Chapter 5 Overview: Derivatives and Rates of Change

Previously, we considered the main context for derivatives: graphing. And while the slope of a tangent line is actually a rate of change of y in terms of x, rates of change are generally thought of a change over time. In this chapter, we will consider the applications of the derivative to motion and growth rates. Some of this is a review of material covered in PreCalculus. Key topics include:

- Position, Velocity, and Acceleration
- Implicit Differentiation
- Related Rates
- Exponential, Bounded, and Logistic Growth

Several multiple-choice questions and at least one full free response question (often parts of others), have to do with the topic of motion.

5.1: Intro to AP: Rectilinear Motion

A key application of the derivative as a rate of change is the application to motion. Typically, we refer to horizontal position in terms of $x(t)$ and vertical position in terms of y(t). Since the derivative is a rate of change of a function and the rate of change of position is velocity, it should be pretty obvious that the derivative of position of velocity. Likewise, the derivative of velocity is acceleration since the rate of change of velocity is acceleration.

Remember:

Position = $x(t) \text{ or } y(t)$

Velocity = $x'(t) \text{ or } y'(t)$

Acceleration = $x''(t) \text{ or } y''(t)$

Position = $\int x'(t)dt \text{ or } \int y'(t)dt$

Velocity = $\int x''(t)dt \text{ or } \int y''(t)dt$

Three things are implied in the definitions:

Also remember from PreCalculus

1. The sign of the velocity determines the direction of the movement:

   Velocity > 0 means the movement is to the right (or up)
   Velocity < 0 means the movement is to the left (or down)
   Velocity = 0 means the movement is stopped.

2. Speeding up and slowing down is not determined by the sign of the acceleration.

   An object is speeding up when $v(t)$ and $a(t)$ have the same sign.

   An object is slowing when $v(t)$ and $a(t)$ have opposite signs.

$$Displacement = \int_a^b v\,dt \qquad Total\ Distance = \int_a^b |v|\,dt$$

$$Position \text{ at } x = a \quad = x(a) + \int_a^b v\,dt$$

Summary of Key Phases

When = solve for *t*
Where = solve for position
Which direction = is the velocity positive or negative
Speeding up or slowing down = are the velocity and acceleration in the same direction or opposite (do they have the same sign or not)

The major emphasis in this section is AP style FRQs that present the given information in either algebraic, tabular, or graphical form.

OBJECTIVES

Use the derivative and antiderivative to make conclusions about motion.
Analyze motion information presented in algebraic, graphical, or tabular format.

Algebraic Format

Ex1 Suppose that a particle is moving along the *x*-axis such that the velocity is described by $v(t) = t\cos t^2$. At $t = 0$, the position is $x(0) = 4$.

a) For what values of $t \in [0, 3]$ is the particle moving left.
b) What is the acceleration at $t = 3$?

c) Find the particular position equation. What is the position of the particle at the first <u>positive</u> time when it stops to switch directions?

a) For what values of $t \in [0,4]$ is the particle moving left.

$$v(t) = t\cos t^2 = 0 \rightarrow t = 0, \sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}, \sqrt{\frac{5\pi}{2}}$$

v	0	+	0	−	0	+	0	−
t	0		$\sqrt{\frac{\pi}{2}}$		$\sqrt{\frac{3\pi}{2}}$		$\sqrt{\frac{5\pi}{2}}$	

$$t \in \left(\sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{2}}\right), \left(\sqrt{\frac{5\pi}{2}}, 3\right)$$

b) What is the acceleration at $t = 3$?

$$a(t) = v'(t) = t(-\sin t^2)(2t) + \cos t^2(1)$$
$$a(3) = -18\sin 9 + \cos 9 = -8.329$$

c) Find the particular position equation. What is the position of the particle at the first <u>positive</u> time when it stops to switch directions?

$$x(t) = \int t\cos t^2\,dt$$
$$= \frac{1}{2}\int \cos t^2 (2t\,dt)$$
$$= \frac{1}{2}\sin t^2 + c$$
$$x(0) = 4 \rightarrow 4 = \frac{1}{2}\sin 0 + c \rightarrow c = 4$$

$$x(t) = \frac{1}{2}\sin t^2 + 4$$

$$x\left(\sqrt{\frac{\pi}{2}}\right)=\frac{1}{2}\sin\frac{\pi}{2}+4=4.5$$

Ex 2 Two particles move along the *x*-axis. For $0\le t\le 8$, the position of particle *P* at time *t* is given by $x_P(t)=\ln\left(t^2-2t+10\right)$, while the velocity of particle Q at time t is given by $v_Q(t)=t^2-8t+15$. Particle Q is at position $x=5$ at time $t=0$.

(a) For $0\le t\le 8$, when is particle *P* moving to the left?

(b) For $0\le t\le 8$, find all times *t* during which the two particles travel in the same direction.

(c) Find the acceleration of particle *Q* at time *t* = 2. Is the speed of particle *Q* increasing, decreasing, or neither at time *t* = 2? Explain your reasoning.

(d) Find the position of particle *Q* the first time it changes direction.

(a) For $0\le t\le 8$, when is particle P moving to the left?

$$x_P(t)=\frac{d}{dt}\ln\left(t^2-2t+10\right)=\frac{2t-2}{t^2-2t+10}=0\rightarrow t=1$$

v: $-$ 0 $+$

t: ⟵ 1 ⟶

$0\le t<1$

(b) For $0\le t\le 8$, find all times *t* during which the two particles travel in the same direction.

$$v_Q(t)=t^2-8t+15=(t-3)(t-5)=0\rightarrow t=3\ and\ 5$$

v: $+$ 0 $-$ 0 $+$

t: ⟵ 3 5 ⟶

$0\le t\le 1$ and $5\le t\le 8$

(c) Find the acceleration of particle Q at time $t=2$. Is the speed of particle Q increasing, decreasing, or neither at time $t=2$? Explain your reasoning.

$$a_Q(t)=2t-8 \rightarrow a_Q(2)=-4$$

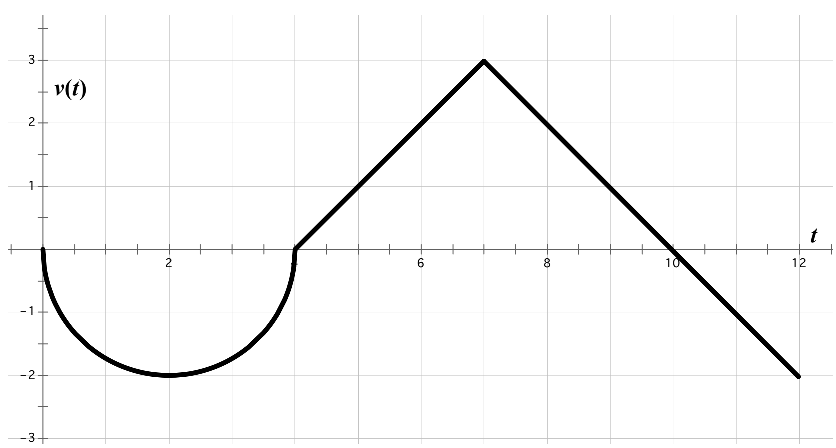
Since $v_P(2)>0$ and $a_Q(2)=-4$, the speed is slowing down because the velocity and ascceleration are opposite signs.

(d) Find the position of particle Q the first time it changes direction.

$$x_Q(t)=5+\int_0^3 \left(t^2-8t+15\right) dt=5+\left(\frac{1}{3}t^3-4t^2+15t\right)\Bigg|_0^3=23$$

Graphical Format

Ex 3 A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t\in[0,\ 12]$ is shown below.



(a) At what times, if any, does the particle switch directions?

(b) At what time on $t \in [0,\ 12]$ is the *speed* the greatest?

(c) What is the total distance traveled by the particle on $t \in [0,\ 12]$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

(a) At what times, if any, does the particle switch directions?

$v(t) = 0$ and switches signs at $t = 4$ and 10.

(b) At what time on $t \in [0,\ 12]$ is the *speed* the greatest?

The range of $v(t)$ is $v \in [-2,\ 3]$. Speed is $|v(t)|$ so the greatest speed is 3

(c) What is the total distance traveled by the particle on $t \in [0,\ 12]$

Total distance traveled = $\int_0^{12} |v(t)| dt$.

$$\int_0^{12} |v(t)| dt = -\int_0^4 v(t)dt + \int_4^{10} v(t)dt - \int_{10}^{12} v(t)dt$$
$$= -(-2\pi) + 9 - (-2)$$
$$= 11 + 2\pi$$

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

$$x(8) = 6 + \int_0^8 v(t)dt = 6 + (-2\pi) + \frac{9}{2} = \frac{21}{2} - 2\pi$$

Tabular Format

t	0	.3	.7	1.3	1.7	2.2	2.8	3.3	4
$v(t)$	0	14.1	9.5	17.1	13.3	15.6	12.7	13.7	12.0

Ex 4: Pat takes her bike on a 4-hour ride. She records her velocity $v(t)$, in miles per hour, for selected values of t over the interval $0 \le t \le 4$ hours, as shown in the table above. For $0 \le t \le 4$, $v(t) > 0$.

(a) Use the data in the table to approximate Pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\dfrac{3\sin(2\pi t) + 8t}{t^2 + 2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time t = 1.7? Give a reason for your answer.

(a) Use the data in the table to approximate pat's acceleration at time t = 1.5 hours. Show the computations that lead to your answer. Indicate units of measure.

$$a(1.5) \approx \frac{v(1.7) - v(1.3)}{1.7 - 1.3} = \frac{13.3 - 17.1}{1.7 - 1.3} = -\frac{3.8}{.4} = -9.5\ {mi}/{hr^2}$$

(b) Using the correct units, explain the meaning of $\int_0^4 v(t)\,dt$ in the context of the problem. Approximate $\int_0^4 v(t)\,dt$ using a left-hand Riemann sum using the values from the table.

$\int_0^4 v(t)\,dt$ would be the approximate number of miles Pat traveled during her four-hour ride.

$$\int_0^4 v(t)\,dt \approx .3(0)+.4(14.1)+.5(9.5)+.4(17.1)+.5(13.3)+.6(15.6)+.5(12.7)+.7(13.7)$$
$$= 49.18\ miles$$

(c) For $0 \le t \le 4$ hours, Pat's velocity can be modeled by the function g given by $f(t) = 9\sqrt{\dfrac{3\sin(2\pi t)+8t}{t^2+2}}$. According to the model, what was Pat's average velocity during the time interval $0 \le t \le 4$?

$$AveVelocity = \frac{1}{4-0}\int_0^4 9\sqrt{\frac{3\sin(2\pi t)+8t}{t^2+2}}\,dt = 13.350\ mph$$

(d) According to the model given in part (c), is Pat's speed increasing or decreasing at time $t = 1.7$? Give a reason for your answer.

$a(1.7) = f'(1.7) = -3.288\ {}^{mi}/_{hr^2}$

Since it was stated that "For $0 \le t \le 4$, $v(t) > 0$," Pat's speed is decreasing because the velocity and acceleration have opposite signs at $t = 1.7$.

5.1 Free Response Homework

1. Consider the velocity equation of a certain object is $v(t)=3t^2-8t+4$ on $t\in[0,5]$.

a) For what values of *t* is the particle moving right.
b) What is the acceleration at $t=0.6$? Show the derivative work.
c) Find the particular position equation if $y(1)=3$.

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2. Consider the velocity equation of a certain object is $v(t)=100-90e^{-0.4t}$ for $0\le t$.

a) For what values of *t* is the particle moving right.
b) What is the acceleration at $t=3$? Show the derivative work.
c) Find the particular position equation if $x(0)=8$.

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3. Consider the velocity equation of a certain particle is $v(t)=12te^{-3t^2}$ on $t\in[-1.5,\ 1.5]$.

a) For what values of *t* is the particle moving left.
b) What is the acceleration at $t=.06$? Show the derivative work.
c) Find the interval(s) on which the particle slowing down.
d) Find the particular position equation if $x(0)=-1$.

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4. Consider the velocity equation of an object is $v(t)=t^2\cos t^3$ on $t\in[0,2]$.

a) For what values of *t* is the particle moving right.
b) What is the acceleration at $t=0.6$? Show the derivative work.
c) Find the particular position equation if $x(0)=3$.

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5. A particle is moving along the x-axis so that its velocity is given by $E(t)=5-416\left(\frac{t}{5}\right)^4\left(1-\frac{t}{10}\right)^5$ on $0\le t\le 10$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 10]$, if any, does the particle switch directions?

b) Find the acceleration at $t=7.3$.

c) What is the total displacement of the particle on $t\in[1,\ 6]$.

d) What is the total distance traveled by the particle on $t\in[2,\ 9]$.

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6. A particle is moving along the x-axis so that its velocity is given by $v(t)=2.6\sqrt{t}\cos t-\frac{e^{0.5t}}{t+6}$ on $0\le t\le 9$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 9]$, if any, does the particle switch directions?

b) Find the acceleration equation at $t=3.4$.

c) What is the total distance traveled by the particle on $t\in[3,\ 8]$.

d) What is the position of the particle on $t=6.9$?

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7. A particle is moving along the x-axis so that its velocity is given by $v(t)=\ln(t+3)-e^{\frac{t}{2}-1}\cos t$ on $t\in[0,\ 8]$. The position of the particle at $t=0$ is $x=-1.6$.

a) At what time(s) on $t\in[0,\ 8]$, if any, does the particle switch directions?

b) Where is the particle when it is furthest to the left?

c) What is the total distance traveled by the particle on $t\in[3,\ 6]$.

d) Find the acceleration equation. At what time interval within $t\in[0,\ 8]$, if any, is the acceleration negative?

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8. A particle is moving along the x-axis so that its velocity is given by $v(t)=3-\frac{3}{20}t-.9\sin\left(\frac{\pi}{7}t\right)-\frac{2.1x}{\sqrt{x^2+1}}$ on $t\in[0,\ 12]$. The position of the particle at $t=0$ is $x=2.4$.

a) At what time(s) on $t\in[0,\ 12]$, if any, does the particle switch directions?

b) Find the acceleration at $t=7.3$.

c) What is the total distance traveled by the particle on $t\in[0,\ 10]$.

d) What is the position of the particle on $t=10.4$.

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t	0	1	4	6	9	10	13	15	18
$v(t)$	55	70	68	55	40	38	46	50	70

9. A car is traveling on a straight road. Values of the continuous and differentiable function $v(t)$ are given on the table above. $v(t)$ is measured in feet per minute and time t is measured in minutes.

a) Approximate the acceleration at $t=7$. Indicate the units.

b) Using left-hand rectangles, approximate $\int_0^{18} v(t)\,dt$. Using the correct units, explain the meaning of the approximation.

c) How many times on the interval $0\le t\le 18$ is $a(t)=0$? Explain your reasoning.

d) Assume the data are modeled by $P(t)=.05t^3-1.07t^2+3.89t+62$. Use the model to find the average velocity of the car on the interval $0\le t\le 18$.

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t in minutes	0	8	15	23	33	45	53
$v(t)$ in mph	0	8.5	7.5	10.1	9.3	7.1	4.1
$v(t)$ in mi/min	0	0.142	0.125	0.168	0.155	0.118	0.068

10. Mr. Evans has run the 7.4-mile Bay-to-Breakers many times (always with cloths on). His best time was 53 minutes. The table above shows estimates of his velocity at different times along the course from The Embarcadero to Ocean Beach. Assume the data represents a continuous and differentiable function.

a) Approximate Mr. Evans' acceleration at $t = 30$.
b) Given you result in a), was his speed increasing or decreasing at $t = 10$? Explain, using the correct units.
c) Find a trapezoidal approximation for $\int_0^{53} v(t)\,dt$. Why isn't the answer 7.4 miles?
d) Mr. Lannan did the same run. (He likes to take his time and gawk at the costumes.) His velocity is modeled by $L(t) = 0.023\left(5 + 4\sin\frac{\pi}{16}t\right)$. According to this model, does Mr. Lannan finish in under 65 minutes? Explain your reasoning.

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11. Mr. Alverado takes the Cross Country team out for a morning run and tracks his pace. The data table below shows his pace $p(t)$ in minutes per mile and his velocity $v(t)$ miles per minute at 15-minute intervals.

t (in minutes)	0	15	30	45	60
$p(t)$ (in min/mile)	8:07	7:34	8:16	8:07	7:14
$v(t)$ (in mi/min)	0.123	0.132	0.121	0.123	0.138

Both $p(t)$ and $v(t)$ are continuous and differentiable functions.

a) Find an approximation for $\int_0^{60} v(t)\,dt$ using midpoint rectangles. Explain the meaning of the result, using the correct units.

b) Using correct units, explain the meaning of $\frac{1}{60}\int_0^{60} p(t)\,dt$.

c) Approximate the acceleration at $t = 37$ minutes.

d) Is there a time during which the pace reaches a maximum? Explain your reasoning.

