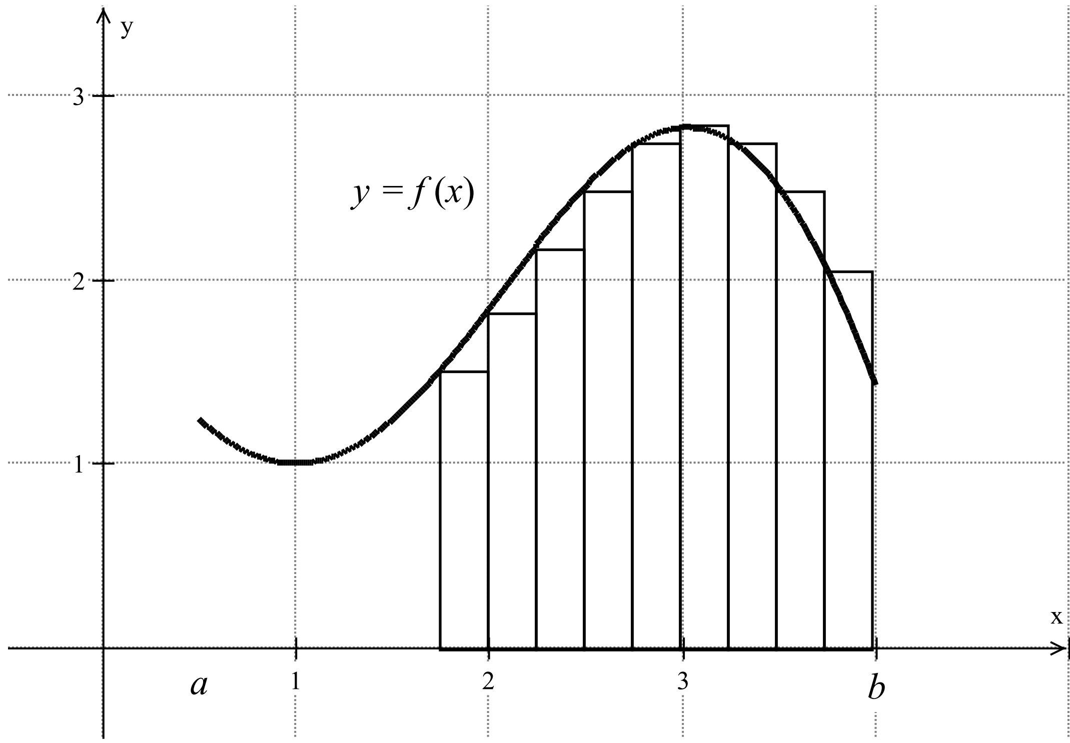
Chapter 3 Overview: Definite Integrals

In this chapter, we will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on the mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

As noted in the overview, Anti-derivatives are known as Indefinite Integrals and this is because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an Analytic Geometry context of area. Though it is not necessary to know the theory behind this to be able to do it, the theory is a major subject of Integral Calculus, so we will explore it briefly.

We know, from Geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider a figure where one side is the curve $y = f(x)$ and the other sides are the x-axis and the lines $x = a$ and $x = b$.



As we can see above, the area can be approximated by rectangles whose height is the *y*-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. The area of each rectangle would be $f(x)\cdot\Delta x$ and the total area of *n* rectangles would be $A=\sum_{i=1}^{n} f(x_i)\cdot\Delta x$. If we could make an infinite number of rectangles (which would be infinitely thin), we would have the exact area. The rectangles can be drawn several ways--with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But once they become infinitely thin, it will not matter how they were drawn--they will have virtually no width (represented by *dx* instead of the Δx) and a height equal to the *y*-value of the curve.

We can make an infinite number of rectangles mathematically by taking the Limit as *n* approaches infinity, or

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\cdot\Delta x.$$

This limit is rewritten as the Definite Integral:

$$\int_a^b f(x)\ dx$$

b is the "upper bound" and *a* is the "lower bound," and would not mean much if it were not for the following rule:

3.1 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

If $f(x)$ is a continuous function on $[a,b]$, then

1) $\frac{d}{dx}\int_c^x f(t)\ dt = f(x)$ or $\frac{d}{dx}\int_c^u f(t)\ dt = f(u) \bullet D_u$

2) If $F'(x) = f(x)$, then $\int_a^b f(x)\ dx = F(b) - F(a)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know--that an integral is an anti-derivative. The second part of the Fundamental Theorem says the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

This idea of the integral meaning the area may not make sense initially, mainly because we are used to Geometry, where area is always measured in square units. But that is only because the length and width are always measured in the same kind of units, so multiplying length and with must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the *x*-axis is time in seconds and the *y*-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet/sec--that is, feet. So the area under the curve equals the distance traveled in feet. In other words, the integral of velocity is distance.

Objectives

Evaluate Definite Integrals
Find the average value of a continuous function over a given interval
Differentiate integral expressions with the variable in the boundary

Let us first consider Part 2 of the Fundamental Theorem, since it has a very practical application. This part of the Fundamental Theorem gives us a method for evaluating definite integrals.

Ex 1 Evaluate $\int_2^8 (4x+3)\,dx$

$\int_2^8 (4x+3)\,dx = 2x^2+3x\Big|_2^8$ — The antiderivative of $4x + 3$ is $2x^2 + 3x$. We use this notation when we apply the Fundamental Theorem

$=\left[(128+24)-(8+6)\right]$ — Plug the upper limit of integration into the antiderivative and subtract it from the lower limit when plugged into the antiderivative

$= 138$ — Check with Math 9.

Ex 2 Evaluate $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

$$\int_1^4 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^4$$
$$=\left[(4)-(2)\right]$$
$$=2$$

Ex 3 Evaluate $\int_0^{\pi/2} \sin x dx$

$$\int_0^{\pi/2} \sin x dx = -\cos x\Big|_0^{\pi/2}$$
$$=-\left[0-1\right]$$
$$=1$$

Ex 4 Evaluate $\int_1^2 \frac{4+u^2}{u^3}\,du$

$$\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 \left(4u^{-3}+u^{-1}\right)du$$
$$= -2u^{-2}+\ln|u|\Big|_1^2$$
$$= \left[\left(-2\cdot\frac{1}{2^2}+\ln 2\right)-\left(-2+\ln 1\right)\right]$$
$$= \frac{3}{2}+\ln 2$$

Ex 5 Evaluate $\int_{-5}^{5} \frac{1}{x^3}\,dx$

This is a trick! We use the Fundamental Theorem on this integral because the curve is not continuous on $x\in[a,b]$. When $x = 0$, $\frac{1}{x^3}$ does not exist, so the Fundamental Theorem of Calculus does not apply.

One simple application of the definite integral is the Average Value Theorem. We all recall how to find the average of a finite set of numbers, namely, the total of the numbers divided by how many numbers there are. But what does it mean to take the average value of a continuous function? Let's say you drive from home to school – what was your average velocity (velocity is continuous)? What was the average temperature today (temperature is continuous)? What was your average height for the first 15 years of your life (height is continuous)? Since the integral is the sum of infinite number of function values, the formula below answers those questions.

Properties of Definite Integrals

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a < c < b$

Ex 6 If $\int_{-5}^{2} f(x)\ dx = -17$ and $\int_{5}^{2} f(x)\ dx = -4$, then $\int_{-5}^{5} f(x)\ dx =$

a) -21 b) -13 c) 0 d) 13 e) 21

$$\int_{-5}^{5} f(x)\ dx = \int_{-5}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx$$

$$= \int_{-5}^{2} f(x)\,dx - \int_{5}^{2} f(x)\,dx$$

$$= -17 - (-4)$$

$$= -13$$

The correct answer is b.

3.1 Free Response Homework

Use Part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

1. $\int_{-1}^{3} x^5 dx$

2. $\int_{2}^{7} (5x-1) dx$

3. $\int_{-5}^{5} \frac{2}{x^3} dx$

4. $\int_{-3}^{-1} \left(\frac{x^7 - 4x^3 - 3}{x} \right) dx$

5. $\int_{1}^{2} \frac{3}{t^4} dt$

6. $\int_{\pi/4}^{3\pi/4} \csc y \cot y \, dy$

7. $\int_{0}^{\pi/4} \sec^2 y \, dy$

8. $\int_{1}^{9} \frac{3}{2z} dz$

9. $\int_{1}^{8} \left(\frac{x^2 - 4}{\sqrt[3]{x}} \right) dx$

10. $\int_{\pi}^{\frac{5\pi}{4}} \sin y \, dy$

11. $\int_{1}^{4} \left(\frac{x^4 - 4x^2 - 5}{x^2} \right) dx$

12. $\int_{3}^{5} (x^2 + 5x + 6) dx$

13. $\int_{\pi}^{\frac{3\pi}{4}} \cos y \, dy$.

