac{4x^4+2x^3+x^2+1}{3x^3-9x^2+4x+15} =$

a) 0 b) $\dfrac{3}{4}$ c) $\dfrac{4}{3}$ d) 3 e) DNE

---

4. $\lim_{x\to\infty} \dfrac{x^2}{e^x}$

a) 0 b) ∞ c) $-\infty$ d) 1 e) e^x

---

5. If the graph of $y=\dfrac{ax+b}{x+c}$ has a horizontal asymptote $y=2$ and vertical asymptote $x=-3$, then $a+c=$

a) –5 b) –1 c) 0 d) 1 e) 5

---

6. $\lim_{n\to\infty} \frac{4n^2}{n^2+10{,}000n}$

a) 0 b) $\frac{1}{2500}$ c) 1 d) 4 e) DNE

---

7. $\lim_{x\to\infty} \left(4x-x^2\right)e^{-x}$

a) 0 b) $+\infty$ c) 1 d) $-\infty$

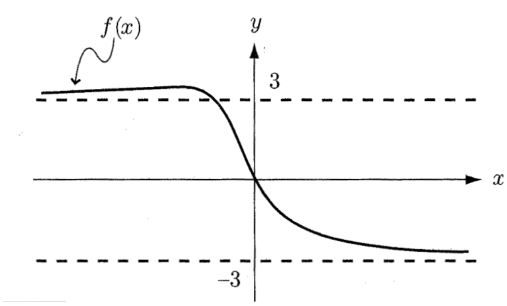
---

8. $\lim_{x\to-\infty} \left(4x-x^2\right)e^{-x}$

a) 0 b) $+\infty$ c) 1 d) $-\infty$

---

9. The figure below shows the graph of a function $f(x)$ which has horizontal asymptotes of $y=3$ and $y=-3$. Which of the following statements are true?



I. $f'(x)<0$ for all $x\geq 0$

II. $\lim_{x\to+\infty} f'(x)=0$

III. $\lim_{x\to-\infty} f'(x)=3$

a) I only b) II only c) III only d) I and II only e) I, II, and III

Limit and Continuity Test

1. The function f is differentiable at $x = b$. Which of the following statements could be false?

a) $\lim_{x \to b} f(x)$ exists b) $\lim_{x \to b} f(x) = f(b)$ c) $\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)$

d) $\lim_{x \to b^-} f'(x) = \lim_{x \to b^+} f'(x)$ e) None of these

---

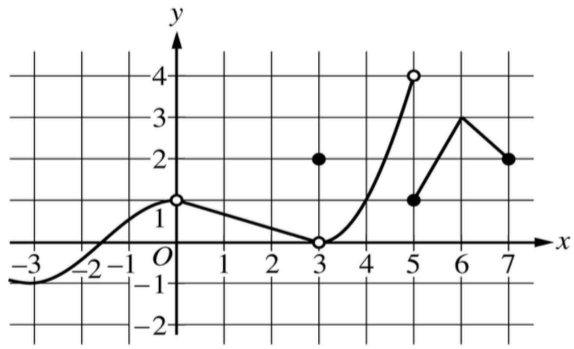
2. The function f is defined for all Reals such that $f(x) = \begin{cases} x^2 + kx & \text{for } x < 5 \\ 5\sin\frac{\pi}{2}x & \text{for } x \geq 5 \end{cases}$.

For which value of k will the function be continuous throughout its domain?

a) -2 b) -1 c) $\frac{2}{3}$ d) 1 e) None of these

---

3. The graph of the function f is shown below. At which value of x is f continuous, but not differentiable?



Graph of f

a) 0 b) 2 c) 3 d) 5 e) 6

---

4. If $f(x)=\begin{cases} x+2 & \text{for } x \leq 3 \\ 4x-7 & \text{for } x > 3 \end{cases}$, which of the following statements are true?