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t in hours	0	12	24	36	48
$v(t)$ in km/hr	21	26.3	31.4	36.8	41.5

12. A Gravitational Slingshot Effect is sometimes used by space probes like Voyager 2 in order to increase its velocity without expending fuel. By flying close to the planet Saturn in a parabolic arc, the velocities on the table above were

achieved by a probe. (In the original *Star Trek* episode "Tomorrow is Yesterday," the Enterprise used this effect around a black hole to time-travel to 1967.)

a) Approximate the probe's acceleration at $t = 30$.

b) Use a trapezoidal approximation for $\int_0^{48} v(t)\,dt$. Using the correct units, explain the meaning of this result.

c) Using your answer in b), approximate the average velocity of the probe between $t = 0$ and $t = 48$? Indicate the correct units.

d) The data on the table can be approximated by the equation $v(t) = 0.000027x^2 + 0.4396x + 21$. Based on this equation, find the total distance traveled by the probe between $t = 0$ and $t = 48$ hours. Indicate the units.

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13. Below is a chart of your speed driving to school in meters/second. Use the information below to find the values in a) and b) below.

t (in seconds)	0	30	90	120	220	300	360
v(*t*) (in m/sec)	0	21	43	38	30	24	0

a) Approximate your acceleration at $t = 100$.

b) Given you result in a), are you speeding up or slowing down at $t = 100$? Explain, using the correct units.

c) Find an approximation for $\int_0^{360} v(t)\,dt$ using right Riemann rectangles. Using the correct units, explain the meaning of your result.

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t	0	0.08	0.2	0.33	0.55	0.75	0.86	1
$H(t)$	60	5	63	25	75	28	70	6

14. Traffic flow on HWY 280 from San Francisco to San Jose during rush hour is notoriously variable because of bottlenecks caused by intersecting with HWYs 380, 92, and 85. The table above shows the average speed of traffic, in miles per hour, at time *t*, where *t* is measured in hours after entering the highway.

a) Approximate $H'(0.6)$. Using the correct units, explain the meaning of $H'(0.6)$.

b) Set up a left-hand Reimann Sum to approximate $\int_0^1 H(t)dt$. Using the correct units, explain the meaning of $\int_0^1 H(t)dt$.

c) Assume that $S(t)=40-35\cos\left(\frac{\pi}{9}(t-5.5)\right)$ would accurqtely model the data on the table. Set up, but do not solve, an integral equation that would determine the time at which the car has been driving 50 miles.

d) Set up, but do not solve, an integral equation that would determine the average $S(t)$ between $t=0.33$ and $t=0.55$.

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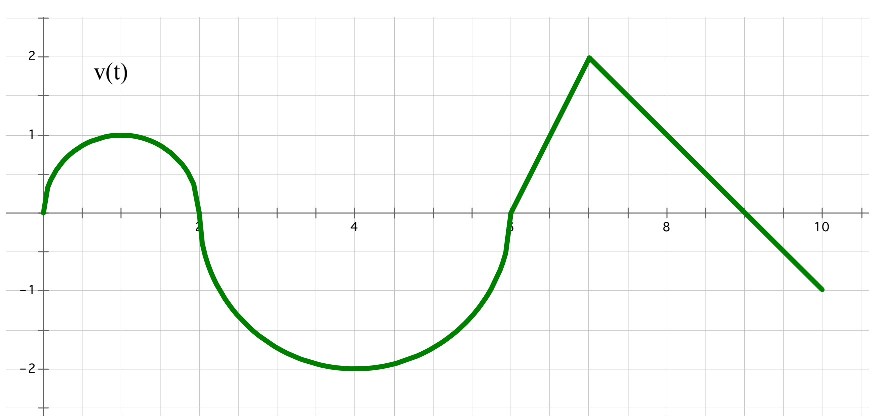
15. A particle is moving along the *x*-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t\in[0,\ 10]$ is shown below.



(a) At what times, if any, does the particle switch directions?

(b) At what time on $t \in [0, 10]$ is the *speed* the greatest?

(c) What is the total distance traveled by the particle on $t \in [0, 10]$.

(d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

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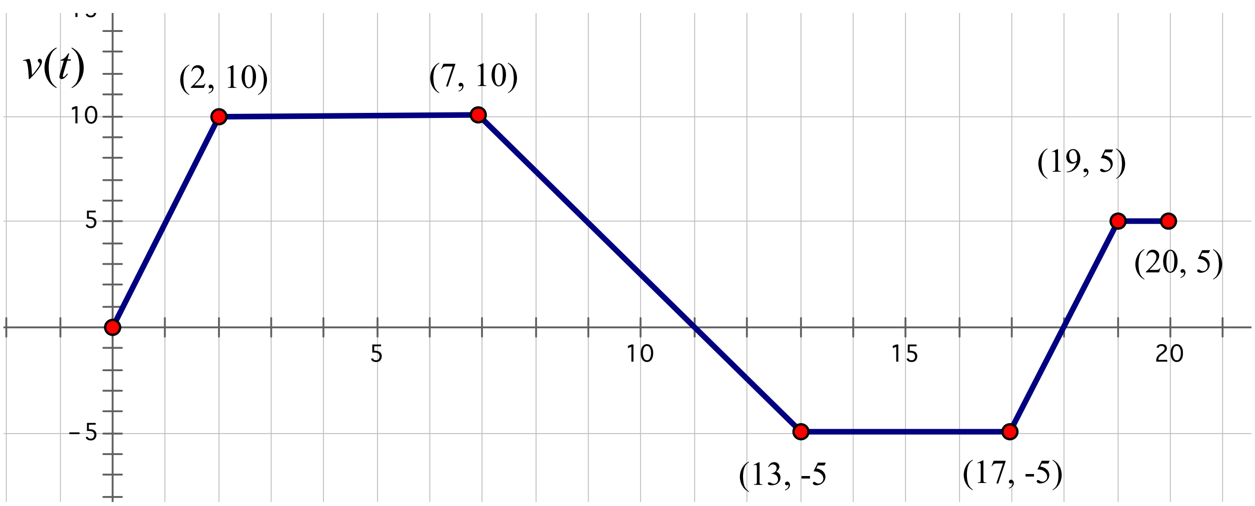
16. A couple take their new dog Skadi to run around at Fort Funston. She immediately runs away and back toward them several times. For $0 \le t \le 20$, Skadi's velocity is modeled by the piecewise-linear function defined by the graph below



where $v(t)$ is measured in feet per second and t is measured in seconds.

a) At what times in the interval $0 \le t \le 20$, if any, does Skadi change direction? Give a reason for your answer.

b) At what time in the interval $0 \le t \le 20$, what is the farthest Skadi gets from the couple?

c) Find the total distance Skadi travels during the time interval $0 \le t \le 20$.

d) Write expressions for Skadi's acceleration $a(t)$, velocity $v(t)$, and distance $x(t)$ from the couple that are valid for the time interval $7 \le t \le 13$.

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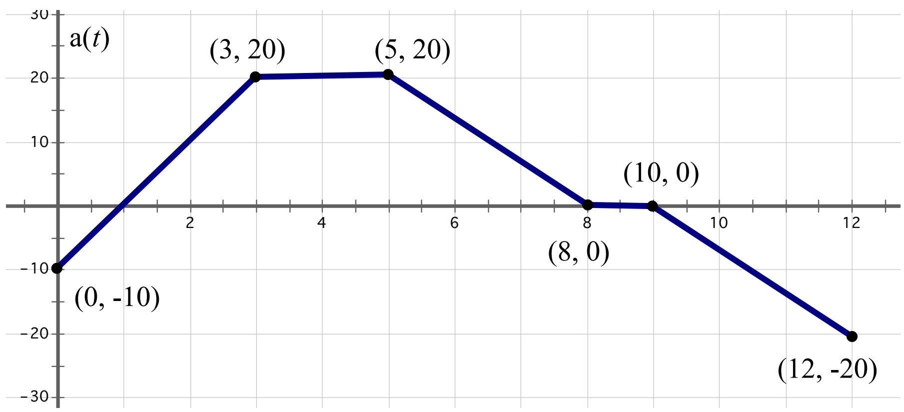
17. A car is driving west on the highway so that its acceleration $a(t)$ is given by the continuous function whose graph at time $t \in [0, 12]$ is shown below.



The graph is comprised of five line segments. $a(t)$ is measured in miles per hour^2 and time t is measured in hours.

a) At what time on $t \in [0, 12]$ is the acceleration the greatest?

b) If $v(0) = 25$, find $a'(6.3)$. Indicate units.

c) If $v(0) = 25$, find $v(3)$.

d) If $v(0) = 25$, what is the maximum velocity on $t \in [0, 12]$?

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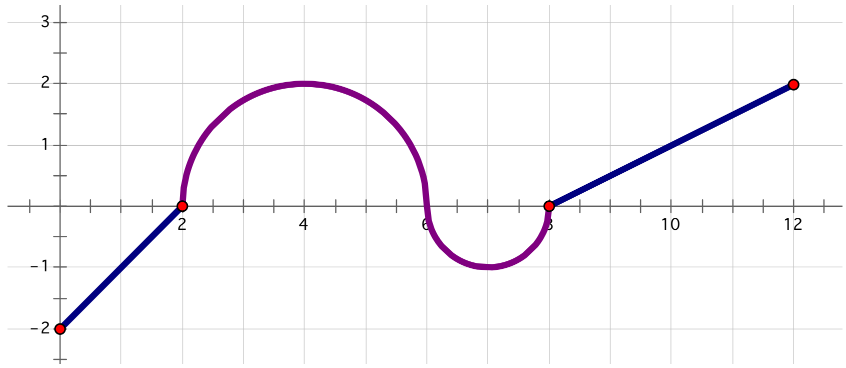
18. A particle is moving along the x-axis so that its velocity $v(t)$ is given by the continuous function whose graph at time $t \in [0,\ 12]$ is shown below.



The graph is comprised of two line segments and two semicircles.

a) At what times, if any, does the particle switch directions?

b) At what time on $t \in [0,\ 12]$ is the *speed* the greatest?

c) What is the total distance traveled by the particle on $t \in [0,\ 12]$.

d) If the initial position of the particle is $x(2) = 6$, what is the position at $t = 8$?

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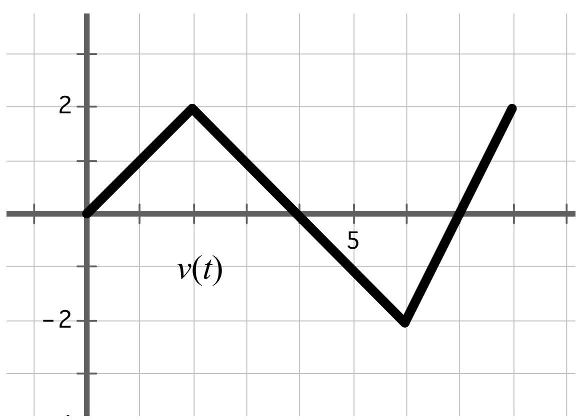
19. A car's velocity $v(t)$, in miles per minute, is modeled by the continuous function whose piecewise-linear graph at time $t \in [0,\ 8]$ is shown below.



a) Find the acceleration at $t = 3.5$. Indicate the units.

b) At what time(s) on $t \in [0,\ 8]$ does the car switch directions? Explain.

c) Find the total distance traveled by the particle on $t \in [0, 8]$.

d) Find the car's average rate of change of the velocity on $t \in [3, 8]$. How many times on $t \in [3, 8]$ is the instantaneous velocity equal to the average velocity? [extra credit is you find the time(s).]

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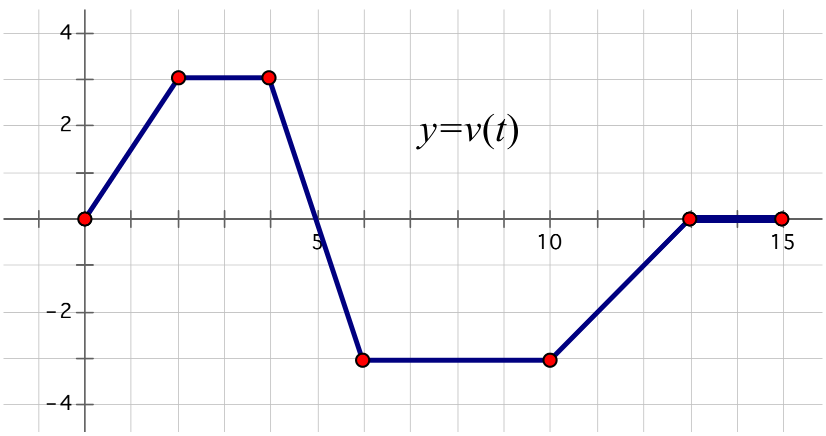
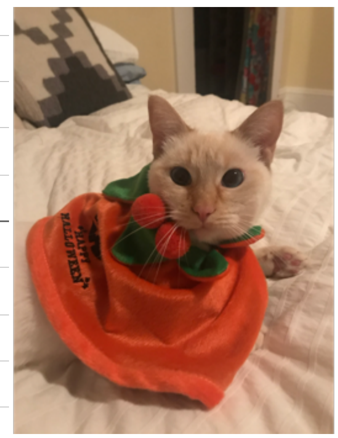
20. A cat named Dolly chases her favorite mouse toy back and forth across the floor. Her velocity, in m/sec, as a function of time in seconds is shown in the graph below.





a) What is Dolly's acceleration between $t = 4$ seconds and $t = 6$ seconds?

b) At what time (after she starts moving) is Dolly's displacement equal to zero?

c) Dolly catches her toy after 13 seconds. What is the total distance that she travels while chasing her toy?

d) When is Dolly's acceleration zero?

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21. AP Packet: AB13 #2, AB14 #4, AB15 #3, AB16 #2, AB 19 #2

5.1 Multiple Choice Homework

1. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Find the value of t at which the acceleration is zero.

a) $-\frac{2}{3}$ b) $-\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{3}$ e) $-\frac{1}{2}$

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2. A particle moves on the x-axis with velocity given by $v(t) = 3t^4 - 11t^2 + 9t - 2$ for $-3 \le t \le 3$. How many times does the particle change direction as t increases from –3 to 3?

a) Zero b) One c) Two

d) Three e) Four

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3. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

4. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the acceleration of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

5. Find the acceleration at time $t = 9$ seconds if the position (in cm.) of a particle moving along a line is $s(t) = 6t^3 - 7t^2 - 9t + 2$.

a) 310 cm/sec^2 b) 310 cm/sec c) 1323 cm/sec^2

d) 1323 cm/sec e) −1323 cm/sec

---

6. The acceleration of a particle is given by $a(t) = 4e^{2t}$. When $t = 0$, the position of the particle is $x = 2$ and $v = -2$. Determine the position of the particle at $t = \frac{1}{2}$.

a) $e - 3$ b) $e - 2$ c) $e - 1$ d) e e) $e + 1$

---

7. A particle moves along a straight line with its position at any time $t \geq 0$ given by $s(t) = \int_0^t (x^3 - 2x^2 + x)\,dx$, where s is measured in meters and t is in seconds. The maximum velocity attained by the particle on $0 \leq t \leq 3$ is

a) $\frac{1}{3}$ m/s b) $\frac{4}{27}$ m/s c) $\frac{27}{4}$ m/s d) 12 m/s

---

8. A particle moves along the x-axis with acceleration at any time t given as $a(t)=3t^2+4t+6$. If the particle's initial velocity is 10 and its initial position is 2, what is the position function?

a) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+12$

b) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+10t+2$

c) $x(t)=\frac{1}{4}t^4+\frac{2}{3}t^3+3t^2+2$

d) $x(t)=3t^4+t^3+t^2+10t+2$

---

9. A particle moves along a straight line with equation of motion $s=t^3+t^2$. Which of the following statements is/are true?

I. The particle is moving right at $t=\frac{2}{3}$.

II. The particle is paused at $t=\frac{1}{3}$

III. The particle is speeding up at $t=1$.

a) I only b) II only c) III only

d) I and II only e) I and III only

---

10. A particle moves along the y-axis so that at any time $t \geq 0$, it velocity is given $v(t) = \sin(2t)$. If the position of the particle at time $t = \frac{\pi}{2}$ is $y = 3$, the particle's position at time $t = 0$ is

a) –4 b) 2 c) 3 d) 4 e) 6

5.2 Models of Exponential Growth and Decay

There are three models of growth and decay which we will consider:

Vocabulary:

1. ***Exponential Growth*** – the rate is directly proportional to the amount of material present.

2. ***Simple Bounded Growth*** – the rate is directly proportional to the amount of material which has not changed yet.

3. ***Logistic Growth*** – the rate of change of y is jointly proportional to the amount which has changed and to the amount which has not yet been changed.

OBJECTIVES

Understand growth and decay in terms of differential equations.
Solve differential equations in a growth and decay context.
Recognize the carrying capacity in a growth setting.
Determine when the maximum growth rate in a logistic growth setting.
Know the solution to a logistic differential equation.

<u>Exponential Growth</u>

In previous math and science classes, exponential growth had been explored, and it was generally considered any growth that followed an exponential equation, like $y = y_0 e^{kt}$. But this equation actually arises from solving the differential equation $\frac{dy}{dt} = ky$. This equation states that the rate of change of y, $\frac{dy}{dt}$, is directly proportional to y itself. In other words, the rate is determined by the amount of material present.