14. $\int_{1}^{4} \left(\frac{x^3 - 2x^2 - 4x}{x^2} \right) dx$

15. $\int_{1}^{2} \left(\frac{x^2 - 4x + 7}{x} \right) dx$

16. $\int_{1}^{16} \left(\frac{2x^2 - 1}{\sqrt[4]{x}} \right) dx$

Use the following values for problems 17 – 27 to evaluate the given integrals

$\int_{-2}^{5} f(x)dx = -2$	$\int_{1}^{5} f(x)dx = 3$
$\int_{-2}^{1} g(x)dx = 4$	$\int_{5}^{1} g(x)dx = 9$
$\int_{1}^{5} h(x)dx = 7$	$\int_{5}^{-2} h(x)dx = -6$

17. $\int_{-2}^{1} f(x)dx =$

18. $\int_{-2}^{5} g(x)dx =$

19. $\int_{-2}^{1} h(x)dx =$

20. $\int_{1}^{5} [f(x) - g(x)]dx =$

21. $\int_{-2}^{5} [g(x) + h(x)]dx =$

22. $\int_{-2}^{1} [h(x) - f(x)]dx =$

23. $\int_{-2}^{5} [h(x) + f(x)]dx =$

24. $\int_{1}^{5} [2f(x) + 3h(x)]dx =$

25. $\int_{-2}^{1} [2f(x) - 3g(x)]dx =$

26. $\int_{-2}^{5} \left[\frac{1}{2}g(x) + 4h(x)\right]dx =$

27. $\int_{1}^{5} \left[\frac{1}{3}h(x) + 2f(x)\right]dx =$

28. $\int_0^2 f(x)\,dx$ where $f(x) = \begin{cases} x^4, & if\ 0 \le x < 1 \\ x^5, & if\ 1 \le x \le 2 \end{cases}$

Use Part I of the Fundamental Theorem of Calculus to find the derivative of the function.

29. $g(y)=\int_2^y t^2 \sin t\,dt$

30. $g(x)=\int_0^x \sqrt{1+2t}\,dt$

31. $F(x)=\int_x^2 \cos(t^2)\,dt$

32. $h(x)=\int_2^{1/x} \arctan t\,dt$

33. $y=\int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt$

34. If $F(x)=\int_1^x f(t)\,dt$, where $f(t)=\int_1^{t^2} \frac{\sqrt{1+u^4}}{u}\,du$, find $F''(2)$.

35. $\frac{d}{dx}\left[\int_e^{x^2} \ln\left(t^2+1\right)dt\right]$

36. If $h(x)=\int_\pi^{\sqrt{x}} e^{5t}\,dt$, find $h'(x)$

37. $\frac{d}{dx}\int_{10}^{x^2} t\ln(t)\,dt$

38. $h(m)=\int_5^{\cos m} t^2\cos^{-1}(t)\,dt$, find $h'(m)$

39. $h(y)=\int_5^{\ln y} \frac{e^t}{t^4}\,dt$, find $h'(y)$

40. $\frac{d}{dx}\int_{e^x}^{5} \left(t^3+t+1\right)dt$

3.1 Multiple Choice Homework

1. If $\int_{-5}^{2} f(x)\,dx = -17$ and $\int_{5}^{2} f(x)\,dx = -4$, then $\int_{-5}^{5} f(x)\,dx =$

a) -21 b) -13 c) 0 d) 13 e) 21

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2. Let f and g be continuous functions such that $\int_0^6 f(x)\,dx = 9$, $\int_3^6 f(x)\,dx = 5$, and $\int_3^0 g(x)\,dx = -7$. What is the value of $\int_0^3 \left(\frac{1}{2}f(x) - 3g(x)\right) dx =$

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

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3. Given that $\int_2^3 P(t)\,dt = 7$ and $\int_2^7 P(t)\,dt = -2$, what is $\int_7^3 P(t)\,dt =$

a) -9 b) -5 c) 5 d) 9

e) not enough information

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4. If $\int_0^6 f(x)\,dx = 9$, $\int_3^6 f(x)\,dx = 5$, and $\int_3^0 g(x)\,dx = -7$ then $\int_0^3 \left[\frac{1}{2}f(x) - 3g(x)\right] dx =$

a) -23 b) -19 c) $-\frac{17}{2}$ d) 19 e) 23

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5. If $\int_0^6 f(x)\,dx = 9$, $\int_6^3 f(x)\,dx = 5$, and $\int_3^0 g(x)\,dx = 7$, then $\int_0^3 \left[\frac{1}{2} f(x) - 3g(x)\right] dx =$

a) -28 b) -18 c) 0 d) 18 e) 28

$\int_{-2}^{5} f(x)dx = -2$	$\int_{1}^{5} f(x)dx = 3$
$\int_{-2}^{1} g(x)dx = -4$	$\int_{5}^{1} g(x)dx = 9$

6. Based on the information above, $\int_{1}^{-2} \left[g(x) + f(x)\right] dx =$

a) -9 b) -1 c) 0 d) 1 e) 9

$\int_{-2}^{5} f(x)dx = -2$	$\int_{1}^{5} f(x)dx = 3$
$\int_{-2}^{1} g(x)dx = 4$	$\int_{5}^{1} g(x)dx = 9$

7. Based on the information above, $\int_{5}^{-2}[g(x)-f(x)]dx =$

a) -3 b) 3 c) 6 d) -6 e) 14

$\int_{-2}^{5} f(x)dx = -2$	$\int_{1}^{5} f(x)dx = 3$
$\int_{-2}^{1} g(x)dx = 4$	$\int_{5}^{1} g(x)dx = 9$

8. Based on the information above, which of the following cannot be determined?

a) $\int_{5}^{1}[g(x)+f(x)]dx$

b) $\int_{1}^{-2}[g(x)-f(x)]dx$

c) $\int_{-2}^{5}[3g(x)][-4f(x)]dx$

d) $\int_{1}^{5}[3g(x)+4f(x)]dx$

3.2 Definite Integrals and The Substitution Rule

Let's revisit u – subs with definite integrals and pick up a couple of more properties for the definite integral.

Objectives
Evaluate definite integrals using the Fundamental Theorem of Calculus. Evaluate definite integrals applying the Substitution Rule, when appropriate. Use proper notation when evaluating these integrals.

Ex 1 Evaluate $\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt$.

$$\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt = \frac{1}{3}\int_0^2 \left(3t^2\sqrt{t^3+1}\right)dt$$

Let $u = t^3+1$ $du = 3t^2 dt$	$u(0)=1$ $u(2)=9$

$$= \frac{1}{3}\int_1^9 \left(\sqrt{u}\right)du$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^9$$

$$= \frac{2}{9}\left[9^{3/2} - 1^{3/2}\right]$$

$$= \frac{52}{9}$$

Ex 2 Evaluate $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx$.

$$\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx \quad = \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \cos u \; du$$

Let $u = x^2$ $du = 2xdx$	$u(0)=0$ $u\left(\sqrt{\pi}\right)=\pi$

$$= \frac{1}{2}\sin u\Big|_0^{\pi}$$

$$=0$$

Ex 3 $\int_1^2 \frac{e^{1/x}}{x^2}dx$

$$\int_1^2 \frac{e^{1/x}}{x^2}dx \quad = -\int_1^{1/2} e^u du$$

Let $u = 1/x$ $du = -1/x^2\,dx$

$u(1) = 1$ $u(2) = 1/2$

$$= -e^u\Big|_1^{1/2}$$

$$= -e^{1/2} + e$$

Ex 4 $\int_1^{\sqrt{13}} \frac{x}{x^2+3}dx =$

$$u = x^2 + 3 \; u(1) = 4$$

$$du = 2xdx \, u\left(\sqrt{13}\right) = 16$$

$$\int_1^{\sqrt{13}} \frac{x}{x^2+3}dx = \frac{1}{2}\int_1^{\sqrt{13}} \frac{2x}{x^2+3}dx$$

$$= \frac{1}{2}\int_4^{16} \frac{1}{u}du$$

$$= \frac{1}{2}\ln|u|_4^{16}$$

$$= \frac{1}{2}\ln 16 - \frac{1}{2}\ln 4$$

As a multiple-choice question, this would not be one of the answers. The expectation is that you know the Log Rules:

Rules of Logarithms:

1. $\log_a x + \log_a y = \log_a (xy)$
2. $\log_a x - \log_a y = \log_a \dfrac{x}{y}$
3. $\log_a x^n = n \log_a x$

So,

$$\frac{1}{2}\ln 16 - \frac{1}{2}\ln 4 = \frac{1}{2}\ln\frac{16}{4} = \frac{1}{2}\ln 4 = \ln\sqrt{4} = \ln 2$$

Average Value Formula:

The average value of a function *f* on a closed interval $\left[a,b\right]$ is defined by

$$f_{avg} = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it can start to make a little more sense.

$\int_a^b f(x)\ dx = F(b) - F(a)$ The Fundamental Theorem of Calculus

$$\frac{1}{b-a}\int_a^b f(x)\ dx = \frac{1}{b-a}\left[F(b) - F(a)\right]$$
$$= \frac{F(b) - F(a)}{b-a}$$

Notice that this is just the average slope for $F(x)$ on $x \in [a,b]$. The average slope of $F(x)$ would be the average value of $F'(x)$. But since the definition in the Fundamental Theorem of Calculus says that $F'(x) = f(x)$, this is actually just the average value of $f(x)$.