I. $\lim_{x \to 3} f(x)$ exists II. f is continuous at $x=3$ III. f is differentiable at $x=3$

a) None b) I only c) II only

d) I and II only e) I, II, and III

---

5. Which of the following functions is differentiable at $x=0$?

a) $f(x)=\sqrt{1+|x|}$ b) $f(x)=|x|$ c) $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$

d) $f(x)=\begin{cases} \frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$ e) $f(x)=\begin{cases} \cos x & \text{for } x<0 \\ \sin x & \text{for } x\geq 0 \end{cases}$

---

6. $\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}=$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{4}$ c) 0 d) $-\frac{\pi}{4}$ e) DNE

---

7. $\lim_{x\to 0}\frac{\int_0^x \sin t^2 dt}{x^3}$

a) 0 b) 1 c) $\frac{1}{3}$ d) 3 e) DNE

---

8. $\lim_{x\to\infty}\frac{4x^5+3x^4+2x^3+x^2+1}{3x^5-9x^4+4x^3+15}=$

a) 0 b) $\frac{3}{4}$ c) $\frac{4}{3}$ d) 3 e) DNE

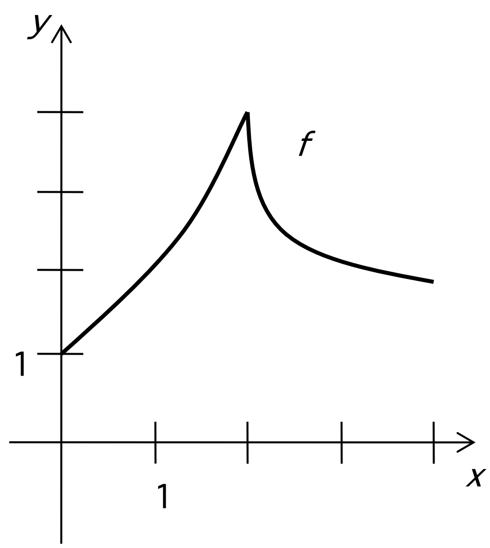
---

9. The graph of a function f is given below. Which of the following statements are true?

I. $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$ dne

II. $\lim_{x\to 2} f(x)=4$

III. $\lim_{x\to 2} f(x)$ dne

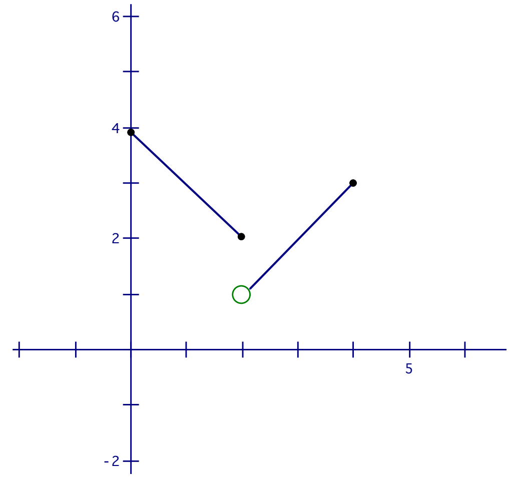


a) I only b) II only c) I and II only

d) I, II, and III e) II and III only

---

10. The graph of a function is shown below.



Which of the following statement(s) is (are) true?