Exponential Growth and Decay are described by the two equations:

Differential Equation: $\frac{dy}{dt} = ky$

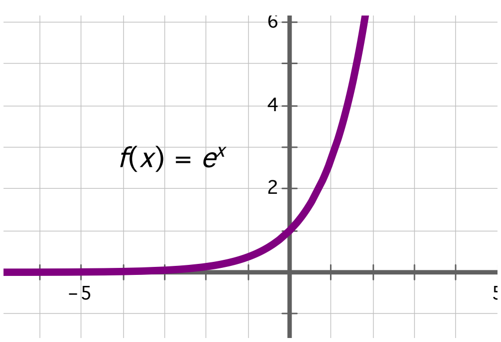
General Solution: $y = Ae^{kt}$ or y_0e^{kt}

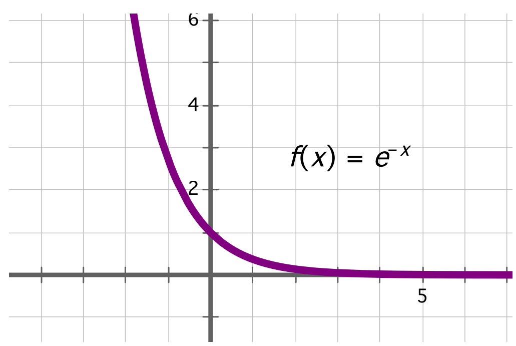
The solution comes from separating the variable to solve the differential equation:

$$\frac{dy}{dt} = ky \to \frac{1}{y}dy = kt \to \int \frac{1}{y}dy = \int kdt \to \ln|y| = kt + c \to |y| = e^{kt+c} \to y = Ae^{kt}.$$

$y = y_0e^{kt}$ is how the solution is given in many science and Algebra 2 courses.

Hopefully, we recall the graphs from PreCalculus:





The difference between growth and decay is the sign of the *k*-value.

EX 1 If there are 100 bacteria in a petri dish and the number doubles every 10 minutes, how long will it take for there to be a million bacteria?

$$t = 0 \to A = 100, \text{ so } y = 100e^{kt}$$

In ten minutes, there will be 200 bacteria, so

$$\begin{aligned} 200 &= 100e^{k(10)} \\ 2 &= e^{k(10)} \\ \ln 2 &= 10k \\ .06931 &= k \end{aligned}$$

$$\text{Therefore, } y = 100e^{.06931t}$$

$$1000000 = 100e^{.06931t}$$
$$10000 = e^{.06931t}$$
$$\ln 10000 = .06931t$$

$$t = 132.878 \text{ min}$$

EX 2 Radium has a half-life of 1690 years. How much of a 75gm. sample will be left in 400 years?

We will skip the solving of the differential equation here as we know it will be $y = Ae^{kt}$

$$.5A = Ae^{k(1690)}$$
$$.5 = e^{k(1690)}$$
$$\ln .5 = k(1690)$$
$$k = \frac{\ln .5}{1690}$$

$$y = Ae^{kt} \rightarrow y = 75e^{\left(\frac{\ln .5}{1690}\right)400} = 63.652gm$$

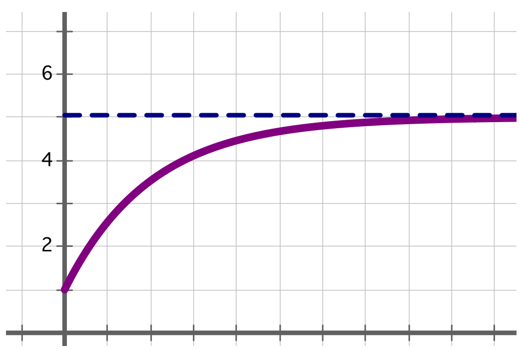
Simple Bounded Growth

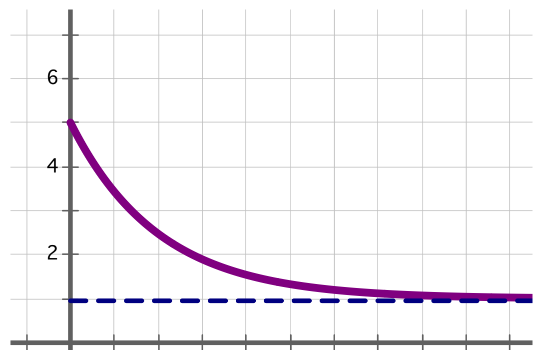
Differential Equation: $\frac{dy}{dt} = k(A - y)$

General Solution: $y = A - Be^{kt}$

We will demonstrate the solving of this differential equation in the examples, as this separation of variables is likely to be the FRQ on the AB Exam.

The Simple Bounded Growth and Decay equations look like this:





The decay curve usually has its boundary at $y = 0$.

NB. These problems are often the Separation of Variables Problems on the AP Calculus FRQ Exam.

EX 3 Suppose that a population of wolves follows the simple bounded growth model. If these wolves have a population limit of 5000, the differential equation would be $\frac{dw}{dt} = k(5000 - w)$.

(a) Find the general solution to the differential equation. Show the steps.

(b) If there are 1000 wolves at time $t = 0$ and 1100 at $t = 1$, find the particular solution to the differential equation.

(c) How many wolves are there at time $t = 20$?

(a)
$$\frac{dw}{dt} = k(5000 - w)$$
$$\frac{1}{5000 - w} dw = kdt$$
$$\int \frac{1}{5000 - w} dw = \int k\, dt$$
$$-\int \frac{1}{5000 - w}(-dw) = \int k\, dt$$
$$-\ln|5000 - w| = kt + c$$
$$\ln|5000 - w| = -kt + c$$
$$|5000 - w| = e^{-kt+c}$$
$$5000 - w = Be^{-kt}$$
$$w = 5000 - Be^{-kt}$$

(b)
$$(0, 1000) \to 1000 = 5000 - Be^{0} \to \text{B} = 4000$$
$$(1, 1100) \to 1100 = 5000 - 4000e^{-k}$$
$$e^{-k} = 0.975$$
$$-k = \ln 0.975 \to k = 0.025317808$$
$$w = 5000 - 4000e^{0.025t}$$

(c) $w(20) = 5000 - 4000e^{0.025(20)} \approx 2589$ wolves.

Ex 4: AP Calculus 2012 #5

Note the use the first and second derivatives to determine the shape of the curve.

A common instance of Simple Bounded Exponential is Newton's Law of Cooling and Heating:

> Newtons' Law of Cooling and Heating states that the rate of change of the temperature of an object is proportional to the difference between the object's temperature and the surrounding atmosphere. For cooling (i.e., the object is warmer than the environment), the differential equation can be written as:
>
> $$\frac{dy}{dt} = k(y - A)$$
>
> where y is the temperature of the object and A is the temperature of the surrounding environment.
>
> If the object is colder than the environment and is warming, the equation is
>
> $$\frac{dy}{dt} = k(A - y)$$

NB. These two equations are actually the same equation, but, in one case, k is a negative number and in the other k is positive.

EX 5 Suppose that some muffins are taken out of the oven and are at a temperature of 250 $F°$. The ambient room is at a pleasant 70 $F°$. In 15 minutes, they have fallen to 125 $F°$.

(a) Find the particular solution to this cooling situation.
(b) When do the muffins reach 75 $F°$?
(c) When do the muffins reach 65 $F°$?

(a) $\frac{dy}{dt} = k(y-70) \rightarrow \frac{1}{y-70}dy = kdt$

$$\int \frac{1}{y-70}dy = \int k\,dt$$
$$\ln|y-70| = kt + c$$
$$|y-70| = e^{kt+c} \rightarrow y-70 = Be^{kt}$$

$$(0,250) \rightarrow 250-70 = Be^{0} \rightarrow B = 180$$
$$(15,125) \rightarrow 125-70 = 180e^{k(15)} \rightarrow k = -0.079$$

$$y = 70 + 180e^{-0.079t}$$

(b) $75 = 70 + 180e^{-0.079t} \rightarrow 5 = 180e^{-0.079t} \rightarrow t = \frac{1}{-0.079}\ln\frac{5}{180} = 4.536 \text{ min}$

(c) $65 = 70 + 180e^{-0.079t} \rightarrow -5 = 180e^{-0.079t} \rightarrow t = error$
The biscuits never reach 65°

Logistic Growth

Logistic growth situations are also bounded, but, unlike Simple Bounded Growth problems, the growth relies on both the amount changed and the amount unchanged. The two most common situations modeled by logistic growth equations are the spread of a disease and the spread of a rumor. Consider a rumor. The rate at which the rumor spreads is much greater in the beginning. But, as more people hear the rumor, fewer new people are available hear about it. So, the rate at which it spreads slows down. A rumor spreading is dependent on how many people have heard the rumor and how many people have not heard the rumor. There is a limit to how many people will hear this rumor. This limit is the carrying capacity.

Three Facts You Need to Know About Logistic Growth:

Given a Logistic growth equation in the forms $\frac{dy}{dt} = Ky(A-y)$ or $\frac{dy}{dt} = ky\left(1-\frac{y}{A}\right)$

NB. K and k will be different numbers, with $K = \frac{k}{A}$

1. $\lim_{t\to\infty} y = A$ -- there is a horizontal asymptote on y of $y = A$ (i.e. there is a limit to how many people will hear the rumor, etc.).

2. The maximum growth rate happens at $y = \frac{A}{2}$.

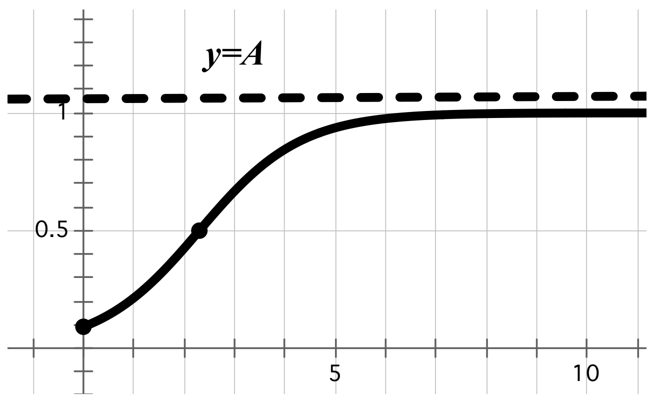
3. The solution to the logistic growth differential equation $\frac{dy}{dt} = ky\left(1-\frac{y}{A}\right)$ is $y = \frac{A}{1+Be^{-kt}}$.

Note: The AP Biology Exam uses $\frac{dN}{dt} = r_{\max} N\left(\frac{K-N}{K}\right)$ where $r_{\max}$ is the maximum growth rate and K is the carrying capacity (what is usually denoted as A in Calculus).

Differential Equation: $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$

General Solution: $y = \frac{A}{1 + Be^{-kt}}$

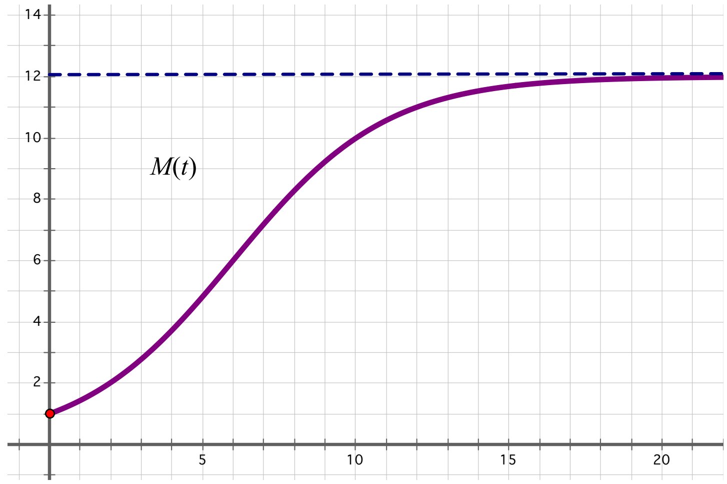
Logistic growth curves look like this:



Note that it looks like an exponential growth curve at the beginning (concave up) and a simple bounded growth curve at the end (concave down).

The solution is a little complicated and involves a technique beyond the scope of this class which is called Partial Fractions. For further information and examples, see the Readable Calculus (BC version) 8.2 and 8.3. **Logistic Growth is not a part of the AP Calculus AB curriculum.** On the BC Calculus Exam, logistic growth is usually a multiple-choice question.

Ex 5 Consider the logistic growth curve below.



Which of the following statements is false?

a) $\lim_{t\to\infty} M(t)=12$

b) The fastest rate of growth occurs when $M(t)=6.5$.

c) The solution equation is $M(t)=\dfrac{12}{1+11e^{-.04t}}$.

d) At $t=10$, $M'(t)>0$ and $M''(t)<0$.

$\lim_{t\to\infty} M(t)=12$ is true. The end behavior is the horizontal asymptote $y=A=12$.

"The fastest rate of growth occurs when $M(t)=6.5$" is false. The fastest rate of growth occurs when $M(t)=\dfrac{A}{2}=6$.

$M(t)=\dfrac{12}{1+11e^{-.04t}}$ is the solution equation.

$M'(t)>0$ and $M''(t)<0$ at $t=10$, because the curve is increasing and concave up.

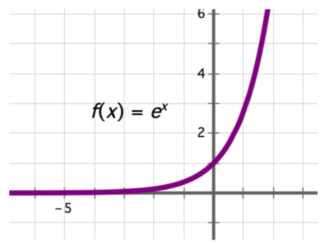
So the answer is B.

In summary,

Exponential Growth

Differential Equation: $\frac{dy}{dt} = ky$

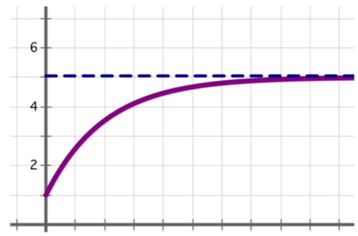
General Solution: $y = Ae^{kt}$ *or* y_0e^{kt}



Simple Bounded Growth

Differential Equation: $\frac{dy}{dt} = k(A - y)$

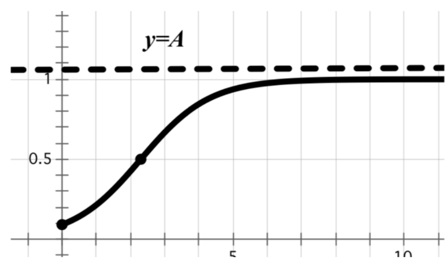
General Solution: $y = A - Be^{kt}$



Logistic Growth

Differential Equation: $\frac{dy}{dt} = ky\left(1 - \frac{y}{A}\right)$

General Solution: $y = \frac{A}{1 + Be^{-kt}}$



5.2 Free Response Homework

1. If the number of bacteria in a petri dish doubles every 30 minutes, how long will it take for the number to triple?

2. If a town's population doubles every ten years and is 20,000 this year, how long will it take for the population to reach 100,000?

3. If a town's population decreases exponentially and drops from 50,000 to 44,000 between 2010 and 2020, what will the population be in 2035?

4. If 30% of a radio-active substance disappears in 15 years, what is the half-life of the substance?

---

5. The Chinook Salmon Problem

The fishing industry is a major part of California's economy. A catch-and-release study of Chinook salmon on the Sacramento Delta near Rio Vista was undertaken in 2008. Over 60 days, the rate at which new fish were caught and released followed the equation $\frac{dF}{dt} = .004(100 - F)$, where $\frac{dF}{dt}$ was measured in number of smolt (young salmon) caught per day and $F(0) = 10$.

a) Find the particular solution to $\frac{dF}{dt} = .004(100 - F)$, if $F(0) = 10$.

b) How many smelt are captured and released when $t = 1.2$?

c) Find $\lim_{t \to \infty} F(t)$. Using the correct units, explain $\lim_{t \to \infty} F(t)$.

---

6. The Philosopher's Stone Problem

Medieval alchemist Pol Maychrowitz believed that the Philosopher's Stone would help them to convert lead into gold. The Stone was never found, but, if it had been found and worked, Pol assumed he could convert 12 pounds of lead over a 72-hour period and that the conversion rate would follow a separable differential growth equation $\frac{dG}{dt} = 1.2\left(3 - \frac{G}{4}\right)$. (By the way, if he had succeeded, Pol would probably have been burned at the stake.)

a) If $G(0)=1$, state the particular solution to the logistic differential equation.

b) If $G(0)=1$, what is $\lim_{t\to\infty} G$.

---

7. The Black Hole Accretion Problem

Research into supermassive black holes (SMBH) seems to indicate that their accretion rate (rate at which they gain mass) follows the differential equation $\frac{dM}{dt} = .256(1000 - M)$, where M is the mass of the SMBH in what we will call Kellartons. $M = f(t)$ would be the particular solution to the differential equation where $M(0)=600$.

(a) Find the equation of the line tangent to $M = f(t)$ at $M(0)=600$.

(b) Use the line tangent found in (a) to approximate the amount of the mass at time $t = 10$ seconds.

(c) Find the particular solution $M = f(t)$ by solving the differential equation $\frac{dM}{dt} = .256(1000 - M)$ with the initial condition $M(0)=600$.

(d) Find $\lim_{t\to\infty} M(t)$.