Ex 5 Find the average value of $f(x) = x^2 + 1$ on $[0,5]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$f_{avg} = \frac{1}{5-0}\int_0^5 \left(x^2+1\right)dx$$ Substitute in the function and interval

$$f_{avg} = {}^{28}\!/_{3}$$ Use Math 9 (or integrate analytically) to calculate answer

Ex 6 Find the average value of $h(\theta) = \sec\theta\tan\theta$ on $\left[0, \frac{\pi}{4}\right]$.

$$\begin{aligned} h_{avg} &= \frac{1}{b-a}\int_a^b h(\theta)\,d\theta \\ &= \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta \\ &= .524 \end{aligned}$$

3.2 Free Response Homework

Evaluate the definite integral, if it exists.

1. $\int_0^1 x^2\left(1+2x^3\right)^5 dx$

2. $\int_0^1 x^3\left(x^4+5\right)^3 dx$

3. $\int_{-1}^1 x\sqrt{4-x^2}\, dx$

4. $\int_1^2 \frac{x^2}{\sqrt[3]{9-x^3}} dx$

5. $\int_0^3 \frac{10t+15}{\sqrt[4]{t^2+3t+1}} dt$

6. $\int_1^2 \frac{x+1}{\sqrt{x^2+2x+3}} dx$

7. $\int_1^3 \frac{5t}{t^2+1} dt$

8. $\int_{-1}^2 \frac{dx}{2x+5}$

12. $\int_3^{e^2+2} \frac{1}{x-2} dx$

14. $\int_5^{e^3+4} \frac{1}{x-4} dx$

$\int_0^{e^2-1} \frac{1}{x+1} dx$

9. $\int_{\sqrt{3}}^{\sqrt{4}} ye^{y^2-3} dy$

10. $\int_0^1 \frac{v^2}{8-v^3} dv$

11. $\int_3^{e^2+2} \frac{1}{x-2} dx$

12. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta} d\theta$

13. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

14. $\int_0^{\pi} \sec^2\left(\frac{t}{4}\right) dt$

15. $\int_0^{\pi} \frac{\sin x}{2-\cos x} dx$

16. $\int_2^4 \frac{dx}{x\ln x}$

17. $\int_0^{Ln2} \frac{e^x}{1+e^{2x}} dx$

18. $\int_{\pi/6}^{\pi/2} \cos^5 x \sin x dx$

19. $\int_0^{\frac{\pi}{8}} \sec^2(2x) dx$

20. $\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y} dy$

21. $\int_0^{\pi} \frac{\cos x}{2+\sin x} dx$

22. $\int_0^{\pi} \frac{\sin y}{2+\cos y} dy$

23. $\int_0^{\sqrt{\frac{\pi}{4}}} m\sec(m^2)\tan(m^2) dm$

24. $\int_0^{\frac{\pi}{4}} \sec^2 x \tan^3 x dx$

25. $\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x) dx$

26. $\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx$

27. $\int_{\pi}^{2\pi} \cos\frac{1}{2}\theta \, d\theta$

28. $\int_2^{e^3+1} \frac{(\ln(x-1))^4}{x-1} dx$

29. $\int_0^{e^2-1} \frac{1}{x+1} dx$

30. $\int_5^{e^3+4} \frac{1}{x-4} dx$

Find the Average Value of each of the following functions.

31. $F(x) = (x-3)^2$ on $x \in [3, 7]$

32. $H(x)=\sqrt{x}$ on $x\in[0, 3]$

33. $F(x)=\sec^2 x$ on $x\in\left[0, \frac{\pi}{4}\right]$

34. $F(x)=\frac{1}{x}$ on $x\in[1, 3]$

35. $f(t)=t^2-\sqrt{t}+5$ on $t\in[1,4]$

36. $f(t)=t^2-\sqrt{t}+5$ on $t\in[4,9]$

37. $f(x)=\cos x\sin^4 x$ on $x\in[0, \pi]$

38. $g(x)=xe^{-x^2}$ on $x\in[1, 5]$

39. $G(x)=\frac{x}{(1+x^2)^3}$ on $x\in[0, 2]$

40. $h(x)=\frac{x}{(1+x^2)^2}$ on $x\in[0, 4]$

41. If a cookie taken out of a 450˚F oven cools in a 60˚F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t)=60+390e^{-.205t}.$$

What is the average value of the cookie during its first 10 minutes out of the oven?

42. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the given equation

$$L(t)=10-3\cos\left(\frac{\pi t}{182}\right),$$

where t the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

43. During one summer in the Sunset, the temperature is modeled by the function $T(t)=50+15\sin\frac{\pi}{12}t$, where T is measured in F° and t is measured in hours after 7 am. What is the average temperature in the Sunset during the six-hour Chemistry class that runs from 9 am to 3 pm?

<u>3.2 Multiple Choice Homework</u>

1. $\displaystyle\int_1^4 \frac{dx}{\left(1+\sqrt{x}\right)^2\sqrt{x}}$

a) $\frac{6}{5}$ b) $\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{4}{9}$ e) $\frac{3}{2}$

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2. If $\int_1^4 h(x)\,dx = 6$, then $\int_1^4 h(5-x)\,dx =$

a) -6 b) -1 c) 0 d) 3 e) 6

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3. $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}dx =$

a) ln(ln(2)) b) $\frac{2}{e^2}$ c) $\ln 2$ d) $\frac{1-2e}{2e^2}$ e) DNE

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4. The average value of $y = e^{6x}$ on $x \in [0, 4]$ is

a) $\frac{e^{24}-1}{4}$ b) $\frac{e^{24}-1}{6}$ c) $\frac{e^{24}}{24}$ d) $\frac{e^{24}}{6}$ e) $\frac{e^{24}-1}{24}$

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5. $\displaystyle\int\left(2-\sin\frac{t}{5}\right)^2\cos\frac{t}{5}\,dt$

a) $-\frac{5}{3}\left(2-\sin\frac{t}{5}\right)^3+c$ b) $\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3+c$

c) $\frac{1}{3}\left(2-\sin\frac{t}{5}\right)^3+c$ d) $5\left(2-\sin\frac{t}{5}\right)^3+c$

e) $-\frac{5}{3}\left(2-\cos\frac{t}{5}\right)^3+c$

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6. The average value of $g(x)=(2x+3)^2$ on $x\in[-3,-1]$ is

a) $\frac{7}{3}$ b) -4 c) 5 d) $\frac{14}{3}$ e) 3

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7. The average value of $g(x)=e^{7x}$ on $x\in[0,2]$ is

a) $\frac{1}{14}e^{14}$ b) $\frac{1}{7}\left(e^{14}-1\right)$ c) $\frac{1}{14}\left(e^{14}-1\right)$

d) $\frac{1}{2}\left(e^{14}-1\right)$ e) $\frac{1}{7}e^{14}$

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8. If the function $y=x^3$ has an average value of 9 on $x\in[0,k]$, then $k=$

a) 3 b) $\sqrt{3}$ c) $\sqrt[3]{18}$ d) $\sqrt[4]{36}$ e) $\sqrt[3]{36}$

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9. Find the average rate of change of $y=x^2+5x+14$ on $x\in[-1,2]$

a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

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10. If the average of the function $f(x) = |x - a|$ on $[-1, 1]$ is $\frac{5}{4}$, what is/are the values of a?

a) ± 1 b) $\pm\frac{1}{2}$ c) $\pm\frac{1}{4}$ d) 0 e) None of these

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11. What is the average rate of change of the function $f(x) = x^4 - 5x$ on the closed interval [0, 3] ?

a) 8.5 b) 8.7 c) 22 d) 33 e) 66

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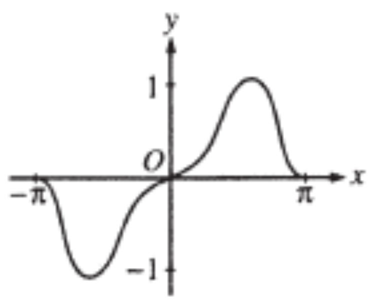
12. The average value of $y = e^x \cos x$ on $x \in \left[0, \frac{\pi}{2}\right]$ is

a) 0 b) 1.213 c) 1.905 d) 2.425 e) 3.810

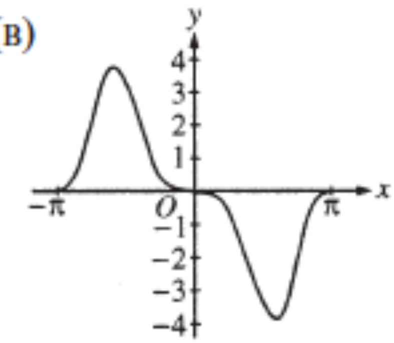
---

13. The graphs of five functions are shown below. Which function has a nonzero average value over the closed interval $x \in [-\pi, \pi]$?