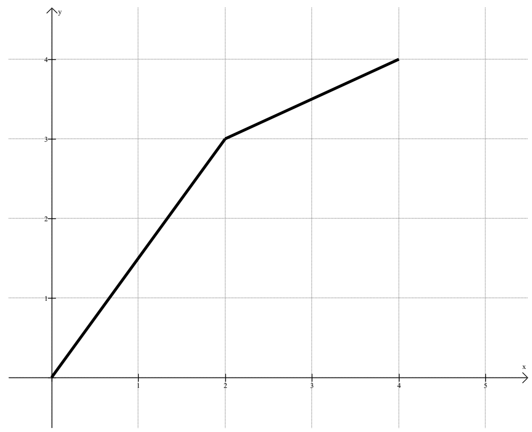
I. $\lim_{x \to 2^-} f(x)$ exists

II. $\lim_{x \to 2^+} f(x)$ exists

III. $\lim_{x \to 2} f(x)$ exists

a) I only b) II only c) I and II only

d) I and III only e) I, II, and III



11. At which x-value is f (graphed above) differentiable but not continuous?

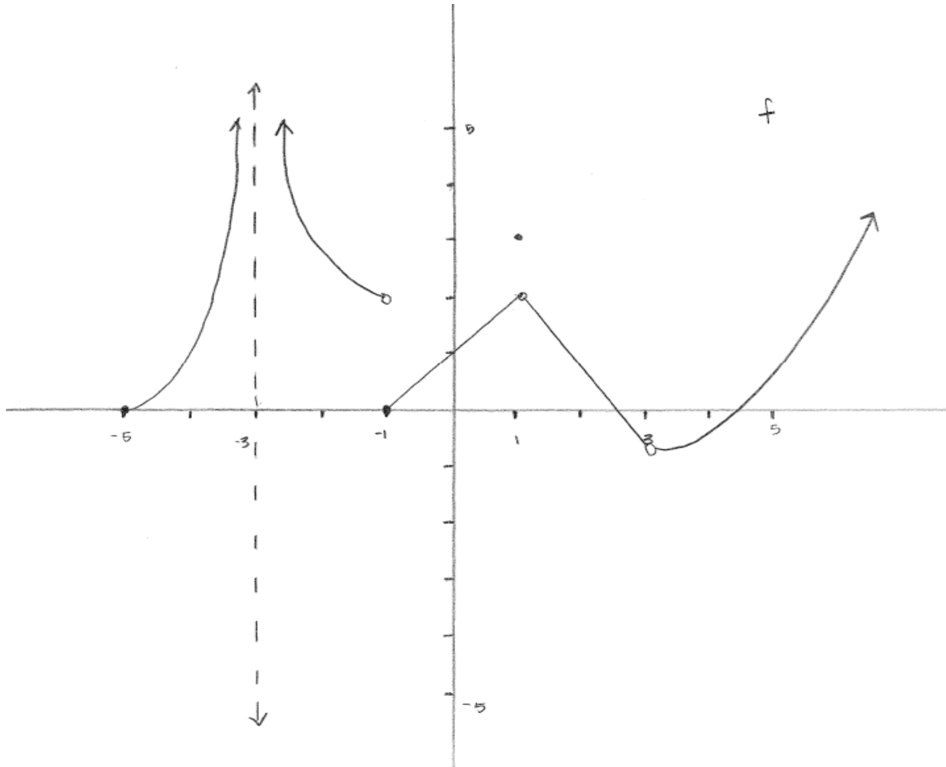
a) 0 b) 1 c) 2 d) 4 e) nowhere

12. $$f(x)=\begin{cases} \sin^{-1}\left[\pi(x-1)\right], & if\ x<1 \\ 0\ , & if\ x=1 \\ \ln x^2, & if\ x>1 \end{cases}$$

a) Is $f(x)$ continuous? Why/Why not?

b) Is $f(x)$ differentiable? Why/Why not?

13. Given the graph of $f(x)$ below, find the values of the following:



a. $\lim_{x \to -3^-} f(x) =$

b. $\lim_{x \to -3^+} f(x) =$

c. $\lim_{x \to -3} f(x) =$

d. $f(-3) =$

e. $\lim_{x \to -1^-} f(x) =$

f. $\lim_{x \to -1^+} f(x) =$

g. $\lim_{x \to -1} f(x) =$

h. $f(-1) =$

i. $\lim_{x \to 1^-} f(x) =$

j. $\lim_{x \to 1^+} f(x) =$

k. $\lim_{x \to 1} f(x) =$

l. $f(1) =$

m. $\lim_{x \to 3^-} f(x) =$

n. $\lim_{x \to 3^+} f(x) =$

o. $\lim_{x \to 3} f(x) =$

p. $f(3) =$

7.1 Free Response Answer Key

1.	$-\frac{1}{6}$	2.	$\frac{1}{2}$	3.	0
4.	$\frac{1}{3}$	5.	1	6.	0
7.	$\frac{3}{5}$	8.	1	9.	0
10.	$\frac{1}{9}$	11.	$\frac{8\sqrt{2}}{2+\sqrt{2}}$	12.	$\frac{-1}{108}$
13.	$\frac{2}{9}$	14.	2	15.	2
16.	0	17.	0	18.	-1
19.	$e^{1/2}$	20.	$-\frac{2}{5}$	21.	1

22.	0	23.	27	24.	-1	25.	e^3
26.	$\frac{2}{71}$	27.	0	28.	$\frac{1}{2}$	20.	4
30.	$3a^2$	31.	$\frac{3}{2}a$	32.	216		

7.1 Multiple Choice Answer Key

1.	C	2.	D	3.	A	4.	E	5.	A	6.	E
7.	C	8.	A	9.	D	10.	D	11.	B	12.	C

13. B 14. D

7.2 Free Response Answer Key

1. 27 2. -1 3. e^3 4. 0 5. 0 6. $\frac{1}{2}$

7. 0 8. $3a^2$ 9. $\frac{3}{2}a$ 10. 216

7.2 Multiple Choice Answer Key

1. D 2. C 3. D 4. C 5. D 6. AC

7. B 8. C 9. D 10. E

7.3 Free Response Answer Key

1a) $\lim_{x\to -4^-} f(x)=4$ b) $\lim_{x\to -4^+} f(x)=1$ c) $\lim_{x\to 3} f(x)=2$