---

8. The Body Farm Problem

Research at the University of Tennessee Anthropological Research Facility, (aka The Body Farm) indicates that the decomposition rate of a certain body might follow the separable differential equation $\frac{dN}{dt} = 0.2(200 - N)$, where N is the number of pounds of flesh which has decomposed in *t* days. $N = f(t)$ is the particular solution to the differential equation where $N(0) = 20$.

(a) Find the equation of the line tangent to $N = f(t)$ at $N(0) = 20$.

(b) Use the line tangent found in (a) to approximate the amount of the substance that has decomposed at time $t = 5$ days.

(c) Find the particular solution $N = f(t)$ by solving the differential equation $\frac{dN}{dt} = .2(200 - N)$ with the initial condition $N(0) = 20$.

(d) Determine whether the amount of the flesh is decomposing at an increasing or a decreasing rate at $t = 5$. Explain your reasoning.

---

9. A cup of coffee is made with boiling water at a temperature of 100 C°, in a room at temperature 20 C°. After two minutes, it has cooled to 80 C°.

(a) Set up the differential equation for this Newton's Law problem and find the particular solution.
(b) What is its temperature after five minutes?
(c) When will the coffee drop below 40 C° and taste cold?

---

10. Dr. and Mrs. Quattrin enjoy the occasional pint of Ben & Jerry's Chocolate Fudge Brownie, but they prefer to let it soften before digging in. Research shows that ice cream tempers (softens) more evenly by putting it in the refrigerator rather than on the kitchen counter. When the ice cream comes out of the freezer, its temperature is -1 F°. The temperature inside of the refrigerator is 37 F°.

(a) Set up the differential equation for this Newton's Law problem. If the temperature of the ice cream is 2.5 F° after 20 minutes, find the particular solution to the differential equation.

(b) What was its temperature after 10 minutes?

(c) When will the temperature of the ice cream reach 4 F° (the ideal tempered temperature)?

(d) Dr. Quattrin cannot wait that long. He leaves the ice cream on the counter in the 67 F° kitchen for 10 minutes. If $k = 0.006$, what will the temperature be after 10 minutes? [Note: The ice cream softens more quickly but not evenly, so this will be an average temperature and the top inch of ice cream will be as much as 1.5° warmer and, thus, softened enough to scoop.]

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11. AP Calculus AB 2012 #5

5.3: Implicit Differentiation

Implicit differentiation is a technique that might be better suited to the Derivative Review chapter, but it is considered here because of its direct impact on Related Rates in the next section. **Implicit Differentiation is an application of the Chain Rule where the y-function is not easily defined explicitly.**

One of the more useful aspects of the Chain Rule is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of y with respect to x – that is, $\frac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra. When y cannot easily be isolated, we can treat y like we treat $g(x)$. In other words,

$\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot\left[g'(x)\right]$ is the same as $\frac{d}{dx}\left[f(y)\right]=f'(y)\cdot\left[\frac{dy}{dx}\right]$.

OBJECTIVES

Take derivatives of relations implicitly.
Use implicit differentiation to find higher order derivatives.

Ex 1 Find if $x^2 + y^2 = 25$

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right] \to \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

We can now isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

With this function, notice that y could have been isolated and $\frac{dy}{dx}$ cpould be found **explicitly**.

$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \sqrt{25 - x^2}$$

$$\frac{dy}{dx} = \frac{-x}{\sqrt{25 - x^2}}$$

Notice that this is the same answer as we found with implicit differentiation. You could substitute y for $\sqrt{25 - x^2}$ in the denominator and come up with the same derivative, $\frac{dy}{dx} = -\frac{x}{y}$.

Ex 2 Find the derivative of $x^2 - 3y^2 + 4x - 12y - 2 = 0$ both explicitly and implicitly.

$$\frac{d}{dx}\left[x^2 - 3y^2 + 4x - 12y - 2 = 0\right]$$

$$2x - 6y\frac{dy}{dx} + 4 - 12\frac{dy}{dx} = 0$$
$$(-6y - 12)\frac{dy}{dx} = -2x - 4$$
$$\frac{dy}{dx} = \frac{-2x - 4}{-6y - 12}$$

When considering functions, implicit differentiation may not seem to be a particularly powerful tool, because it is often simple to isolate *y*. But consider a non-function, like this circle, ellipse, or hyperbola, where *y* is not so easily isolated.

Ex 3 Find $\frac{dy}{dx}$ for the hyperbola $x^2 - 3xy + 3y^2 = 2$

It would be very difficult to solve for *y* here, so implicit differentiation is really our only option.

$$\frac{d}{dx}\left[x^2 - 3xy + 3y^2 = 2\right]$$

Note that $-3xy$ is a product. It will require the Product Rule:

$$\frac{d}{dx}\left[x^2 - 3xy + 3y^2 = 2\right]$$
$$2x - 3x\frac{dy}{dx} - 3y + 6y\frac{dy}{dx} = 0$$
$$2x - 3y = 3x\frac{dy}{dx} - 6y\frac{dy}{dx}$$
$$2x - 3y = (3x - 6y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x - 3y}{3x - 6y}$$

Ex 4 Find the equation of the line Tangent to $x^3 - y^2 + 6y = -3$ at $y = 1$.

a) $3x^2 - 2y = -6$ b) $3x - y = -7$

c) $3x + y = -5$ d) $x + 3y = 1$

e) $x - 3y = -5$

$$x^3 - y^2 + 6y = -3 \rightarrow x^3 - 1^2 + 6(1) = -3 \rightarrow x^3 = -8 \rightarrow x = -2$$

$$\frac{d}{dx}\left[x^3 - y^2 + 6y = -3\right]$$

$$3x^2 - 2y\frac{dy}{dx} + 6\frac{dy}{dx} = 0$$

At (-2, 1), $3(-2)^2 - 2\frac{dy}{dx} + 6\frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = 3$

$$y - 1 = 3(x + 2)$$
$$y - 1 = 3x + 6$$
$$-7 = 3x - y$$

The answer is B.

Ex 5 Given $\dfrac{dy}{dx} = \dfrac{x+2}{3y+6}$, find $\dfrac{d^2y}{dx^2}$

$$\frac{d}{dx}\left[\frac{dy}{dx} = \frac{x+2}{3y+6}\right]$$

$$\frac{d^2y}{dx^2} = \frac{(3y+6) - (x+2)\left(3\frac{dy}{dx}\right)}{(3y+6)^2}$$

Since we already know $\dfrac{dy}{dx}$, we can substitute

$$\frac{d^2y}{dx^2} = \frac{(3y+6) - (x+2)(3)\left(\frac{x+2}{3y+6}\right)}{(3y+6)^2}$$

$$= \frac{\left[(3y+6) - (x+2)(3)\left(\frac{x+2}{3y+6}\right)\right](3y+6)}{(3y+6)^2(3y+6)}$$

$$= \frac{(3y+6)^2 - 3(x+2)^2}{(3y+6)^3}$$

5.3 Free Response Homework

Find $\frac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for y and differentiating. Show that $\frac{dy}{dx}$ is the same in both cases.

1. $x^2+y^2=1$
2. $x^3+4y^2=16$
3. $\frac{1}{x}+\frac{1}{y}=1$
4. $\sqrt{x}+\sqrt{y}=4$

Find $\frac{dy}{dx}$ for each of these equations by implicit differentiation.

5. $x^2+xy-4y-1=0$
6. $xy+2x+3x^2=4$
7. $x^2+4xy-5y^2=4$
8. $3x^2+xy-4y^2=5$
9. $x^3+10x^2y+7y^2=60$
10. $x^2y^2+x\sin(y)=4$
11. $4\cos(x)\sin(y)=1$
12. $e^{x^2y}=x+y$
13. $\tan(x-y)=\frac{y}{1+x^2}$
14. $y^2=\frac{x-1}{x+1}$
15. $x^2=\frac{x-y}{x+y}$
16. $x^2+y^2=\frac{x}{y}$
17. $y^2=\frac{x-y}{x+y}$
18. $y^2=\frac{x^2-1}{x+2}$

Find the equation of the line tangent to each of the following relations at the given point.

19. Find the equation of the line tangent $x^2-y^2-6y-3=0$ at $\left(\sqrt{3},0\right)$
20. Find the equation of the line tangent $9x^2+4y^2+36x-8y-32=0$ at $(0,-2)$

21. Find the equation of the line **normal** $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $(-1, -3)$.

22. Find the equation of the line tangent to $x^3 + \frac{y}{x} + y^2 = 7$, through the point (1,2).

23. Find the equation of the line normal to $x^2 + 3xy + y^2 = 11$, through the point (1,2).

24. Find the equation of the lines tangent and normal to $1 + y\cos\frac{\pi}{x} = x - \sin\frac{\pi}{y}$ at $(2, 2)$.

25. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y\sin x} + y^3 - 3$ at the point $\left(\frac{\pi}{2}, 0\right)$.

26. Find $\frac{d^2y}{dx^2}$ if $xy + y^2 = 1$

27. Find $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$

28. Find $\frac{d^2y}{dx^2}$ if $x^2 + y^2 = 1$

29. Find $\frac{d^2y}{dx^2}$ if $x^3 + 4y^2 = 16$

5.3 Multiple Choice Homework

1. Use Implicit Differentiation to find the points on $x^3 - y^2 + x^2 = 0$ has vertical tangent lines.

a) $(0, 0)$ only
b) $(-1, 0)$ only
c) $(1, \sqrt{2})$ only
d) $(-1, 0)$ and $(0, 0)$
e) The tangent line is never vertical

---

2. If $x^2 + xy = 10$, then when $x = 2$, $\frac{dy}{dx} =$

a) $-\frac{7}{2}$ b) -2 c) $\frac{2}{7}$ d) $\frac{3}{2}$ e) $\frac{7}{2}$

---

3. What is the slope of the line tangent to the curve $y^2 + x = -2xy - 5$ at the point (2,1)?

a) $-\frac{4}{3}$ b) $-\frac{3}{4}$ c) $-\frac{1}{2}$ d) $-\frac{1}{4}$ e) 0

---

4. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x:

a) $\frac{6x - 4y}{4x + 4}$ b) $\frac{9x^2 - 4}{4x + 8y}$ c) $\frac{9x^2 - 4}{4 + 8y}$

d) $\frac{9x^2 - 4y}{4x + 8y}$ e) $\frac{9x^2 - 4y}{4 + 8y}$

---

5. Given $x + xy + 2y^2 = 6$, then $\left.\frac{dy}{dx}\right|_{(2,\,1)} =$

a) $\frac{2}{3}$ b) $\frac{1}{3}$ c) $-\frac{1}{3}$ d) $-\frac{1}{5}$ e) $-\frac{3}{4}$

---

6. Consider the closed curve in the *x-y* plane given by $2x^2 + 5x + y^2 + y = 8$. Which of the following is correct?

a) $\frac{dy}{dx} = -\frac{4x+5}{8x+2y+1}$ b) $\frac{dy}{dx} = \frac{4x+5}{2y+1}$

c) $\frac{dy}{dx} = -\frac{4x+5}{8x+2y}$ d) $\frac{dy}{dx} = \frac{4x+5}{8x+2y}$

e) $\frac{dy}{dx} = -\frac{4x+5}{2y+1}$

---

7. The slope of the line tangent to $xy - y^3 + 6 = 0$ at (1,2) is

a) 0 b) $-\frac{1}{12}$ c) $\frac{2}{11}$ d) $\frac{1}{6}$ e) $\frac{1}{4}$

---

8. Find the equation of the line tangent to the curve $\sec(x^2) + xy^3 = 2 - y$ at $x = 0$.

a) $y = -x$ b) $y - 1 = -x$ c) $y - 2 = -x$

d) $y - 1 = x$ e) $y - 2 = x$

---

9. If $\sin^{-1}x = \ln y$, then $\frac{dy}{dx} =$

a) $\frac{y}{\sqrt{1-x^2}}$ b) $\frac{xy}{\sqrt{1-x^2}}$ c) $\frac{y}{1+x^2}$ d) $e^{\sin^{-1}x}$

e) $\frac{e^{\sin^{-1}x}}{1+x^2}$

---

10. If $x^2y + yx^2 = 6$, then, at $(1,\ 3)$, $\dfrac{d^2y}{dx^2} =$

a) -18 b) -6 c) 6 d) 12 e) 18

---

11. If $y = x + \sin(xy)$, then $\dfrac{dy}{dx} =$

a) $1 + \cos(xy)$ b) $1 + y\cos(xy)$ c) $\dfrac{1}{1 - \cos(xy)}$

d) $\dfrac{1}{1 - x\cos(xy)}$ e) $\dfrac{1 + y\cos(xy)}{1 - x\cos(xy)}$

---

12. If $\sin(xy) = x^2$, then $\dfrac{dy}{dx} =$

a) $2x\sec(xy)$ b) $\dfrac{\sec(xy)}{x^2}$ c) $2x\sec(xy) - y$

d) $\dfrac{2x\sec(xy)}{y}$ e) $\dfrac{2x\sec(xy) - y}{x}$

---

13. Given $y = \ln(x^2 + y^2)$, find $\dfrac{dy}{dx}$ at the point $(1,\ 0)$.

a) 0 b) 0.5 c) 1 d) 2 e) undefined

5.4: AP-Style Implicit Differentiation Problems

Common Sub-Topics:

- Equation of a tangent line
- Horizontal and/or vertical tangent lines
- Points on a curve with a particular slope
- Second Derivative Test
- Finding the particular solution

Remember:

The 2nd Derivative Test

For a function *f*,

1) If $f'(c)=0$ and $f''(c)>0$, then f has a relative minimum at c.
2) If $f'(c)=0$ and $f''(c)<0$, then f has a relative maximum at c.

This is needed because one cannot create a sign pattern without an **explicitly** stated function, so the First Derivitaive Test will not work on problems that need Implicit Differentiation to find the derivative.

OBJECTIVES

Take derivatives of relations implicitly.
Use implicit differentiation to find higher order derivatives.
Use the Second Derivative Test to etermine whether a point is at a maximum, minimum or neither.
Use separation of variables to find the particular solution to a differential equation.

Ex 1 Consider the curve given by $x^2 + 4xy + y^2 = -12$.

(a) Show that $\frac{dy}{dx} = -\frac{x+2y}{2x+y}$.

(b) Find the equations of all the tangent lines which are horizontal.

(c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

(a) Show that $\frac{dy}{dx} = -\frac{x+2y}{2x+y}$.

$$\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right]$$

$$2x + 4x\frac{dy}{dx} + 4y(1) + 2y\frac{dy}{dx} = 0$$

$$4x\frac{dy}{dx} + 2y\frac{dy}{dx} = -2x - 4y$$

$$(4x+2y)\frac{dy}{dx} = -2x - 4y$$

$$\frac{dy}{dx} = \frac{-2x-4y}{4x+2y} = -\frac{x+2y}{2x+y}$$

Note the use of the Product Rule for $\frac{d}{dx}[4xy]$.

(b) Find the equations of all the tangent lines which are horizontal.

Horizontal lines have a slope = 0, so $\frac{dy}{dx} = -\frac{x+2y}{2x+y} = 0 \rightarrow x + 2y = 0 \rightarrow x = -2y$

To be on the curve, $x = -2y$ must make the original equation true:

$$(-2y)^2 + 4(-2y)y + y^2 = -12$$
$$4y^2 - 8y^2 + y^2 = -12$$
$$-3y^2 = -12$$
$$y^2 = 4$$
$$y = \pm 2$$

$$\text{x} = -2\text{y} \rightarrow (-4,\ 2)\ \textit{and}\ (4,\ -2)$$

(c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

What is really asked here is to apply the Second Derivative Test, because we cannot create a sign pattern for non-functions. The *y* is not isolated in the eqation.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx} = -\frac{x+2y}{2x+y}\right] = -\frac{(2x+y)\left(1+2\frac{dy}{dx}\right)-(x+2y)\left(2+\frac{dy}{dx}\right)}{(2x+y)^2}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-4,\ 2)} = -\frac{(2(-4)+2)(1+2(0))-(-4+4)(2+(0))}{(2(-4)+2)^2} = \frac{6}{(-6)^2} > 0$$

$$\left.\frac{d^2y}{dx^2}\right|_{(4,\ -2)} = -\frac{(2(4)-2)(1+2(0))-(4-2(2))(2+(0))}{(2(4)-2)^2} = \frac{-6}{(6)^2} < 0$$

$(-4, 2)$ will be at a minimum because the second derivative is positive.
$(4, -2)$ will be at a maximum because the second derivative is negative

Be Careful! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don't make careless mistakes.

Ex 2 Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ and suppose the point $(0,\ -3)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

b) Determine if the point $(0,\ -3)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{y+1}{x^2+9}$ at $(0,\ -3)$.

---

a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{y+1}{x^2+9}\right] = \frac{\left(x^2+9\right)\frac{dy}{dx} - (y+1)(2x)}{\left(x^2+9\right)^2}$$

$$= \frac{\left(x^2+9\right)\left[\frac{y+1}{x^2+9}\right] - (y+1)(2x)}{\left(x^2+9\right)^2} = \frac{(y+1)-(y+1)(2x)}{\left(x^2+9\right)^2} = \frac{(y+1)(1-2x)}{\left(x^2+9\right)^2}$$

---

b) Determine if the point $(0,\ -3)$ is at a maximum, a minimum, or neither.