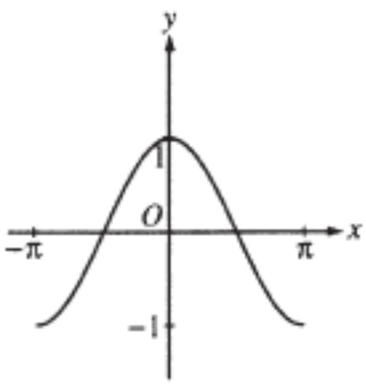
(A)



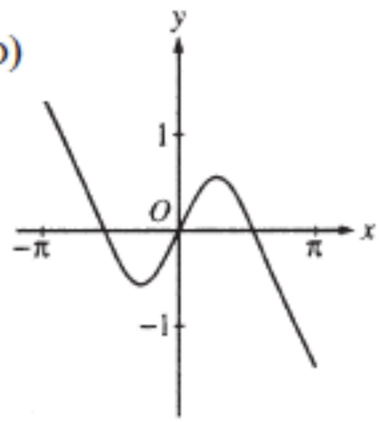
(B)



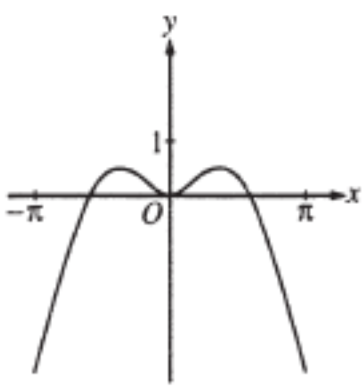
(C)



(D)



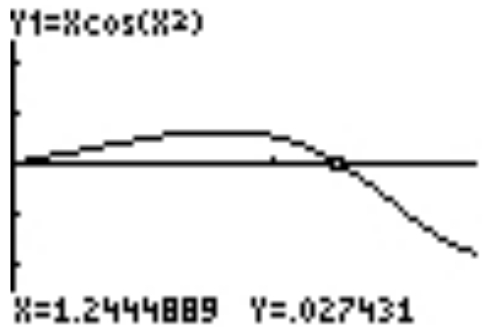
(E)



3.3 Context for Definite Integrals: Area, Displacement, and Net Change

Since we originally defined the definite integrals in terms of "area under a curve," we need to consider what this idea of "area" really means in relation to the definite integral.

Let's say that we have a function, $y = \left(x\cos\left(x^2\right)\right)$ on $x \in \left[0, \sqrt{\pi}\right]$. The graph looks like this:



In the last section, we saw that $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)\, dx = 0$. But we can see there is area under the curve, so how can the integral equal the area and equal 0? Remember that the integral was created from rectangles with width *dx* and height *f(x)*. So the area below the x-axis would be negative, because the *f(x)*-values are negative.

Ex 1 What is the area under $y = \left(x\cos\left(x^2\right)\right)$ on $x \in \left[0, \sqrt{\pi}\right]$?

We already know that $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)\, dx = 0$, so this integral cannot represent the area. As with example 1b, we really are looking for the positive number that represents the area (total distance), not the difference between the positive and negative "areas" (displacement). The commonly accepted context for area is a positive value. So,

$$\begin{aligned} Area &= \int_0^{\sqrt{\pi}} \left| x\cos\left(x^2\right) \right| \, dx \\ &= \int_0^{1.244} x\cos\left(x^2\right) \, dx - \int_{1.244}^{\sqrt{\pi}} x\cos\left(x^2\right) \, dx \\ &= 1 \end{aligned}$$

We could have used our calculator to find this answer.

Objectives:

Relate definite integrals to area under a curve.
Understand the difference between displacement and total distance.
Extend that idea to understanding the difference between the two concepts in other contexts.

When we use the phrase "area under the curve, we really mean the area between the curve and the *x*-axis. CONTEXT IS EVERYTHING. The area under the curve is only equal to the definite integral when the curve is completely above the *x*-axis. When the curve goes below the x-axis, the definite integral is negative, but the area, by definition, is positive.

Remember:

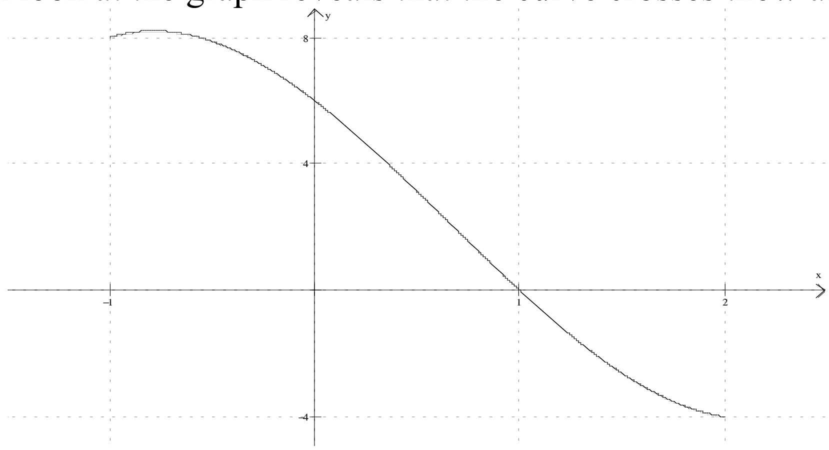
1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

Ex 2 Find the area under $y = x^3 - 2x^2 - 5x + 6$ on $x \in [-1, 2]$.

A quick look at the graph reveals that the curve crosses the *x*-axis at *x*=1.



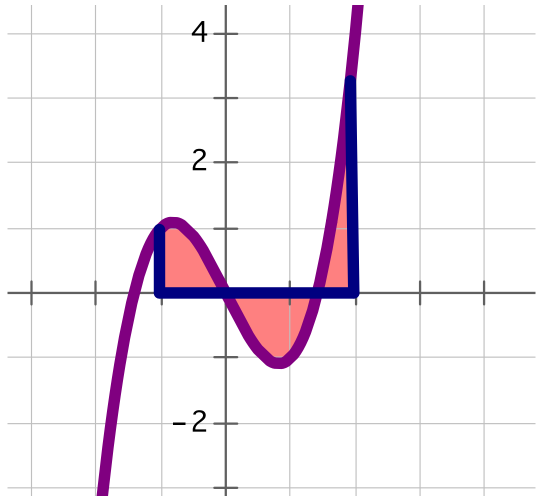
If we integrate y on $x \in [-1, 2]$, we will get the **difference** between the areas, not the sum. To get the total area, we need to set up two integrals:

$$\begin{aligned}
Area &= \int_{-1}^{1} \left(x^3 - 2x^2 - 5x + 6\right)\, dx + \left(-\int_{1}^{2} \left(x^3 - 2x^2 - 5x + 6\right)\, dx\right) \\
&= \left.\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{-1}^{1} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{1}^{2} \\
&= \frac{32}{3} + \frac{29}{12} \\
&= \frac{157}{12}
\end{aligned}$$

<u>Steps to Finding Total Area:</u>

1. Draw the function.
2. Find the zeros between $x = a$ and $x = b$.
3. Set up separate integrals representing the areas above and below the *x*-axis.
4. Change the sign on those integrals which represent the negative values (i.e., those where the curve is below the *x*-axis).
5. Solve the integral expression.

Ex 3 Find the area under $y = x^3 - 2x$ on $x \in [-1, 2]$.



Note that there are three regions here, therefore there will be three integrals:

$$Area = \int_{-1}^{0} \left(x^3 - 2x\right)\,dx + \left(-\int_{0}^{\sqrt{2}} \left(x^3 - 2x\right)\,dx\right) + \int_{\sqrt{2}}^{2} \left(x^3 - 2x\right)\,dx$$

$$= \frac{x^4}{4} - x^2\Bigg]_{-1}^{0} - \left[\frac{x^4}{4} - x^2\right]_{0}^{\sqrt{2}} + \left[\frac{x^4}{4} - x^2\right]_{\sqrt{2}}^{2}$$

$$= \frac{3}{4} - (-1) + 1$$

$$= \frac{11}{4}$$

F could be a function that describes anything – volume, weight, time, height, temperature. F' represents its rate of change. The left side of that equation accumulates the rate of change of *F* from *a* to *b* and the right side of the equation says that accumulation is difference in *F* from *a* to *b*.

Imagine if you were leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your calculus homework (gasp!). You head back home, pick up your assignment, and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? You are six miles from where you started but you have traveled 12 miles. These are the two different ideas behind displacement and total distance.

Vocabulary:

Displacement – How far apart the starting position and ending position are. (It can be positive or negative.)
Total Distance – how far you travel in total. (This can only be positive.)

$$Displacement = \int_a^b v dt \qquad Total\ Distance = \int_a^b |v| dt$$

$$Position \text{ at } x = a \quad = x(a) + \int_a^b v\, dt$$

Ex 4 A particle moves along a line so that its velocity at any time t is $v(t) = t^2 + t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during the time period $1 \le t \le 4$.

$$(a) \quad \int_a^b v\, dt = \int_1^4 \left(t^2 + t - 6\right) dt$$

$$= \frac{t^3}{3} + \frac{t^2}{2} - 6t \Big|_1^4$$

$$= -4.5$$

(b) $\int_a^b |v|\,dt = \int_1^4 \left(t^2+t-6\right)dt$

$$= -\int_1^2 \left(t^2+t-6\right)dt + \int_2^4 \left(t^2+t-6\right)dt$$

$$= -\frac{t^3}{3} - \frac{t^2}{2} + 6t\Big|_1^2 + \frac{t^3}{3} + \frac{t^2}{2} - 6t\Big|_2^4$$

$$= 10\frac{1}{6}$$

Note that we used the properties of integrals to split the integral into two integrals that represent the separate positive and negative distanced and then made the negative one into a positive value by putting a – in front. We split the integral at $t=2$ because that would be where $v(t)=0$.