d) $\lim_{x\to -4} f(x)=dne$ e) $\lim_{x\to 5^+} f(x)=-\infty$ f) $\lim_{x\to 5^-} f(x)=\infty$

g) $f(-4)=4$ h) $\lim_{x\to 0} f(x)=-3$ i) $f(0)=dne$

j) $f(3)=5$ k) $\lim_{x\to 0^+} f(x)=-3$ l) $\lim_{x\to 0^-} f(x)=-3$

m) $\lim_{x\to 3} f(x)=2$ n) $f(5)=dne$

2a. $\lim_{x\to -2^-} f(x)=-1$ b. $\lim_{x\to -2^+} f(x)=1$ c. $\lim_{x\to -2} f(x)=dne$

d. $f(-2)=1$ e. $\lim_{x\to 0^+} f(x)=2$ f. $\lim_{x\to 0^-} f(x)=2$

g. $\lim_{x\to 0} f(x) = 2$ h. $f(0) = 2$ i. $\lim_{x\to 3^+} f(x) = 3$

j. $\lim_{x\to 3^-} f(x) = -2$ k. $\lim_{x\to 3} f(x) = dne$ l. $f(3) = -2$

m. $\lim_{x\to 4^+} f(x) = 3$ n. $\lim_{x\to 4^-} f(x) = 3$

3. $-\infty$ 4. $-\infty$ 5. $-\infty$ 6. ∞

7. $-\infty$ 8. $-\infty$ 9. ∞

10a. 0 10b. 0 10c. $-\infty$ 10d. $+\infty$ 10e. $-\infty$

7.3 Multiple Choice Answer Key

1. A 2. C 3. E 4. B 5. D 6. D

7. C 8. B

7.4 Free Response Answer Key

1. $x = -4, 0, 3, 5$ 2. $x = -2, 3$

3. Lim DNE 4. G(-1) DNE 5. Continuous

6. Continuous 7. Lim DNE 8. Continuous

9. H(-1) not equal to the LIM 10. Continuous

11. Continuous 12. Continuous

7.4 Multiple Choice Answer Key

1.	A	2.	E	3.	C	4.	E	5.	C	6.	C
7.	D	8.	A	9.	C	10.	E	11.	B	12.	D

7.5 Free Response Answer Key

1. $x = -4$ and $x = -1$

2. $x = \pm 5, \pm 3$

3. $x = \pm 2$

4. $x = -1$

5. Not differentiable, $\frac{dy}{dx}$ *dne*.

6. Not continuous.

7. Not differentiable, $f'(1^-) \neq f'(1^+)$.

8. Differentiable

9. $m = -2,\ k = -2$

10. $m = -6,\ k = \frac{3}{2}$

11. $m = -6,\ k = 3$

12. $m = \frac{4}{3},\ k = \frac{2}{3}$

7.5 Multiple Choice Answer Key

1.	B	2.	E	3.	E	4.	A	5.	A	6.	C
7.	B	8.	B	9.	D	10.	B	11.	C	12.	A
13.	E										

7.6 Free Response Answer Key

1.	$\frac{3}{4}$	2.	0	3.	π	4.	$\ln\left(\frac{\pi}{4}\right)$
5.	0	6.	∞	7.	0	8.	$-\frac{\pi}{2}$
9.	$\frac{\pi}{2}$	10.	$\frac{\pi}{4}$	11.	0		

12. HA $y=-\frac{\pi}{2}$ on Left, Up on the left

13. HA at y = 0

14. HA at y = 0 on Right

15. HA at y = 0 on Left

7.6 Multiple Choice Answer Key

1.	A	2.	E	3.	E	4.	A	5.	E	6.	D
7.	A	8.	D	9.	D						

Chapter 7 Practice Test Answer Key

1.	E	2.	E	3.	A	4.	D	5.	E	6.	C
7.	C	8.	C	9.	C	10.	C	11.	E		

12a. $f(x)$ is continuous

b. $f(x)$ is not differentiable

13a.	$+\infty$	13b.	$+\infty$	13c.	$+\infty$	13d.	DNE
13e.	2	13f.	1	13g.	DNE	13h.	0
13i	2	13j.	2	13k.	2	13l.	3

13m. $-\frac{1}{2}$ 13n. $-\frac{1}{2}$ 13o. $-\frac{1}{2}$ 13p. $-\frac{1}{2}$