At the point $(0,\ -3)$, $\dfrac{dy}{dx} = \dfrac{(-3)+1}{0^2+9} = -\dfrac{2}{9} \neq 0$, therefore, neither.

---

c) Find the particular solution to $\frac{dy}{dx} = \frac{y+1}{x^2+9}$ at $(0, -3)$.

$$\frac{dy}{dx} = \frac{y+1}{x^2+9}$$

$$\frac{1}{y+1}dy = \frac{1}{x^2+9}dx$$

$$\int \frac{1}{y+1}dy = \int \frac{1}{x^2+9}dx$$

$$\ln|y+1| = \frac{1}{3}\tan^{-1}\frac{x}{3} + c$$

$$|y+1| = e^{\frac{1}{3}\tan^{-1}\frac{x}{3} + c}$$

$$y+1 = ke^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

$$(0, -3) \rightarrow -3+1 = ke^{\frac{1}{3}\tan^{-1}\frac{0}{3}} \rightarrow -2 = k$$

$$y+1 = -2e^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

$$y = -1 - 2e^{\frac{1}{3}\tan^{-1}\frac{x}{3}}$$

5.4 Free Response Homework

1. Consider the curve given by $3x^2 - 4xy + 5y^2 = 25$.

   a) Show that $\dfrac{dy}{dx} = \dfrac{3x-2y}{2x-5y}$.

   b) Find point(s) P with x-coordinate 2.

   c) Find the equations of the lines tangent to $3x^2 - 4xy + 5y^2 = 25$ at the points found in b).

   d) Find the points on $3x^2 - 4xy + 5y^2 = 25$ where the tangent line is horizontal.

2. Consider the curve given by $x^2 - xy + y^2 = 4$.

   a) Show that $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$.

   b) Find point(s) P with x-coordinate 2.

   c) Find the equations of the lines tangent to $x^2 - xy + y^2 = 4$ at the points found in b).

   d) Find the points on $x^2 - xy + y^2 = 4$where the tangent line is vertical.

3. Consider the curve given by $2x^2 - xy + y^2 = 44$.

   a) Show that $\dfrac{dy}{dx} = \dfrac{4x-y}{x-2y}$.

   b) Find point(s) P with x-coordinate 5.

   c) Find the equations of the lines tangent to $2x^2 - xy + y^2 = 44$ at the points found in b).

   d) Find the points on $2x^2 - xy + y^2 = 44$ where the tangent line is vertical.

4. Consider the curve given by $x^2 - 3xy + 4y^2 = 7$.

a) Show that $\frac{dy}{dx} = \frac{3y - 2x}{8y - 3x}$.

b) Show that there is a point P with x-coordinate 3 at which the tangent line is horizontal. Find the y-coordinate of point P.

c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

5. Consider the curve given by $x^2 + xy + y^2 = 12$.

(a) Show that $\frac{dy}{dx} = \frac{-y - 2x}{2y + x}$.

(b) Find point(s) P Where the tangent line is horizontal.

(c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

6. Consider the curve given by $x^3y^4 - 5 = x^3 - x^2 + y$.

(a) Show that $\frac{dy}{dx} = \frac{x^2 - 2x - 3x^2y}{4x^3y^3 - 1}$.

(b) Find the equation of the line tangent to the above curve at the point $(2, -1)$.

(c) Find the value of $\frac{d^2y}{dx^2}$ at $(0, -5)$. Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

7. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = y^2(6-2x)$, and suppose the point $\left(3,\ -\dfrac{1}{3}\right)$ is on the graph of $y = f(x)$.

a) Use implicit differation to find $\dfrac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $\left(3,\ -\dfrac{1}{3}\right)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\dfrac{dy}{dx} = y^2(6-2x)$ at $\left(3,\ -\dfrac{1}{3}\right)$.

8. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = xy + y$, and suppose the point $(-1,\ 2)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(-1,\ 2)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\dfrac{dy}{dx} = xy + y$ at $(-1,\ 2)$.

9. Let $y = f(x)$ be a differentiable function such that $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$ and suppose the point $(0,\ 1)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\dfrac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(0,\ 1)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\dfrac{dy}{dx} = \dfrac{3x^2}{y+2}$ at $(0,\ 1)$.

10. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = (x-1)(y+2)$ and suppose the point $(1, 0)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(1, 3)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = (x-1)(y+2)$ at $(1, 3)$.

11. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = y^3(x+1)$, and suppose the point $\left(0, -\frac{1}{2}\right)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$ in terms of x and y.

b) Determine if the point $\left(0, -\frac{1}{2}\right)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = y^3(x+1)$ at $\left(0, -\frac{1}{2}\right)$.

12. Let $y = f(x)$ be a differentiable function such that $\frac{dy}{dx} = \frac{x-1}{y}$ and suppose the point $(1, -2)$ is on the graph of $y = f(x)$.

a) Use implicit differentiation to find $\frac{d^2y}{dx^2}$.

b) Use the solution to a) to determine if the point $(1, -2)$ is at a maximum, a minimum, or neither.

c) Find the particular solution to $\frac{dy}{dx} = \frac{x-1}{y}$ at $(1, -2)$.

13. AP Packet: AB 2000 #5, AB08B #6, AB15 #6

5.5: Related Rates

In this course, derivatives have primarily been interpreted as the slope of the tangent line. But, as with rectilinear motion, there are other contexts for the derivative. One overarching concept is that a derivative is a **Rate of Change**. The tendency is to think of rates as distance per time unit, like miles/hour or feet/second, but even slope is a rate of change—it is just that rise and run are both measured as distances.

The idea behind related rates is two-fold. First, change is occurring in two or more measurements that are related to each other by the geometry (or algebra) of the situation. Second, an implicit chain rule situation exists in that the x and y-values are functions of time, which may or may not be a variable in the problem. Therefore, when taking the derivative of an x or y, an **Implicit Rate Term** $\left(\frac{dx}{dt} \text{ or } \frac{dy}{dt}\right)$ occurs.

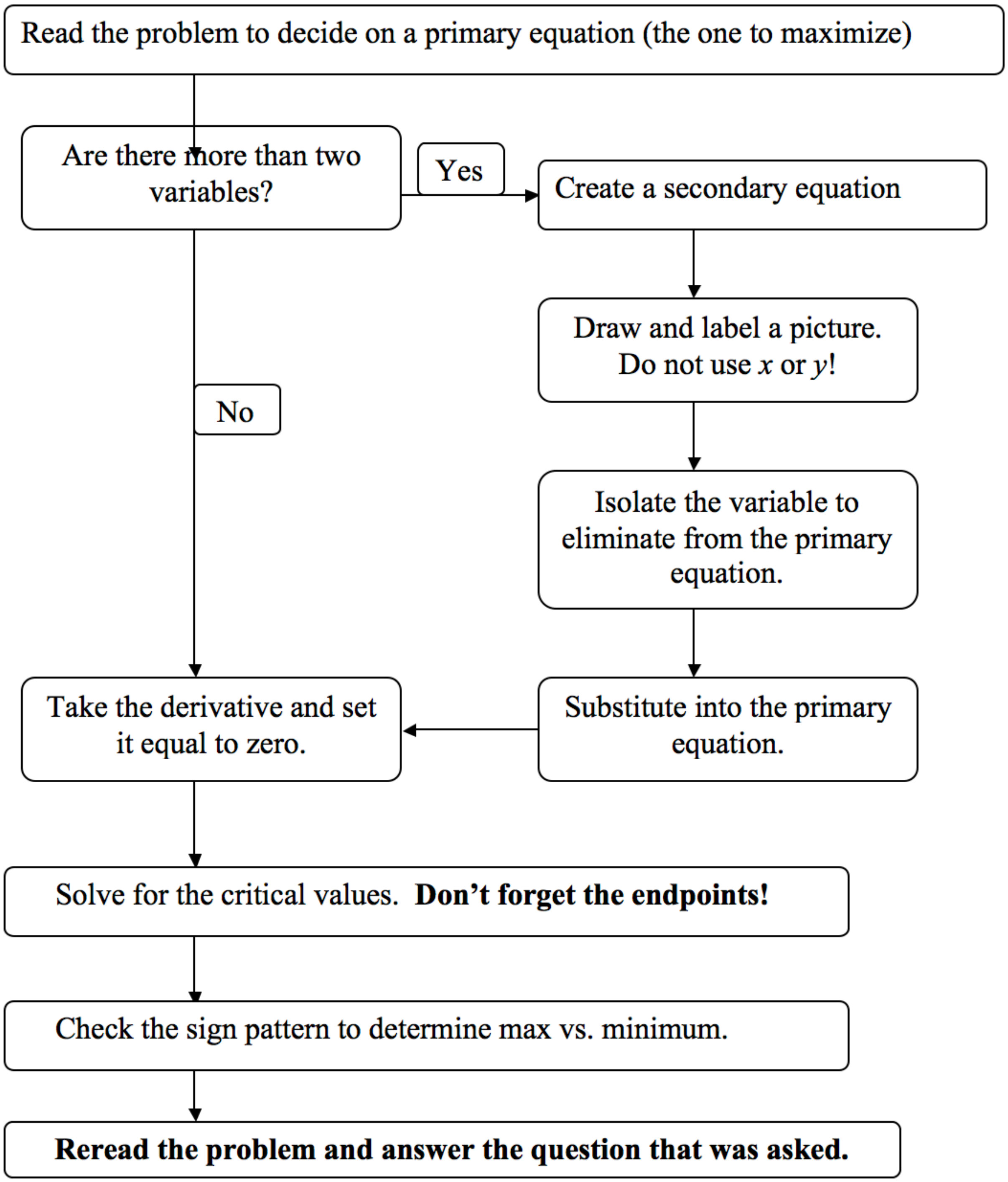
LEARNING OUTCOME
Solve related rates problems.

At first glance, Related Rates Problems might look like Optimization Problems. Consider the Cola Can Problem from Chapter 4:

EX 2 (from section 4.2) The volume of a cylindrical cola can is $32\pi \text{ in}^3$**. What is its minimum surface area for such a can?**

The word "minimum" tells us we have an optimization problem. Remember how we attacked Optimization problems:

Strategy for Approaching Optimization Problems with Calculus



So,

EX 2 The volume of a cylindrical cola can is $32\pi \text{ in}^3$. What is its minimum surface area for such a can?

The problem asks to minimize the surface area, which is determined by:

$$S = 2\pi r^2 + 2\pi rh$$

Either *r or h* need to be eliminated in this formula before differentiating. The volume is $V = \pi r^2 h = 32\pi$, so $h = \dfrac{32}{r^2}$ and

$$S = 2\pi r^2 + 2\pi r\left(\frac{32}{r^2}\right) = 2\pi r^2 + \left(\frac{64\pi}{r}\right)$$

$$S' = 4\pi r - \left(\frac{64\pi}{r^2}\right) = 0$$

$$4\pi r = \frac{64\pi}{r^2}$$

$$r^3 = 16$$

$$r = 2.5198$$

S'	$-$	0	$+$
r		2.520	

$$S(2.5198...) = 2\pi(2.5198...)^2 + \left(\frac{64\pi}{(2.5198...)}\right)$$

$$= 119.687 \text{ in}^2$$

The minimum surface area of the cola can is 119.687 in^2.

A Related Rates Problem is characterized by various **measurements that are changing** in relation to one another. The relationship between the variables is till geometry. The major difference is we will be differentiating in terms of time instead of in terms of x. In other words,

Optimization Problems:	Apply $\frac{d}{dx}$
Related Rates Problems:	Apply $\frac{d}{dt}$

A Different (Related Rates) Cola Can Problem:

Ex 1 The volume of a cylindrical cola can is $32\pi \text{ in}^3$. The height of the can is changing at $\frac{1}{4}\,{}^{in}\!/\!_{\sec}$. If the radius changes at the same time so as to maintain the volume, how fast is the radius shrinking when the can is 4 inches tall?

$$V = \pi r^2 h = 32\pi \rightarrow h = 32r^{-2}$$

$$\frac{d}{dt}\left[h = 32r^{-2}\right]$$
$$\frac{dh}{dt} = -64r^{-2}\frac{dr}{dt}$$

$$h = 4 \rightarrow 4 = 32r^{-2}$$
$$r^2 = 8$$

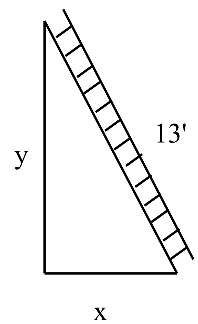
$$\frac{1}{4} = -64\left(\frac{1}{8}\right)\frac{dr}{dt}$$
$$\frac{dr}{dt} = -\frac{1}{32}\,{}^{in}\!/\!_{\sec}$$

Process for Related Rates Problems:

1. Draw a picture.
2. Label the picture with what is being given and what is being asked.
   a. Use variables for any sizes that are **changing**
   b. Pay particular attention to the units.
3. Determine the equation(s) that relates the variables to each other.
   a. Decide which equation will be differentiated
   b. If there is a product of two variables, eliminate the product by either multiplying the equation out or by substituting a secondary equation. [Note: This is only because we have not learned the Product Rule yet.]
4. **Differentiate in terms of time**. This is the key step!
   a. Do not forget the implicit fractions.
5. Substitute the given information and solve for the missing variable.
6. Reread the problem and make sure to answer the question that was asked.

A classic problem is the falling ladder.

EX 2 A 13-foot-tall ladder is leaning against a wall. The bottom of the ladder slides away from the wall at 4 ft/sec. How fast is the top of the ladder moving down the wall when the ladder is 5 feet from the wall?



As can be seen in the picture, the height of the top of the ladder and the distance the bottom of the ladder is from the wall are related by the Pythagorean theorem. Both are variables, because the ladder is moving. Therefore,

$$x^2 + y^2 = 13^2$$

4 ft/sec is the rate at which the x-value is changing—i.e. $\frac{dx}{dt}$. To find $\frac{dy}{dt}$, differentiate $x^2 + y^2 = 13^2$ to get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

This is essentially an equation in four variables. But x and $\frac{dx}{dt}$ are known, and $y = 12$ (by the Pythagorean theorem).
So,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(5)(4) + 2(12)\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{5}{3} \text{ ft / sec}$$

It should make sense that $\frac{dy}{dt}$ is negative since the top of the ladder is falling.

REMEMBER: Common Formulas for Optimization Problems:

Pythagorean Theorem

$$x^2 + y^2 = r^2$$

Area Formulas

Circle: $A = \pi r^2$ Rectangle: $A = lw$

Triangle: $A = \frac{1}{2}bh$ Trapezoid: $A = \frac{1}{2}h\left(b_1 + b_2\right)$

Volume Formulas

Sphere: $V = \frac{4}{3}\pi r^3$ Right Prism: $V = Bh$

Cylinder: $V = \pi r^2 h$ Cone: $V = \frac{1}{3}\pi r^2 h$ Right Pyramid: $V = \frac{1}{3}Bh$

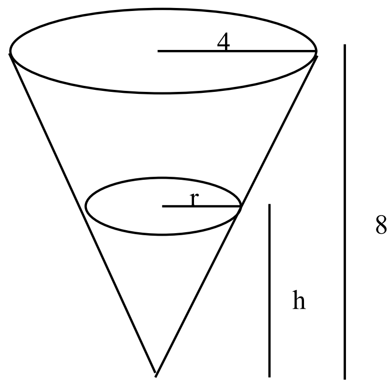
Surface Area Formulas

Sphere: $S = 4\pi r^2$ Cylinder: $S = 2\pi r^2 + 2\pi rh$

Cone: $S = \pi r^2 + \pi rl$ Right Prism*: $S = 2B + Ph$

Another common related rate problem is where a tank of a particular shape is filling or draining.

EX 3 A tank shaped like an inverted cone 8 feet in height and with a base diameter of 8 feet is filling at a rate of 10 ft^3/minute. How fast is the height changing when the water is 6 feet deep?



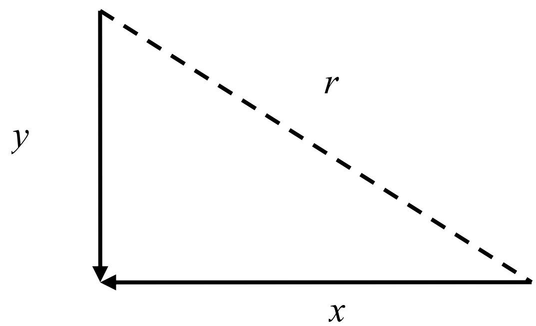
The units on the 10 tell us that it is the change in volume, or $\frac{dV}{dt}$. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. But this equation has too many variables for us to differentiate it as it stands. Since the rate of change of the height—i.e., $\frac{dh}{dt}$—was what the question is, eliminate the r from the equation. By similar triangles, $\frac{r}{h} = \frac{4}{8}$ and $r = \frac{1}{2}h$. Substitution gives a volume equation in terms of height only:

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h \\ &= \frac{\pi}{12}h^3 \end{aligned}$$

Differentiate and plug into to solve for $\frac{dh}{dt}$.

$$\begin{aligned} V &= \frac{\pi}{12}h^3 \\ \frac{dV}{dt} &= \frac{\pi}{4}h^2\frac{dh}{dt} \\ 10 &= \frac{\pi}{4}(6)^2\frac{dh}{dt} \\ \frac{dh}{dt} &= \frac{10}{9\pi} \text{ ft/min} \end{aligned}$$

Ex 4 Two cars approach an intersection, one traveling south at 20 mph and the other traveling west at 30 mph. How fast is the direct distance between them decreasing when the westbound car is .6 miles and the southbound car is .8 miles from the intersection?