Ex 5 AB 1997 # 1

<u>Steps to solving Integrals in Context</u>

1. Graph the function or find the sign pattern for the velocity.
2. Identify the zeros (if any) between $x=a$ and $x=b$.
3. Set up separate integrals to represent the sections above the *x*-axis and below the *x*-axis.
4. Change the signs on the integrals representing the regions below the *x*-axis .
5. Perform the integration and solve.

3.3 Free Response Homework

Find the area between the curve of the given equation and the x-axis on the given interval.

1a. $y=x^3$ on $x\in[0, 2]$

1b. $y=x^3$ on $x\in[-1, 2]$

2a. $y=4x-x^3$ on $x\in[0, 2]$

2b. $y=4x-x^3$ on $x\in[-1, 2]$

3a. $y=\sin x$ on $x\in[0, \pi]$

3b. $y=\sin x$ on $x\in[-\pi, \pi]$

4a. $y=2x^2-x^3$ on $x\in[0, 2]$

4b. $y=2x^2-x^3$ on $x\in[-1, 2]$

5. $y=x^3-2x^2-3x$ on $x\in[-2, 2]$

6. $y=x^3-4x^2+4x$ on $x\in[-1, 2]$

7. $y=x^3-2x^2-x+2$ on $x\in[-3, 3]$

8. $y=\frac{\pi}{2}\cos x\left(\sin\left(\pi+\pi\sin x\right)\right)$ on $x\in\left[-\frac{\pi}{2}, \pi\right]$

9. $y=\frac{-x}{x^2+4}$ on $x\in[-2, 2]$

10. $y=\frac{4-x^2}{x^2+4}$ on $x\in[-3, 3]$

11. $y=\frac{\sin\sqrt{x}}{\sqrt{x}}$ on $x\in\left[.01, \pi^2\right]$

12. $y = x\sqrt{18 - 2x^2}$ on $x \in [-2,\ 1]$

13. $y = 3\sin x\ \sqrt{1 - \cos x}$ on $x \in \left[-\frac{\pi}{2},\ \frac{\pi}{3}\right]$

14. $y = x^2 e^{x^3}$ on $x \in [0,\ 1.5]$

15. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \le t \le 3$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

16. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \le t \le 6$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

For problems # 17 - 20, show the setup to determine the area described and use Math 9 to solve.

17. Find the area under the curve $f(x) = e^{-x^2} - x$ on $x \in [-1,\ 2]$ (do not use absolute values in your setup, break it into multiple integrals).

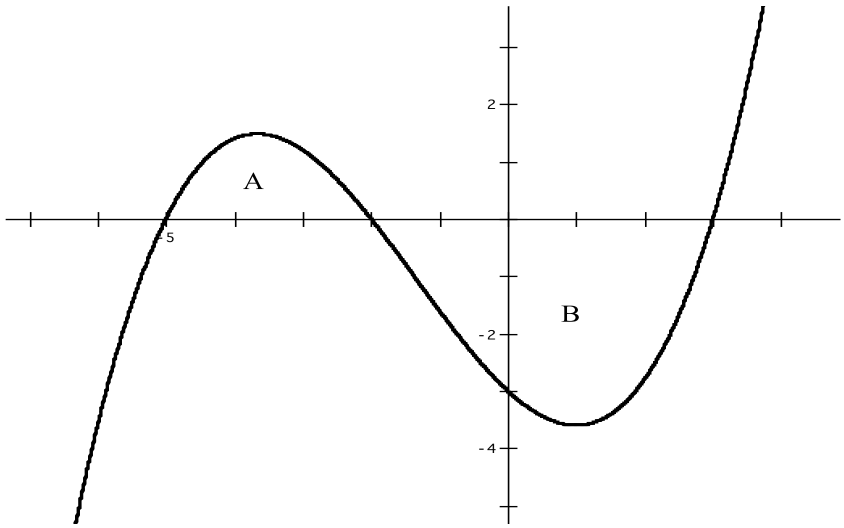
18. Find the area under the curve $f(x) = e^{-x^2} - 2x$ on $x \in [-1,\ 2]$ (do not use absolute values in your setup, break it into multiple integrals).

19. Find the area under the curve $f(x) = \frac{x}{x^2 + 1} + \cos(x)$ on $x \in [0, \pi]$

20. Find the area under the curve $g(x) = -1 - x\sin x$ on $x \in [0, 2\pi]$

3.3 Multiple Choice Homework

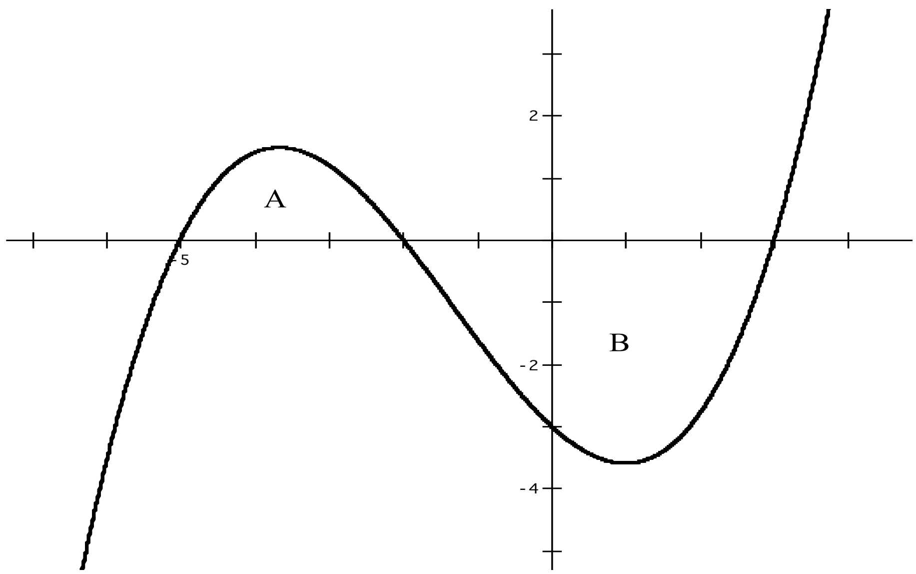
1. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent he areas between the curve and the x-axis.



In terms of A and B, $\int_{-5}^{3} f(x)\,dx + \int_{-2}^{3} f(x)\,dx =$

a) A b) $A-B$ c) $2A-B$ d) $A+B$ e) $A-2B$

2. The graph of $y = f(x)$ is shown below. A and B are positive numbers that represent the areas between the curve and the x-axis.

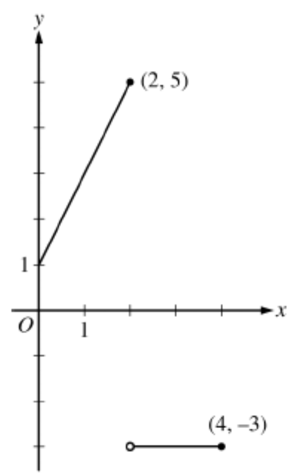


In terms of A and B, $2\int_{-5}^{3} f(x)\,dx - \int_{-2}^{3} f(x)\,dx =$

a) A b) $A-B$ c) $2A-B$ d) $A+B$ e) $A+2B$

---

3. The graph of $f(x)$ on $0 \le x \le 4$ is shown.



What is the value of $\int_0^4 f(x)dx$?

a) -1 b) 0 c) 2 d) 6 e) 12

---

4. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. The total distance traveled by the particle from 0 to 3 is

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

---

5. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2} \sin(2t)$ meters per second. What is the total distance, in meters, traveled by the particle during the time interval $0 \leq t \leq 2$ seconds?

a) 0.835 b) 1.625 c) 1.661 d) 2.261

---

6. A particle moves along the x-axis so that at any time $t \geq 0$ its velocity is given by $v(t) = \ln(t+1) - 2t + 1$. The total displacement of the particle from t=0 to t=3 is

a) -3.455 b) 0.704 c) 1.540 d) 2.667 e) 4.291

---

7. A particle travels along a straight line with a velocity of $v(t) = 3e^{-t^2} \sin(2t)$ meters per second. What is the total displacement, in meters, of the particle during the time interval $0 \leq t \leq 2$ seconds?

a) 0.835 b) 1.625 c) 1.661 d) 2.261

3.4 Accumulation of Rates

As we saw with the Riemann sums, in $\int R(t)\,dt$, $R(t)\cdot dt$ is the area of a rectangle (height times base). The $\int$ is the sum of the areas. The concept in these "Accumulation of Rates" problems is that, since a definite integral is a sum of values, then **an integral of a rate of change equals the total change.**

Objective
Analyze the relationship between rates of change and integrals.