As we can see in the picture, the distance between the two cars are related by the Pythagorean Theorem.

$$x^2 + y^2 = r^2$$

We know several pieces of information. The southbound car is moving at 20 mph; i.e. $\frac{dy}{dt} = -20$. By similar logic we can deduce each of the following:

$$\frac{dy}{dt} = -20 \qquad \frac{dx}{dt} = -30$$
$$y = 0.8 \qquad x = 0.6$$

And, by the Pythagorean Theorem, $r = 1.0$

Now we take the derivative of the Pythagorean Theorem and get

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

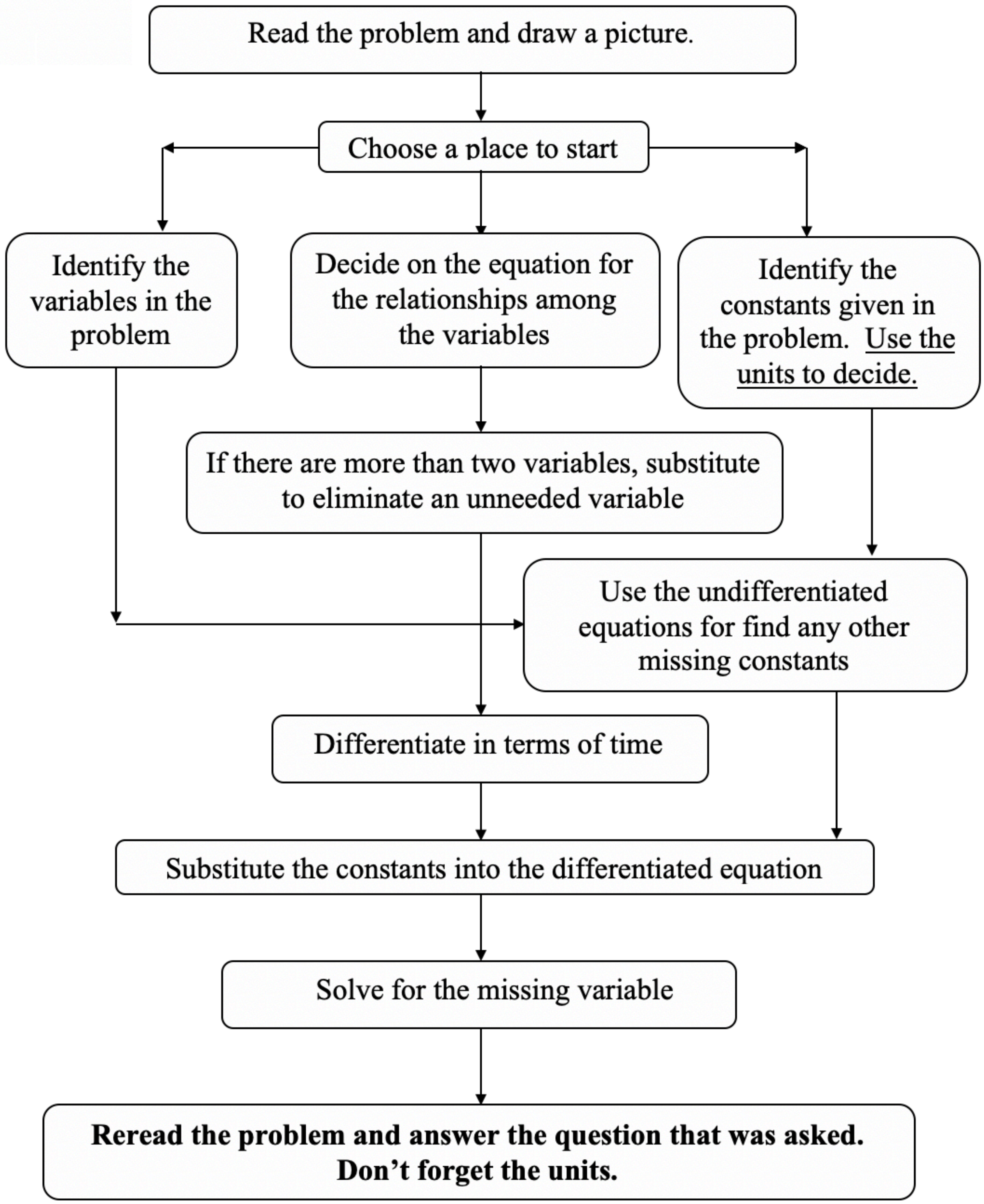
This is essentially an equation in six variables. But we know five of those six variables, so just substitute and solve.

$$2(0.8)(-30) + 2(0.6)(-20) = 2(1.0)\frac{dr}{dt}$$

$$\frac{dr}{dt} = -36 \ miles/hour$$

It should make sense that $\frac{dr}{dt}$ is negative since the two cars are approaching one another. We know the units based on the fraction, . Since *r* was in miles and *t* was in hours, our final units must be miles/hour.

Strategy for Approaching Related Rates Problems



5.5 Free Response Homework

1. Two boats leave an island at the same time, one heading north and one heading east. The northbound boat is moving at 12 mph and the eastbound boat is traveling at 5 mph. At $t = 0.2$ hours, the northbound boat is 1.4 miles away from the island and the eastbound boat is 1 mile from the island.

a) Draw a picture of the situation at any time *t*.

b) What variables are present in the problem?

c) What known quantities are given? What quantities can be determined directly from the given information? And what is the unknown for which to solve?

d) What equation(s) relates the quantities? And which one will be differentiated?

e) How fast is the distance between the two ships increasing at $t = 0.2$ hours?

2. A railroad track and a road cross at right angles. An observer stands on the road 70 meters south of the intersection and watches an eastbound train traveling 60m/sec.

a) Draw a picture of the situation at any time *t*.

b) What variables are present in the problem?

c) What known quantities are given? And what is the unknown for which to solve?

d) What equation(s) relates the quantities? And which one will be differentiated?

e) At how many m/sec is the train moving away from the observer 4 second after it passes the intersection?

3. A circular ink stain is spreading (i.e. the radius is changing) at half an inch per minute.

a) Draw a picture of the situation at any time *t*.

b) What variables are present in the problem?

c) What known quantities are given? And what is the unknown for which to solve?

d) What equation(s) relates the quantities? And which one will be differentiated?

e) How fast is the area changing when the stain has a 1-inch diameter?

4. A screensaver has a rectangular logo that expands and contracts as it moves around the screen. The ratio of the sides stays constant, with the long side being 1.5 times the short side. At a particular moment, the long side is 3 cm, and perimeter is changing by .25 cm/sec.

a) Draw a picture of the situation at any time *t*.
b) What variables are present in the problem?
c) What known quantities are given? And what is the unknown for which to solve?
d) What equation(s) relates the quantities? And which one will be differentiated?
e) How fast is the area of the rectangle changing at that moment?

5. Sand is dumped onto a pile at 30π ft^3/min. The pile forms a cone with the height always equal to the base diameter.
a) Draw a picture of the situation at any time *t*.
b) What variables are present in the problem?
c) What known quantities are given? And what is the unknown for which to solve?
d) What equation(s) relates the quantities? And which one will be differentiated?
e) How fast is the height changing when the pile is 5 feet high?

6. A cylindrical oil tank of height 30' and radius 10' is leaking at a rate of 300 ft^3/min.

a) What variables are present in the problem?
b) What known quantities are given? And what is the unknown for which to solve?
c) What equation(s) relates the quantities? And which one will be differentiated?
d) How fast is the oil level dropping?

7. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min. If the tank is 8 m tall and has a diameter of 4 m.
a) Draw a picture of the situation at any time *t*?
b) What variables are present in the problem?
c) What known quantities are given? And what is the unknown for which to solve?
d) What equation(s) relates the quantities? And which one will be differentiated?
e) Find the rate at which the height is decreasing when the water level is at 3 m.
f) Then find the rate of change of the radius at that same instant.

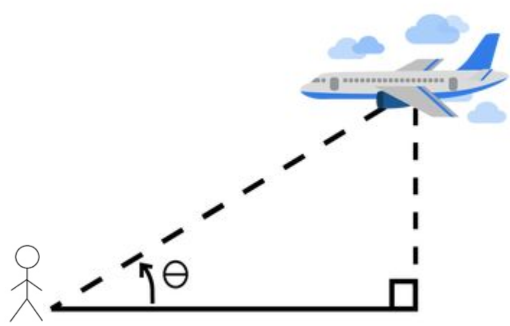
8. Sand is dumped onto a pile at 30π ft^3/min. The pile forms a cone with the height always equal to the base diameter. How fast is the base area changing when the pile is 10 feet high?

9. A spherical balloon is being inflated so that its volume is increasing at a rate of 6 ft^3/min. How fast is the radius changing when $r = 10$ ft?

10. The edge of a cube is expanding at a constant rate of 6 inches per second. What is the rate of change of the volume, in in^3 per second, when the total surface area of the cube is 54 in^2?

11. A 25-foot tall ladder is leaning against a wall. The bottom of the ladder is pushed toward the wall at 5 ft/sec. How fast is the top of the ladder moving up the wall when it is 7 feet up?

12. You are standing outside. A plane flies overhead, approaching you at a constant altitude, and constant speed of 600 miles per hour. When the plane flies over a house 13 miles away from where you are standing, the angle of elevation θ is 0.647 radians. How quickly is the direct (diagonal) distance between you and the plane changing at that moment?



13. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm^2/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm^2?

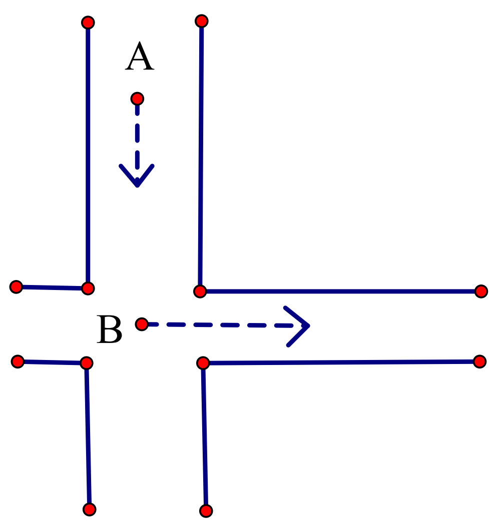
14. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

15. According to Boyle's Law, gas pressure varies directly with temperature and inversely with volume $\left(\text{or } P = \frac{kT}{V}\right)$. Suppose that the temperature is held constant while the pressure increases at 20 kPa/min. What is the rate of change of the volume when the volume is 600 in^3 and the pressure is 150 kPa?

16. Adiabatic Law: According to the adiabatic law for expansion of air, $P \cdot V^{1.4} = \frac{4}{81}$, where P is pressure and V is volume. If, at a specific instant, P = 108 lb/in^2 and is increasing at 27 lb/in^2 per second, what is the rate of change of the volume?

---

17. Person A is 220 feet north of an intersection and walking toward it at $10\ {}^{ft}\!/_{sec}$. Person B starts at the intersection and walks east at $5\ {}^{ft}\!/_{sec}$.



a) At $t = 10$ seconds, how far is each person from the intersection?

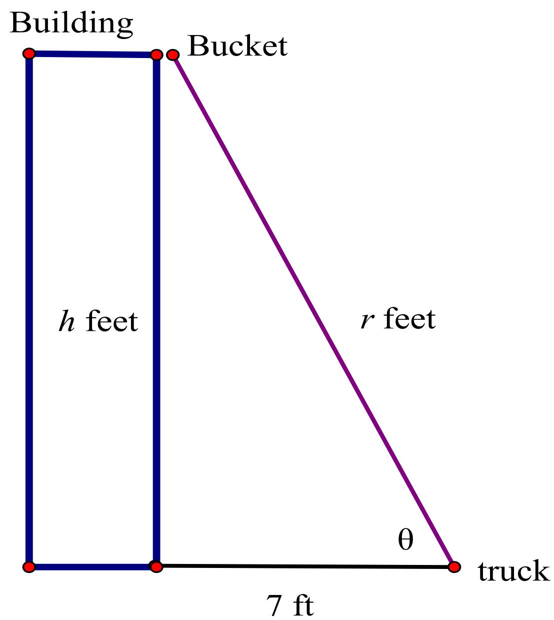
b) At $t = 10$ seconds, how far apart are the two people?

c) How fast is the distance between the two people changing at $t = 10$ seconds?

d) If person A looks at person B when $t = 10$ seconds. How fast is the angle changing?

---

18. A fire truck is parked 7 feet away from the base of a building and its ladder is extended to the top of the building. The ladder retracts at a rate of 0.5 feet per second, while the angle of the ladder changes such that the bucket at the end of the ladder comes down vertically.



a) How far is the ladder extended when the bucket is 10 feet above the ground?

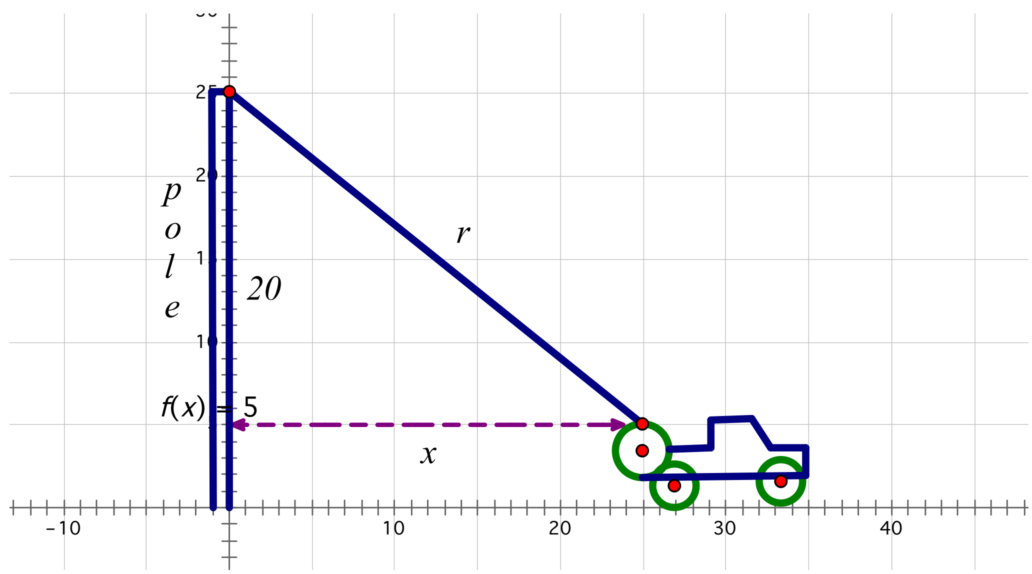
b) Find the rate at which the bucket is dropping vertically when the bucket is 10 feet above the ground.

c) What is the relationship between the angle θ and the height of the bucket? Find θ, in radians, when the bucket is 10 feet above the ground.

d) Find the rate, in radians per second, at which the angle the ladder forms with the ground is changing when the bucket is 10 feet above the ground.

---

19. A telephone crew is replacing a phone line from one telephone pole to the next. The line is on a spool on the back of a truck, and one end is attached to the top of a 25’ pole. The vertical distance from the top of the pole to the level of the spool is 20’.



The truck moves down the street at $20\ {}^{ft}\!/_{sec}$.

---

a) Find the length of line that has been rolled out when $t = 15$ sec.

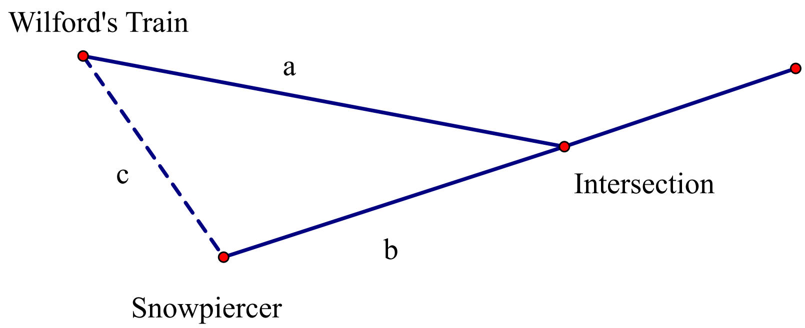
b) Find the rate at which the telephone line is coming off the spool when the truck is 50 feet from the pole.

c) What is the relationship between the angle θ and the truck’s distance from the pole? Find θ, in radians, when the truck is 40 feet from the pole.

d) Find the rate, in radians per second, at which the angle the line forms with horizontal is changing when the truck is 40 feet from the pole.

---

20. At the end of the first season of the series *Snowpiercer*, a second train, Big Alice) controlled by the industrialist Mr. Wilford), came down another track to intercept and stop Snowpiercer. If the two train tracks meet at a 30° angle, then the Law of Cosines would apply such that $c^2 = a^2 + b^2 - 1.969ab$ as in the figure below.



Snowpiercer is described as being two-and-a-half stories tall and 1,001 cars long, with an average speed of 100 kilometers per hour. Wilford's train Big Alice is much shorter, so it can average 120 km per hour.

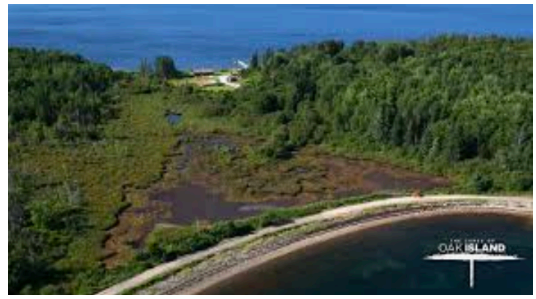
a) If Snowpiercer is 38 km from the junction, how soon will it reach the junction?

b) If Wilford's train is 45 km from the junction, which train will reach the junction first?

c) How fast is the distance c between the two trains changing when Snowpiercer is 38 km from the junction and Big Alice is 45 km from the junction.

---

21. The triangle-shaped Swamp on Oak Island has been proven to have been man-made sometime around 1250 AD, possibly by the Knights Templar. It has been drained several times, revealing a stone-paved wharf and paths, an ancient clay mine, and the remains of a galleon which had been destroyed by fire and sunk in the Swamp to conceal it.



The Triangle is roughly isosceles, with legs measuring 730 feet and a base of 640 feet. The apex (top angle) of the triangle measures 0.79 radians.

a) Based on the Law of Sines, the area of a triangle can be determined by the equation $Area = \frac{1}{2}ab\sin C$, where a and b are the length of the legs and $\angle C$ is the included angle. Find the surface area of the swamp before it was drained. Indicate the units.

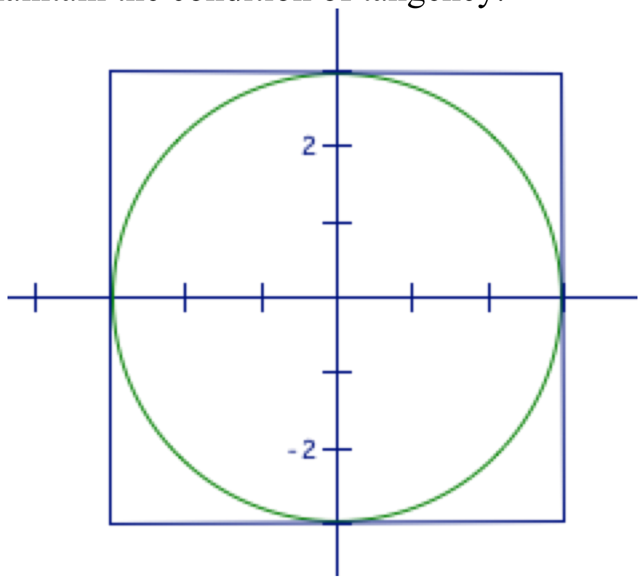
b) The length of third side of the Triangle can be found using the Law of Cosine: $c^2 = a^2 + b^2 - 2ab\,cosC$, where c is the third side and $\angle C$ is the apex. How long is the third side when the legs are each 370 feet?

c) At $t = 24\,hrs$, the legs are 370 feet and their rate of change is -15.2 feet per hour. How fast is the third side of the Triangle changing?

d) Using $Area = \frac{1}{2}ab\sin C$, find the rate of change of the surface area at $t = 24\,hrs$. Indicate the units.

---

22. A circle is inscribed in a square as shown. The circumference of the circle is increasing at a constant rate of 4 inches per second. As the circle expands, the square expands to maintain the condition of tangency.

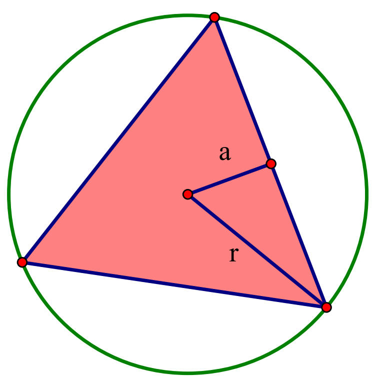


a) Find the rate of change of the perimeter of the square.

b) At the instant when the area of the circle is 16π square inches, find the rate at which the area between the square and the circle is increasing.

---

23. An equilateral triangle is inscribed in a circle. The circle's circumference is expanding at 6π in/sec and the triangle maintains the contact of its corners with the circle.



Given that the area of an equilateral triangle is equal to half the apothem a times the perimeter p, find out how fast the area inside the circle but outside the triangle is expanding when the area of the circle is $64\pi\ in$. [Hint: Find p and a in terms of r.]

5.5 Multiple Choice Homework

1. The side of a cube is expanding at a constant rate of 6 inches per second. What is the rate of change of the volume, in in^3 per second, when the total surface area of the cube is 54 in^2?

a) 324 b) 108 c) 18 d) 162 e) 54

---

2. Gravel is being dumped from a conveyor belt at a rate of 35 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 15ft high?

a) 0.27 ft/min b) 1.24 ft/min c) 0.14 ft/min

d) 0.2 ft/min e) 0.6 ft/min

---

3. Two cars start moving from the same point. One travels south at 28 mi/h and the other travels west at 70 mi/h. At what rate is the distance between the cards increasing 5 hours later?

a) 75.42 mi/h b) 75.49 mi/h c) 76.4 mi/h

d) 75.39 mi/h e) 75.38 mi/h

---

4. The radius of a sphere is decreasing at a rate of 2 centimeters per second. At the instant when the radius of the sphere is 3 cm, what is the rate of change, in square centimeters per second, of the surface area of the sphere? (The surface area S of a sphere with radius r is $S = 4\pi r^2$.)

a) -108π b) -72π c) -48π d) -24π e) -16π

---

5. Water is flowing into a spherical tank with 6-foot radius at the constant rate of 30π cu ft per hour. When the water is *h* feet deep, the volume of the water in the tank is given by $V = \frac{\pi h^2}{3}(18 - h)$. What is the rate at which the depth of the water in the tank is increasing the moment when the water is 2 feet deep?

a) 0.5 ft/hour b) 1.0 ft/hour c) 1.5 ft/hour

d) 2.0 ft/hour e) 2.5 ft/hour

---

6. The radius of a circle is in increasing at a constant rate of 0.2 meters per second. What is the rate of increase in the area of the circle at the instant when the circumference of the circle is 20π meters?

a) 0.04π m^2/sec b) 0.4π m^2/sec c) 4π m^2/sec

d) 20π m^2/sec e) 100π m^2/sec

---

7. If the rate of change of a number *x* with respect to time *t*, is *x*, what is the rate of change of the reciprocal of the number when $x = -\frac{1}{4}$?

a) –16 b) –4 c) $-\frac{1}{48}$ d) $\frac{1}{48}$ e) 4

---

8. A Golden Rectangle is one where the ratio (called ϕ)of the length to the short side *w* to the long side *l* is equal to the ratio of the long side to the sum of the two sides. In other words, $l = 1.236w$. If a Golden Rectangle changes such that *w* is growing at 2 in/min, how fast is the area changing when *w* is 5 inches?

a) 1.236 in^2/min b) 12.36 in^2/min c) 2.472 in^2/min

d) 24.72 in^2/min e) 30.9 in^2/min

Derivative Applications II Practice Test Part 1

1. The position of a particle moving along a horizontal line is given by

$$x(t)=3(t-4)^3$$

What is the maximum speed of the particle for $0 \le t \le 10$?

a) 108 b) 324 c) 144 d) 576 e) 48

---

2. If the radius of a sphere is increasing at 2 in/second, how fast, in cubic inches per second, is the volume increasing when the radius is 10 inches?

a) 40π b) 80π c) 800

d) 800π e) 3200π

---

3. A model train's velocity is modeled by $v(t)=\sin(2x)+2$ feet per second. What is the train's average velocity, in feet/sec, from $t=0$ to $t=\frac{\pi}{4}$ seconds?

a) $\frac{2}{\pi}+2$ b) $\frac{1}{\pi}+2$ c) 2 d) $\frac{2}{\pi}$ e) $\frac{4}{\pi}$

---

4. A particle is moving along the *x*-axis and its position is given by $x(t)=te^{-2t}$. For what values of *t* is the particle at rest?

a) No values b) 0 only c) $\frac{1}{2}$ only d) 1 only e) 0 and $\frac{1}{2}$

---

5. Given $3x^3 - 4xy - 4y^2 = 1$, determine the change in y with respect to x:

a) $\frac{6x-4y}{4x+4}$ b) $\frac{9x^2-4}{4x+8y}$ c) $\frac{9x^2-4}{4+8y}$

d) $\frac{9x^2-4y}{4x+8y}$ e) $\frac{9x^2-4y}{4+8y}$

---

6. A particle is moving along the x-axis and its position at time $t \geq 0$ is given by $S(t) = (t-2)^2(t-6)$. Which of the following is (are) true?

I. The particle changes direction at $x = 2$ and $x = 6$.
II. The particle is slowing down on [0, 2].
III. The particle is speeding up on [2, 6].

a) I, II and III b) II and III only c) I and III only

d) II only e) I only

---

7. The population of a herd of bison in Yellowstone National Park is modeled by the function B that satisfies the logistic differential equation $\frac{dB}{dt} = 0.2B\left(1 - \frac{B}{900}\right)$, where t is time in years, and $B(0) = 120$. What is the value of B that maximizes the rate of change of B?

a) 120 b) 450 c) 900 d) 4500 e) 9000

---

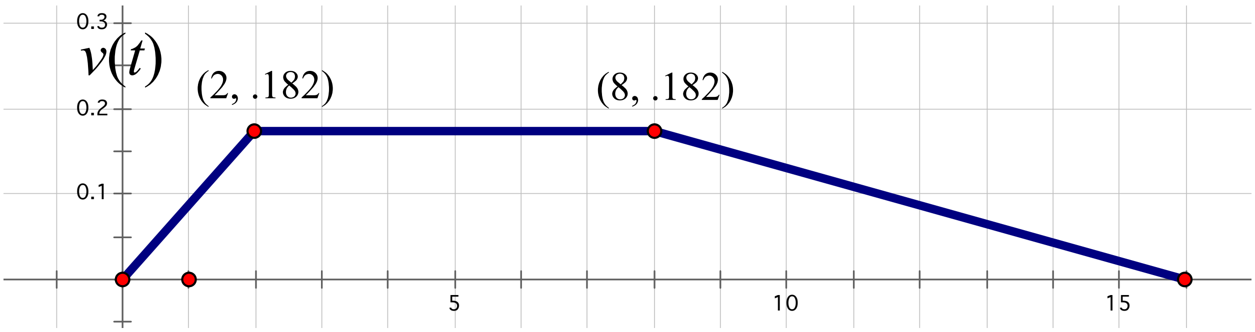
9. An infection spreads through a community modeled by the function I that satisfies the logistic differential equation $\frac{dI}{dt} = 4I\left(1 - \frac{I}{12{,}000}\right)$, where t is time in days, and $I(0) = 100$. Which of the following could be an equation for I?

a) $I = 2I^2 - \frac{I^3}{3000}$ b) $t = 2I^2\left(I - \frac{I^2}{24{,}000}\right)$ c) $I = \frac{12{,}000}{1 + 119e^{-4t}}$

d) $I = \frac{100}{1 + 3000e^{-4t}}$ e) $t = \frac{12{,}000}{1 - 4e^{119I}}$

Derivative Applications II Practice Test Part 2

1. In the 1970s, members of SI's Football Team had to complete a physical fitness test to play. The test included a two-mile run which the backs and receivers had to complete in 12 minutes and the linemen had to complete in 16 minutes. The velocity of Runner A, a lineman, was described by the graph below.



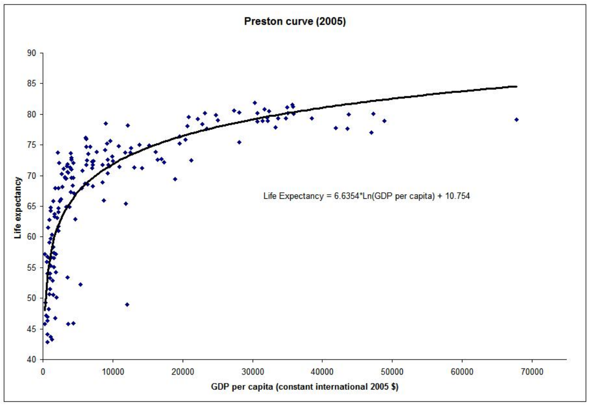
The velocity of Runner B, a receiver, was described by the equation $v(t) = \dfrac{12t - t^2}{10t^2 + 11}$.

---

a) Find the velocity of each runner at $t = 10\ min$. Indicate the units.

b) Find the acceleration of each runner at $t = 9.5\ min$. Indicate the units.

c) Find the total distance run be each runner within their time limit. Did either make their time limit? Explain your reasoning.



2. In 1975, Samuel Preston explored the correlation between life expectancy and real per capita income in the World over the previous 75 years. Show are the data and curve for 2005. The curve that best fits the data is a logarithmic growth curve, but another model might be a simple bounded exponential function. Assume that the 2022 data might be modeled by the differential equation $\frac{dL}{dt} = 0.01(83 - L)$, where L is the average life expectancy in years and i is the per capita income in hundreds of doollars. Also assume that $L(0) = 5$.

(a) If $L(200) = 72$, find the equation of the line tangent to the Preston curve.

(b) Use the tangent line equation to approximate $L(350)$. Explain why this is an overestimate.

(c) Find the particular solution to $\frac{dL}{di} = 0.01(83 - L)$ with the initial condition $L(0) = 5$.

3. Consider the curve given by $x^2+4xy+y^2=-12$.

---

a) Show that $\frac{dy}{dx}=-\frac{x+2y}{2x+y}$.

b) Find all the points where the tangent lines which are horizontal.

c) Find the value(s) of $\frac{d^2y}{dx^2}$ at the point(s) found in part (b). Does the curve have a local maximum, a local minimum, or neither at those points? Justify your answer.

Chapter 5 Answer Key

5.1 Free Response Answers

1a) $t \in \left[0, \frac{2}{3}\right] \cup \left[2, 5\right]$ 1b) 4 1c) $y(t) = t^3 - 4t^2 + 4t + 2$

2a) Always 2b) 10.843

2c) $x(t) = 100(t) + 225e^{-0.4(t)} + 0.036$

3a) $t \in (-\infty, 0]$ 3b) -12.323 3c) $t \in [-0.408, 0] \cup [0.408, 1.5]$

3d) $x(t) = -2e^{-3t^2} + 1$

4a) $x \in [0, 1.162] \cup [1.667, 1.988]$ 4b) -1.089

4c) $x(t) = \frac{1}{3}\sin t^3 + 3$

5a) $t = 2.292 \text{ and } 6.752$ 5b) $a(7.3) = 3.536$

5c) -18.957 5d) 31.831

6a) $t = 1.482, 4.898, \text{and } 7.411$ 6b) $a(3.4) = 0.314$

6c) 15.899 6d) $x = -4.477$

7a) $t = 5.160 \text{ and } 7.718$ 7b) -1.817 7c) 8.455

7d) $a(t) = \frac{1}{t+3} + e^{\frac{t}{2}+1}\left(\sin t - \frac{1}{2}\cos t\right)$; $x \in [3.664, 6.738]$

8a) $t = 2.214, 7.526, \text{and } 11.480$ 8b) 0.245

8c) 4.704 8d) 3.552

9a) $-5\ ^{ft}/_{\min^2}$ 9b) The car traveled 962 feet in these 18 minutes.

9c) Twice 9d) $54.35\ ^{ft}/_{\min}$

10a) $-0.08\ ^{mi}/_{hr^2}$ 10b) Decreasing

10c) 6.672 miles. 10d) Yes

11a) The team ran approximately 7.65 miles in these 60 minutes.

11b) The result would equal the time, on average, it would take to complete one mile.

11c) $\frac{2}{15000}\ ^{mi}/_{\min^2}$ 11d) Yes

12a) $0.45\ ^{mi}/_{\min^2}$ 12b) $1509 km$

12c) $31.438\ ^{km}/_{hr}$ 12d) $1515.415\ km$

13a) $-\frac{1}{6}\ ^{m}/_{\sec^2}$ 13b) Slowing down 13c) 13,190 meters

14a) $-23.5\ mph\ per\ hour$ 14b) 46.97 14c) 50

14d) $\frac{1}{0.55-0.33}\int_{0.33}^{0.55}\left[40-35\cos\left(\frac{\pi}{9}(t-5.5)\right)\right]dt$

15a) $t=4$ and 10 15b) 3 15c) $\frac{5\pi}{2}+\frac{7}{2}$ 15d) $\frac{17}{2}-2\pi$

16a) $t=11$ and 18 16b) $t=11$ 16c) 160 ft

16d) $a(t)=-\frac{5}{2}$, $v(t)=-\frac{5}{2}t+11$, and $x(t)=-\frac{5}{2}t^2+11t+\frac{199}{4}$

17a) $t \in [3,5]$ 17b) $-\frac{20}{3} mi/hr^3$

17c) $40 \text{mi}/\text{hr}$ 17d) $110 \text{mi}/\text{hr}$.