Beginning in 2002, AP shifted emphasis on understanding of the accumulation aspect of the Fundamental Theorem to a new kind of rate problem. Previously, accumulation of rates problems were mostly in context of velocity and distance, though the 1996 Cola Consumption problem hinted at the direction the test would take. The "Amusement Park Problem" of 2002 caught many students and teachers off guard, though. Almost every year since then, the test has included this "Accumulation of Rate" kind of problem. Here is an example similar to the 2002 Amusement Park Problem:

Ex 1 The rate at which people enter a park is given by the function $E(t)=\dfrac{15600}{t^2-24t+160}$, and the rate at which they are leaving is given by $L(t)=\dfrac{9890}{t^2-38t+370}-76$. Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8\le t\le 24$. At $t = 8$ there are no people in the park.

a) How many people have entered the park at 4 pm ($t = 16$)? Round your answer to the nearest whole number.

b) The price of admission is \$36 until 4 pm ($t = 16$). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

c) Let $H(t)=\int_8^t E(x)-L(x)\,dx$ for $8 \le t \le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

d) At what time *t*, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

Notice that several questions are about the meaning of the derivative or integral in context of the word problem and require the use of proper units.

Functions	**Motion**	**Accumulation of Rates**	**Formula**	**Units**
$f(x)$	Position	Total Change	$\int_a^b R(t)\,dt$	Amount
$f'(x)$	Velocity	Rate	$R(t)$	Amount per time
$f''(x)$	Acceleration	Rate increasing or decreasing	$R'(t)$	Amount per time per time

It is important to remember several key phrases:

Key Phrases for decoding Accumulation of Rates questions:

Total change: $\int_a^b R(t)\,dt$ or $\int_a^t [\textit{incoming rate} - \textit{outgoing rate}]\,dx$

Total rate of change: $\textit{incoming rate} - \textit{outgoing rate}$

Total Amount: $\textit{Total}(t) = \textit{initial value} + \int_a^t [\textit{incoming rate} - \textit{outgoing rate}]\,dx$

Instantaneous rate of change: $\frac{dx}{dt}$ or $R(t)$

Average rate of change: $\frac{f(b) - f(a)}{b - a}$

Average value of $f(x)$: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(\textit{Rate of Change})$ is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(\textit{Rate of Change})$ is positive

*Note that it can be a function that is increasing or decreasing, or it could be a derivative that is increasing or decreasing, etc.

Consider the Units!!!

If we consider the units involved in integrating a rate, this becomes more apparent. If $R(t)$ is measured in miles per hour,

$$\int_a^b R(t)dt = \int_a^b \frac{miles}{hour}(hours) = sum\ of\ miles = \text{total miles.}$$

So,

$$\int_a^b R(t)dt = \int_a^b \frac{units}{time}(time) = sum\ of\ units = \text{total units.}$$

Amount increasing or decreasing would be $\frac{d}{dt}(units) = \frac{units}{time}$

Rate would be given in $\frac{units}{time}$

$\frac{f(b)-f(a)}{b-a}$ would be in $\frac{units}{time}$

$f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$ would be $\frac{1}{time}\int_a^b \frac{units}{time}(time) = \frac{units}{time}$

Amount increasing or decreasing would be $(units)\frac{1}{time} = \frac{units}{time}$

Amount increasing or decreasing at an increasing or decreasing rate would be $(units)\frac{1}{time}\cdot\frac{1}{time} = \frac{units}{time^2}$

Rate increasing or decreasing would be $\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{1}{time}\left(\frac{units}{time}\right) = \frac{units}{time^2}$

Now let us try to actually do Ex 1:

Ex 1 The rate at which people enter a park is given by the function $E(t)=\dfrac{15600}{t^2-24t+160}$, and the rate at which they are leaving is given by $L(t)=\dfrac{9890}{t^2-38t+370}-76$. Both $E(t)$ and $L(t)$ are measured in people per hour where t is the number of hours past midnight. The functions are valid for when the park is open, $8\le t\le 24$. At t = 8 there are no people in the park.

a) How many people have entered the park at 4 pm (t = 16)? Round your answer to the nearest whole number.

Since $E(t)$ is a rate in people per hour, the number of people who have entered the park will be an integral from $t=8$ to $t=16$.

$$\text{Total entered} = \int_8^{16} E(t)\,dt = 6126.105 \approx 6126 \text{ people}$$

Note that this is the basic accumulation of rate definition.

b) The price of admission is \$36 until 4 pm ($t=16$). After that, the price drops to \$20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

Since there are different entry fees for different times of day, we need to determine how many people paid each fee:

$$\text{Total entered before 4pm } = \int_8^{16} E(t)\,dt = 6126 \text{ people}$$

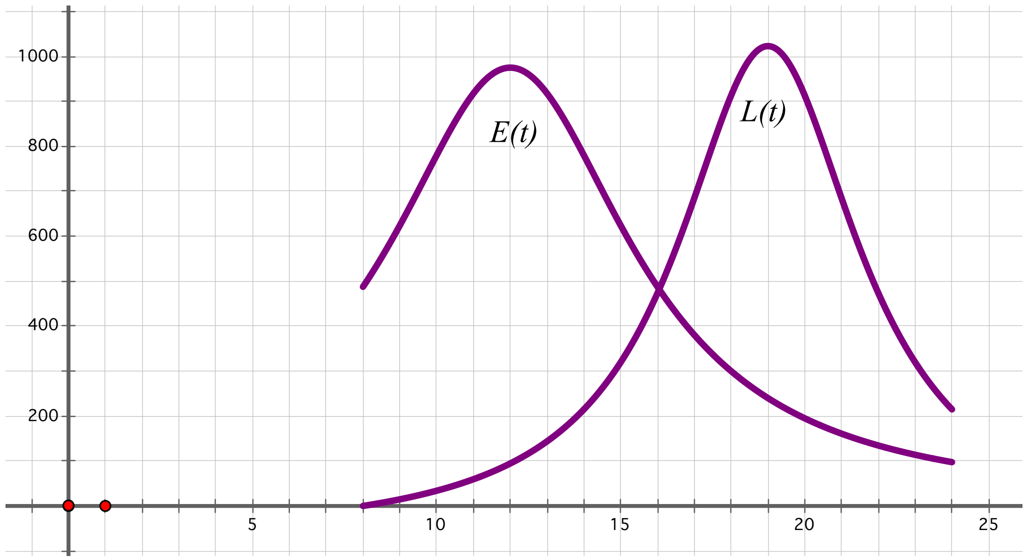
$$\text{Total entered after 4 pm } = \int_{16}^{24} E(t)\,dt = 1808 \text{ people}$$

The total revenue that the park gets from admissions means multiplying the number of people by the admission charge:

Total revenue = \$36(6126) + \$20(1808) = \$256,696

c) Let $H(t)=\int_8^t E(x)-L(x)\,dx$ for $8\le t\le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

While the graph is not necessary for solving this problem, sometimes it helps to visualize the situation. Below are the graphs of $E(x)$ and $L(x)$ on $8\le t\le 24$:



Note how each function increases and then decreases. Since $H(t)=\int_8^t E(x)-L(x)\,dx$, we can use the Fundamental Theorem of Calculus to determine the derivative.

$$\begin{aligned} H'(t) &= \frac{d}{dt}\int_8^t E(x)-L(x)\,dx \\ &= E(t)-L(t) \\ &= E(16)-L(16) \\ &= 14 \end{aligned}$$

But we still need to interpret the meaning of the numbers.

$H(16)=5023$ people – since we know that integrating a rate gives total change, and $H(t)$ was defined as an integral of the difference of two rates,

$H(16)$ is how many people entered the park minus how many people left the park. In other words, $H(16)$ is how many people are **in the park** at 4 pm.

$H'(16)=14$ people per hour. Since the original equations were rates and $H'(t)=E(t)-L(t)$, $H'(16)$ is the rate of change of the number of people in the park at $t = 16$. In other words, the number of people in the park is increasing at 14 people per hour at 4 pm.

d) At what time *t*, for $8 \le t \le 24$, does the model predict the number of people in the park is at a maximum.

We will consider this in the next chapter.

Ex 2 A certain industrial chemical reaction produces synthetic oil at a rate of $S(t)=\frac{15t}{1+3t}$. At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of $R(t)=2+5\sin\left(\frac{4\pi}{25}t\right)$. Both functions have units of gallons per hour, and the reaction runs from $t = 0$ to $t = 6$. At time of $t = 0$, the reaction vessel contains 2500 gallons of oil.

a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measure.

$$\text{Amount removed} = \int_0^6 \left[2+5\sin\left(\frac{4\pi}{25}t\right)\right]dt = 31.816 \text{ gallons}$$

b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time *t*.

$$P(t)=2500+\int_0^t \left[S(x)-R(x)\right]dx$$

Note: Since the variable *t* is the upper boundary, as "dumby variable"—namely, *x*—needs to be used in the integand.

c) Find the rate at which the total amount of oil is changing at $t = 4$.

$$S(4) - R(4) = -1.909 \text{ gal/hr.}$$

SUMMARY

- Realize that these problems are actually three or four different problems with a common source rather than a single problem.
- Let the units dictate the mathematical set-up.
- Slow down and read critically
- These are CALCULATOR problems.

3.4 Free Response Homework

1. If w'(t) is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\,dt$ represent?

2. If oil leaks from a tank at a rate of r(t) gallons per minute at time *t*, what does $\int_0^{120} r(t)\,dt$ represent?

3. If *x* is measured in meters and *f(x)* is measured in newtons, what are the units for $\int_0^{100} f(x)\,dx$?

4. In 1881, the silver mines in Tombstone, Arizona, struck the local aquifer at 520 feet and began to flood. The owners of the Grand Central Mine bought the Cornish engines from the Comstock Mines to pump the water out. On a given day, the water was seeping into the mine at a constant rate of 100 $\frac{gal}{hr}$, and the pumps could drain the water at a rate of $D(t)=414+375\sin\left(\frac{x^2}{72}\right)\frac{gal}{hr}$. When the pumps start, there are 10,000 gallons of water in the mine.