18a) $t = 2, 6$ and 8. 18b) $t = 0,\ 4$ and 12.

18c) $6 + \frac{5\pi}{2}$ 18d) $6 + \frac{3\pi}{2}$

19a) -1 19b) $t = 4$ and 7 19c) 8

19d) $\frac{1}{5}$; twice. 19EC. $t = 3.8$ and 6.1

20a) $-3 m/s^2$ 20b) $t = 9$

20c) $28.5m$ 20d) $t \in [2,4] \cup [6,10] \cup [13,15]$

21. See AP Central

5.1 Multiple Choice Answers

1.	B	2.	C	3.	C	4.	E	5.	A	6.	D
7.	A	8.	B	9.	D	10.	B				

5.2 Free Response Key

1. $t = 47.549\ min$ 2. $t = 23.219\ years$

3. 36,322 or 36,323 4. $t = 29.150\ years$

5a. $F = 100 - 90e^{-.4t}$ 5b. $F(1.2) \approx 70$ 5c. 1000

6a. $G = 12 - 11e^{-3.6t}$ 6b. 12

7a. $\left.\frac{dM}{dt}\right|_{(0,\,600)} = 102.4t$ 7b. $M(10) \approx 1624\ Kellertons$

7c. $M = 1000 - 400e^{-0.256t}$ 7d. 1000

8a. $3.6t$ 8b. $38lbs$ 8c. $N = 200 - 180e^{-0.02t}$

8d. Since the first derivative is positive asnd the second derivative is negative, the body's decomposition is increasing at a decreasing rate.

9a. $y = 20 + 80e^{-0.144t}$ 9b. $58.940\ C°$ 9c. $t = 9.627\min$

10a. $y = 37 + 38e^{\left(\frac{1}{20}\ln\frac{34.5}{38}\right)t} = 37 + 38e^{-0.005t}$ 10b. $0.792\ F°$

10c. $t = 29.391\min$ 10d. $2.187F°$

11. see AP Central

5.2 Multiple Choice Answers

1. C 2. A 3. D 4. D 5. C 6. B

7. B 8. A 9. C

5.3 Free Response Answers

1. Imp: $\frac{dy}{dx} = -\frac{x}{y}$; Exp: $\frac{dy}{dx} = \frac{-x}{\pm\left(1-x^2\right)^{1/2}}$

2. Imp: $\frac{dy}{dx} = \frac{-3x^2}{8y}$; Exp: $\frac{dy}{dx} = \frac{\pm 3x^2}{8\sqrt{4 - \frac{1}{4}x^3}}$

3. Imp: $\frac{dy}{dx} = \frac{-y^2}{x^2}$; Exp: $\frac{dy}{dx} = \frac{-1}{(1-x)^2}$

4. Imp: $\frac{dy}{dx} = \frac{-y^{1/2}}{x^{1/2}}$; Exp: $\frac{dy}{dx} = \frac{-4+\sqrt{x}}{\sqrt{x}}$

5. $\frac{dy}{dx} = \frac{-2x-y}{x-4}$

6. $\frac{dy}{dx} = \frac{-6x-y-2}{x}$

7. $\frac{dy}{dx} = \frac{-2x-4y}{4x-10}$

8. $\frac{dy}{dx} = -\frac{6x+y}{x-8y}$

9. $\frac{dy}{dx} = \frac{-3x^2-20xy}{10x^2+14}$

10. $\frac{dy}{dx} = \frac{-\sin y - 2xy^2}{2x^2y + x\cos y}$

11. $\frac{dy}{dx} = \tan(x)\tan(y)$

12. $\frac{dy}{dx} = \frac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$

13. $\frac{dy}{dx} = \frac{\sec(x-y)(1+x^2)^2 + 2xy}{1+x^2+\sec(x-y)(1+x^2)^2}$

14. $\frac{dy}{dx} = \frac{1}{y(x+1)^2}$

15. $\frac{dy}{dx} = (x+y)^2 - \frac{y}{x}$

16. $\frac{dy}{dx} = \frac{y-2xy^2}{2y^3+x}$

17. $\frac{dy}{dx} = \frac{1}{(x+y)^2-1}$

18. $\frac{dy}{dx} = \frac{x^2+4x+1}{2y(x+2)^2}$

19. $y = \frac{\sqrt{3}}{3}\left(x-\sqrt{3}\right)$

20. $y+2 = \frac{3}{2}x$

21. $y+3=-\frac{6}{5}(x+1)$

22. $y-2=-10/17(x-1)$

23. $y-2=-8/7(x-1)$

24. $x=2$

25. *Tangent*: $y-0=-4/\pi\left(x-\pi/2\right)$ *Normal*: $y-0=\pi/4\left(x-\pi/2\right)$

26. $\frac{2y(x+y)}{(x+2y)^3}$

27. $\frac{-36y^2-16x^2}{81y^2}$

28. $\frac{d^2y}{dx^2}=\frac{x^2-y^2}{y^3}$

29. $\frac{d^2y}{dx^2}=\frac{-3x\left(16y^2-3x^3\right)}{64y^3}$

5.3 Multiple Choice Answers

1.	D	2.	A	3.	A	4.	D	5.	C	6.	E
7.	C	8.	B	9.	A	10.	B	11.	E	12.	A
13.	D										

5.4 Free Response Key

1a. $\frac{dy}{dx}=\frac{3x-2y}{2x-5y}$

1b) $(2,-1)\left(2,\frac{13}{5}\right)$

1c) $y+1=\frac{8}{9}(x-2)$ and. $y-13/5=-\frac{4}{45}(x-2)$

1d) $\left(\frac{10}{\sqrt{33}},\frac{5}{\sqrt{33}}\right)\left(-\frac{10}{\sqrt{33}},-\frac{5}{\sqrt{33}}\right)$

2a) $\frac{dy}{dx}=\frac{y-2x}{2y-x}$

2b) $(2,\ 0),(2,\ 2)$

2c) $y=2(x-2)$ and $y-2=-2(x-2)$

2d) $\left(\frac{4}{\sqrt{3}},\frac{2}{\sqrt{3}}\right),\left(\frac{-4}{\sqrt{3}},\frac{-2}{\sqrt{3}}\right)$

3a) $\frac{dy}{dx}=\frac{4x-y}{x-2y}$

3b) $(5,\ 2),(5,\ 3)$

3c) $y-2=18(x-5)$ and $y-3=-17(x-5)$

3d) $\left(\frac{4\sqrt{11}}{\sqrt{7}},\frac{2\sqrt{11}}{\sqrt{7}}\right),\left(-\frac{4\sqrt{11}}{\sqrt{7}},-\frac{2\sqrt{11}}{\sqrt{7}}\right)$

4a) $\frac{d^2y}{dx^2}=-2y^2+(6-2x)^2\,2y^3$

4b) $\left(3,-\frac{1}{3}\right)$ is a maximum.

4c) $y=\frac{1}{x^2-6x-21}$

5a) $\frac{d^2y}{dx^2}=y\left[1+(x+1)^2\right]$

5b) $(-1,2)$ is a minimum.

5c) $y=2e^{x^2+x}$

6a) $\frac{d^2y}{dx^2}=\frac{(y+2)^2(6x)-9x^4}{(y+2)^3}$.

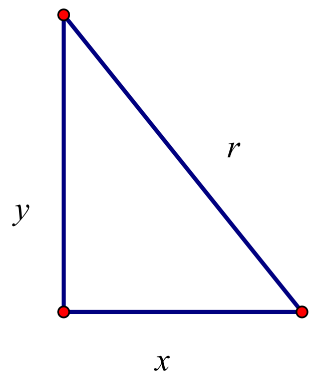
6b) $(0,1)$ is a neither

6c) $y=x^{3/2}-1$

12. See AP Central

5.5 Free Response Answers

1a.



b) $x, y, r, \frac{dx}{dt}, \frac{dy}{dt}, and\ \frac{dr}{dt}$

c) $x = 1, y = 1.4, \frac{dx}{dt} = 5, \frac{dy}{dt} = 12,$

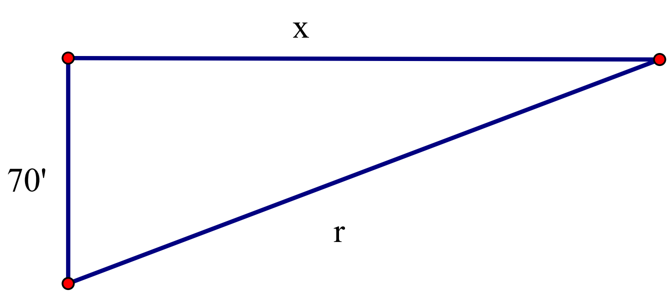
$1^2 + 1.4^2 = r^2 \rightarrow r = 1.720.$

Find $\frac{dr}{dt}$

d) $x^2 + y^2 = r^2$

e) $\frac{288}{5}\ ft/\text{sec}$

2.



b) $x, r, \frac{dx}{dt}, and\ \frac{dr}{dt}$

c) $\frac{dx}{dt} = 60$

$x\Big|_{t=4} = \left(60\frac{ft}{\text{sec}}\right)(4\,\text{sec}) = 240\,ft,$

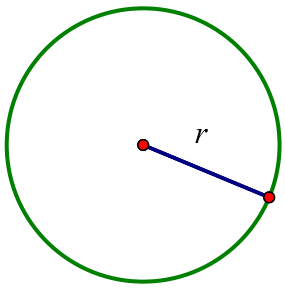
$r = 250.$

Find $\frac{dr}{dt}$

d) $x^2 + 70^2 = r^2$

e) 12.674

3a.



b) $A, d, r, \frac{dA}{dt}, and\ \frac{dr}{dt}$

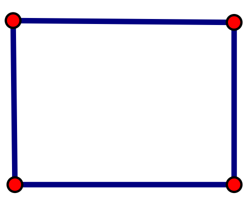
c) $\frac{dr}{dt} = \frac{1}{2}\ \text{in}/\text{min},\ d = 1\ \text{in} \to (r = \frac{1}{2}\ \text{in})$

Find $\frac{dA}{dt}$

d) $A = \pi r^2$

e) $\frac{\pi}{2}\ \text{in}^2/\text{min}$

4a.



b) $l, s, p, A, \frac{dp}{dt}, and\ \frac{dA}{dt}$

c) $\frac{dp}{dt} = .25\ \text{cm}/\text{sec},\ l = 3 \to s = 2$

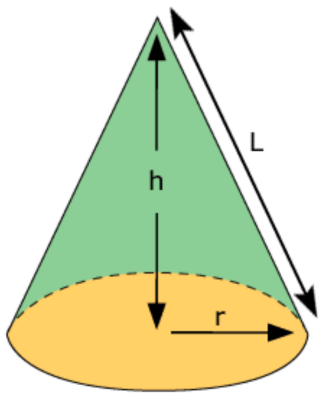
Find $\frac{dA}{dt}$

d) $l = 1.5s$,
$p = 2l + 2s = 2(1.5s) + 2s = 5s,\ A = ls$

e) $1.25\ \text{cm}^2/\text{sec}$

5a.

b) $Volume, diameter, height, radius, \frac{dV}{dt}, and\ \frac{dh}{dt}$



c) $\frac{dV}{dt} = 30\pi \ {ft^3}/{\min}, \ h = 5$

$h = 5 \rightarrow r = 2.5$

Find $\frac{dh}{dt}$

d) $\frac{1}{2}h = r; \ V = \frac{\pi}{3}r^2h = \frac{\pi}{12}h^3;$

$V = \frac{\pi}{3}r^2h = \frac{\pi}{12}h^3$

e) $\frac{6}{5\pi}$

6a) $V, h, r, \frac{dV}{dt}, and \ \frac{dh}{dt}$

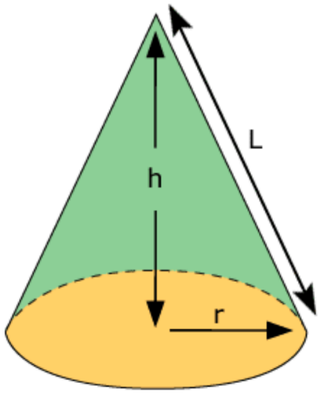
b) $\frac{dV}{dt} = -300 \ {ft^3}/{\min}, \ h = 30, r = 10$. Find $\frac{dh}{dt}$

c) $V = 100\pi h$

d) $-\frac{1}{\pi} \ {ft}/{\min}$

7a)

b) $h, d, r, V, \frac{dh}{dt}, \frac{dr}{dt}, \frac{dV}{dt}$



c) $\frac{dV}{dt} = -5000\ \text{cm}^3/\text{min};\ h = 8;\ d = 4$

$d = 4 \rightarrow r = 2$

Find $\frac{dh}{dt}$ and $\frac{dr}{dt}$

d) $V = \frac{\pi}{3} r^2 h,\ d = 2r$

e) $-\frac{1250}{\pi}\ \text{m}^3/\text{min}$

f) $\frac{dr}{dt} = -\frac{625}{2\pi}\ \text{m}^3/\text{min}$

8. $\frac{6}{5\pi}$

9. $\frac{3}{200\pi}\ \text{ft}/\text{min}$

10 $162\ \text{in}^3/\text{sec}$

11. $-\frac{120}{7}\ ft/\sec$

12. $-478.754\,mph$

13. $-\frac{55}{6}$

14. $65\ mi/h$

15. $-80\ {in^3}/{\text{min}}$

16. $-43.393\ {in^3}/{\text{sec}}$

17a. Person A: $120\,ft$; Person B: $50\,ft$ 17b. $130\,ft$

17c. $-7.308\ {ft}/{\text{sec}}$ 17d. $0.065\ {rad}/{sec}$

18a. $12.207\,ft$ 18b. 0.610 18c. $0.960\ rads$ 18d. $0.029\ {rads}/{\text{min}}$

19a. $300.666\,ft$ 19b. $\frac{dr}{dt} = 19.825\ {ft}/{\text{sec}}$

19c. 0.142 19d. $-0.015\ {rad}/{sec}$

20a. $t_{SP} = 0.38hrs$ 20b. $t_{BA} = 0.375hrs$. Big Alice gets there first

20c. $14.022\ {km}/{hr}$

21a. $189{,}273.629\ sq.ft.$ 21b. $284.758\,ft$

21c. $-5.856\ {\text{ft}}/{\text{hr}}$ 21d. $-3995.027\ \text{ft}^2$

22a. πx 22b. $4\pi - 8$

23. $143.002\ {in^2}/{\text{sec}}$

5.5 Multiple Choice Key:

1.	D	2.	D	3.	D	4.	C	5.	E	6.	C
7.	E	8.	D								

Derivative Applications II Practice Test Answer Key

1.	D	2.	E	3.	C	4.	C	5.	B	6.	D
7.	B	8.	D	9.	D	10.	B	11.	C		

1a. $t \in [0, 1.465] \cup [1.845, 2]$

1b. $a(t) = 3t^4 \cos t^3 + 2t \sin t^3$

1c. $y(t) = -\frac{1}{3}\cos t^3 + \frac{10}{3}$

1d. $y(1.465) = \frac{11}{3}$

2a. $V = \frac{4\pi}{27}h^3$

2b. $\frac{dh}{dt} = 0.458$

2c. 46

3a. $\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right] \to 2x + 4x\frac{dy}{dx} + y(4) + 2y\frac{dy}{dx} = 0$

$(4x + 2y)\frac{dy}{dx} = -2x - 4y \to \frac{dy}{dx} = -\frac{x + 2y}{2x + y}$

3b. $(4, -2)$ *and* $(-4, 2)$

3c. $(4, -2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

$(-4, 2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$

Derivative Applications II Practice Test Answer Key

1.	D	2.	E	3.	C	4.	C	5.	B	6.	D

7. B 8. D 9. D 10. B 11. C

1a. $t \in [0, 1.465] \cup [1.845, 2]$

1b. $a(t) = 3t^4 \cos t^3 + 2t \sin t^3$

1c. $y(t) = -\frac{1}{3}\cos t^3 + \frac{10}{3}$

1d. $y(1.465) = \frac{11}{3}$

2a. $V = \frac{4\pi}{27}h^3$

2b. $\frac{dh}{dt} = 0.458$

2c. 46

3a. $\frac{d}{dx}\left[x^2 + 4xy + y^2 = -12\right] \rightarrow 2x + 4x\frac{dy}{dx} + y(4) + 2y\frac{dy}{dx} = 0$

$(4x + 2y)\frac{dy}{dx} = -2x - 4y \rightarrow \frac{dy}{dx} = -\frac{x + 2y}{2x + y}$

3b. $(4, -2)$ *and* $(-4, 2)$

3c. $(4, -2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

$(-4, 2)$ is at a minimum because $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$