   (a) How many gallons of water were pumped out of the mine during the time interval $0 \le t \le 24$ hours?
   (b) Is the level of water rising or falling at *t* = 6? Explain your reasoning.
   (c) How many gallons of water are in the mine at *t* = 14 hours?

5. In 1920, Dr. Quattrin's grandfather Andrea returned to America from Italy after fighting in World War I. He arrived in New York Harbor on the *SS Pannonia* and, despite having established residency in 1913, had to be processed through the Immigration Center at Ellis Island. There were 1123 non-citizen, third-class passengers on the *Pannonia* that had to go through processing. (First- and second-class passengers passed through without processing.) Immigrants entered the processing line at a rate modeled by the function $E(t)=8843\left(\frac{t}{5}\right)^4\left(1-\frac{t}{10}\right)^5$, where *t* is measured in hours after the ship began opffloading immigrants and $0 \le t \le 10$. The new arrivals were processed out at a rate of 250 people per hour. The

Pannonia was the third ship in port, so there were already 2500 people in line when the *Pannonia* passengers got into line.

(a) How many passengers from the *Pannonia* had gotten in line for processing in the first 6.2 hours?
(b) Is the rate of change of people entering the processing line increasing or decreasing at $t = 6.2$?
(c) How many people were in line at $t = 6.2$?

6. More than 30% of observed star systems have multiple stars, and 70% of those have more than two stars. When stars are close together, they exchange mass in a process known as accretion. Consider a trinary system where S_1 is larger than S_2, and S_2 is larger than S_3. S_3 will lose mass to S_2, and S_2 will lose mass to S_1 While scientific readings are not available because of the time scale, let us suppose that S_2 loses mass to the larger S_1 at a rate of $L(t) = 1 + (.01t)^2 + .23\sin\left(\frac{\pi}{25}t\right)$ and gains mass from the smaller S_3 at a rate of $G(t) = 0.2 + 0.15\sqrt{t}$ where $0 \le t \le 100$ years. $L(t)$ and $G(t)$ are measured in yottatons per year $\left(\mathrm{Y}/_{yr}\right)$. (A yottaton is 10^{26} tons, or 10^{-7} solar masses.)

(a) How much mass does S_2 lose to S_1 on $0 \le t \le 100$? State the units.
(b) At $t = 50$, is the mass S_2 is gaining from S_3 increasing at an increasing rate? Using the correct units, justify your answer.
(c) At what times on $0 \le t \le 100$ is S_2 losing as much mass to S_1 as it is gaining from S_3?

7. A diabetic patient takes Metformin twice a day to control her blood sugar. The medication enters the bloodstream at a rate expressed by $M(t) = 8 - \frac{e^{0.47t}}{t+6}$, where $M(t)$ is measured in centigrams per hour (cg/hr) and *t* is measured in hours for $0 \le t \le 12$. The liver cleans the medication out of the bloodstrream at a rate of $L(t) = 7 - .46t\cos(t)$ cg/hr.

a) How much Metformin enters the bloodstream during this 12-hour time period?

b) After 9 hours, how much Metformin is still in her bloodstream?

c) Find $L'(6)$ and explain the meaning of the answer, using the correct units.

d) Set up, but do not solve, an integral equation that would determine the time when the dose of Metformin has been completely cleaned out of the bloodstream.

8. At an intersection in San Francisco, cars turn left at the rate $L(t) = 50\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour for the time interval $0 \le t \le 18$.

(a) To the nearest whole number, find the total number of cars turning left on the time interval given above.

(b) Traffic engineers will consider turn restrictions if $L(t)$ equals or exceeds 125 cars per hour. Find the time interval where $L(t) \ge 125$, and find the average value of $L(t)$ for this time interval. Indicate units of measure.

(c) San Francisco will install a traffic light if there is a two-hour time interval in which the product of the number of cars turning left and the number of cars travelling through the intersection exceeds 160,000. In every two-hour interval, 480 cars travel straight through the intersection. Does this intersection need a traffic light? Explain your reasoning.

9. Letters arrive at a post office at a rate of $P(t) = 8 + t\sin\frac{t^3}{80}$ hundred letters per hour over the course of a workday. The day begins at 9am ($t = 0$) and ends at 5pm ($t = 8$). There are 3 hundred letters in the office at 9am. Workers send letters out of the office at a constant rate of 5 hundred letters per hour.

a) Find $P'(2)$. Using correct units, interpret the meaning of $P'(2)$ in the context of this problem.

b) Find the total number of letters that arrive at the office between 9am and noon ($t = 3$). Round to the nearest whole number of letters.

c) Write an expression for $L(t)$, the total number of letters in the post office at time *t*.

10. The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(t/3\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

(a) Is the amount of water increasing at $t = 15$? Why or why not?
(b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?
(c) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let *k* be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of *k*.

11. A tank at a sewage processing plant contains 125 gallons of raw sewage at time $t = 0$. During the time interval $0 \le t \le 12$ hours, sewage is pumped into the tank at the rate $E(t) = 2 + \frac{10}{1 + \ln(t+1)}$. During the same time interval, sewage is pumped out at a rate of $L(t) = 12\sin\left(\frac{t^2}{47}\right)$.

(a) How many gallons of sewage are pumped into the tank during the time interval $0 \le t \le 12$ hours?
(b) Is the level of sewage rising or falling at $t = 6$? Explain your reasoning.
(c) How many gallons of sewage are in the tank at $t = 12$ hours?

12. Handout of AP Questions: AB/BC 2002B #2, AB 2005B #2, AB 2006 #2, BC 2011 #2, BC 2015 #1

3.4 Multiple Choice Homework

1. For $t \geq 0$ hours, H is a differentiable function of t that gives the temperature, in degrees Celsius, at an Arctic weather station. Which of the following is the best interpretation of $H'(24)$?

a) The change in temperature during the first day.
b) The change in temperature during the 24^{th} hour.
c) The average rate at which the temperature changed during the 24^{th} hour.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24^{th} day.

---

2. For $t \geq 0$ hours, H is a differentiable function of t that gives the change in temperature, in degrees Celsius per hour, at an Arctic weather station. Which of the following is the best interpretation of $\int_0^t H(x)dx$?

a) The change in temperature during the first t hours.
b) The change in temperature during the first day.
c) The average rate at which the temperature changed during the first t hours.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24^{th} day.

---

3. In the classic 2002 Amusement Park problem, equations $E(t)$ and $L(t)$ were given, representing the rate at which people were entering and leaving the park respectively, for time $9 \leq t \leq 23$, the hours during which the park was open, with t=9 corresponding to 9 am. Let us assume that $F(t) = E(t) - L(t)$. Which of the following is the best interpretation of $F(16)$?

a) The number of people in the park at 4 pm.
b) The number of people entering and leaving the park before 4 pm
c) The average number of people in the park between 9 am and 4 pm.
d) The rate at which the number of people in the park is changing at 4 pm.
e) The rate of change of how quickly the number of people in the park is changing at 4 pm.

---

4. The cost, in dollars, to shred the confidential documents of a company is modeled by C, a differentiable function of the weight of documents in pounds. Of the following, which is the best interpretation of $C'(500) = 80$?

a) The cost to shred 500 pounds of documents is \$80.

b) The average cost to shred documents is $\frac{80}{500}$ dollar per pound.

c) Increasing the weight of documents by 500 pounds will increase the cost to shred the documents by approximately \$80.

d) The cost to shred documents is increasing at a rate of \$80 per pound when the weight of the documents is 500 pounds.

---

5. An ice field is melting at the rate $M(t) = 4 - \sin^3 t$ Acre-feet per day, where t is measured in days. How many acre-feet of this field will melt from the beginning of Day 1 $(t = 0)$ to the beginning of Day 4 $(t = 3)$?

a) 6.846 b) 10.667 c) 10.951 d) 11.544 e) 11.999

---

6. Let $R(t)$ represent the rate in gal/hr at which water is leaking out of a tank, where is measured in hours. Which of the following expressions represents the average rate of change of gallons of water per hour that leaks out in the first three hours?

a) $\int_0^3 R(t)\, dt$ b) $\frac{1}{3}\int_0^3 R(t)\, dt$ c) $\int_0^3 R'(t)\, dt$

d) $R(3) - R(0)$ e) $\frac{R(3) - R(0)}{3 - 0}$

---

7. The rate of natural gas sales for the year 1993 at a certain gas company is given by $P(t)=t^2-400t+160000$, where $P(t)$ is measured in gallons/day and t is the number of days in 1993 from day 0 to 365. To the nearest gallon, what is the average rate of natural gas sales at this company for the 31 days of January 1993?

a) 4,777,730 b) 4,617,930 c) 154,120

d) 148,965 e) 148,561

---

8. The rate at which ice is melting in a pond is given by $\frac{dV}{dt}=\sqrt{1+2^t}$, where V is the volume of the ice in cubic feet and t is the time in minutes. The amount of ice which has melted in the first five minutes is

a) 14.49 ft^3 b) 14.51 ft^3 c) 14.53 ft^3

d) 14.55 ft^3 e) 14.57 ft^3

---

9. The number of parts per million (ppm), *C*(*t*), of chlorine in a pool changes at the rate of $C'(t)=1-3e^{-0.2\sqrt{t}}$ ounces per day, where *t* is measured in days. There are 10 ppm of chlorine in the pool at time $t = 0$. How many ounces of chlorine are in the pool when $t = 9$?

a) -.646 b) 9.354 c) -9.285

d) 9.285 e) 0.715

---

10. The amount of money in a bank account is increasing at the rate of $R(t) = 10000e^{.06t}$ dollars per year, where t is measured in years. If $t = 0$ corresponds to the year 2005, then what is the approximate total amount of increase from 2005 to 2007.

a) $21,250 b) $4,500 c) $18,350

d) $32,560 e) $16,250

---

11. The rate at which water is pumped into a tank is $r(t) = 20e^{0.02t}$, where t is in minutes and $r(t)$ in gallons per minute. Approximately how many gallons of water are pumped into the tank during the first five minutes?

a) 20 b) 22 c) 85 d) 105 e) 150

---

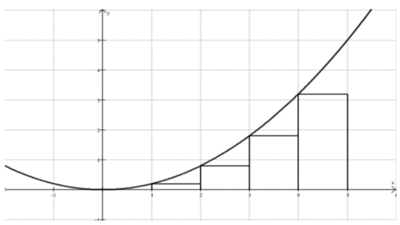
12. Oil is leaking from a tanker at the rate of $R(t) = 2000e^{-0.2t}$ gallons per hour, where t is measured in hours. How much oil leaks out of the tanker from $t = 0$ to $t = 10$?

a) 54 gallons b) 271 gallons c) 865 gallons

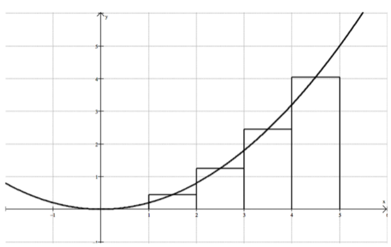
d) 8,647 gallons e) 14,778 gallons

3.5 Integral Approximations: Riemann Rectangles and Trapezoidal Sums

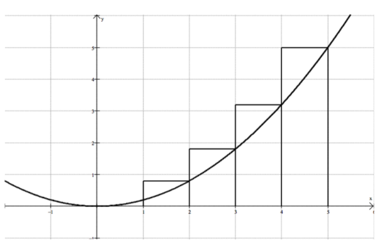
We have been focusing on anti-derivatives of functions where the equation is known. But let us suppose we need to evaluate an integral where either the function is unknown of cannot be anti-differentiated—such as $\int_{-2}^{3} e^{x^2}\,dx$. If we had some exact y-values, we could approximate the area geometrically. This can be done by dividing the area in question into rectangles, and then finding the area of each rectangle. There are three ways to draw the rectangles:



Left-hand Rectangles



Midpoint Rectangles



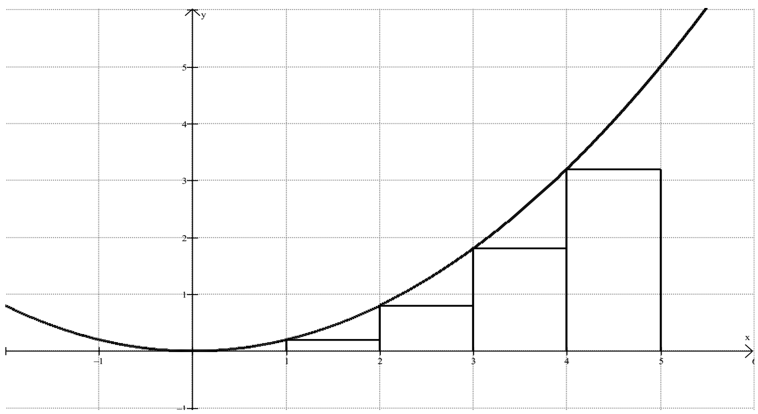
Right-hand Rectangles

Objectives:

Find approximations of integrals using different rectangles.
Use proper notation when dealing with integral approximation.

Ex 1 Use a left end Riemann sum with four equal subintervals to approximate $\int_1^5 f(x)\,dx$ given the table of values below.

x	1	2	3	4	5
$f(x)$	1	4	9	16	25



The heights or each rectangle are the $f(x)$ values on the table, starting from the left, and the widths are the differences between the adjacent x-values.

$$\int_1^5 x^2\,dx \approx 1\cdot(1)+1\cdot(4)+1\cdot(9)+1\cdot(16)=30$$

****Notice the use of ≈ and =**

Also notice that the 25 does not get used in the problem as it is not the height of any of the rectangles drawn.

<u>Steps to Approximating an Integral with Rectangles:</u>

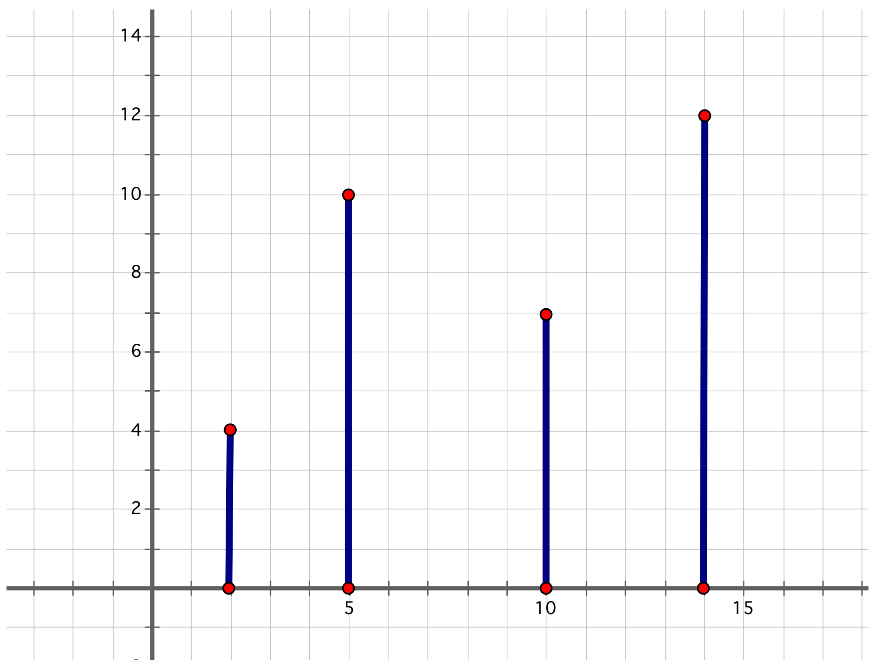
1. Draw the heights represented on the tables.
2. Draw the tops of the rectangles from the left endpoint, right endpoint, or midpoint values.
3. Calculate the areas of each rectangle.
4. Add the areas together for your approximation.
5. State answer using proper notation.

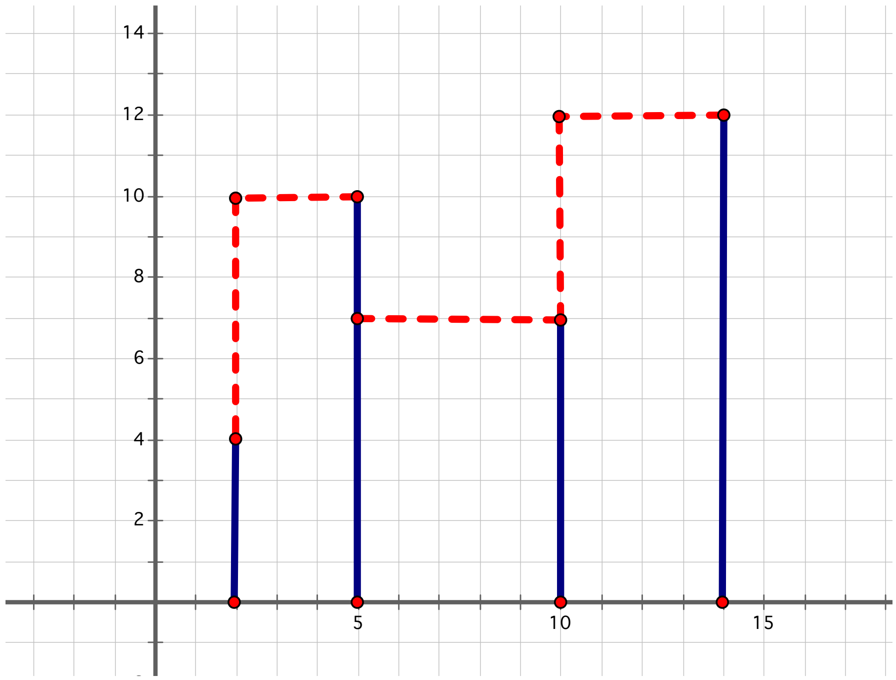
Example 1 had equal width rectangles, but this does not need to be true.

Ex 2   Let f be a differentiable function on the closed interval [2, 14] and which has values as shown on the table below.

x	2	5	10	14
$f(x)$	4	10	7	12

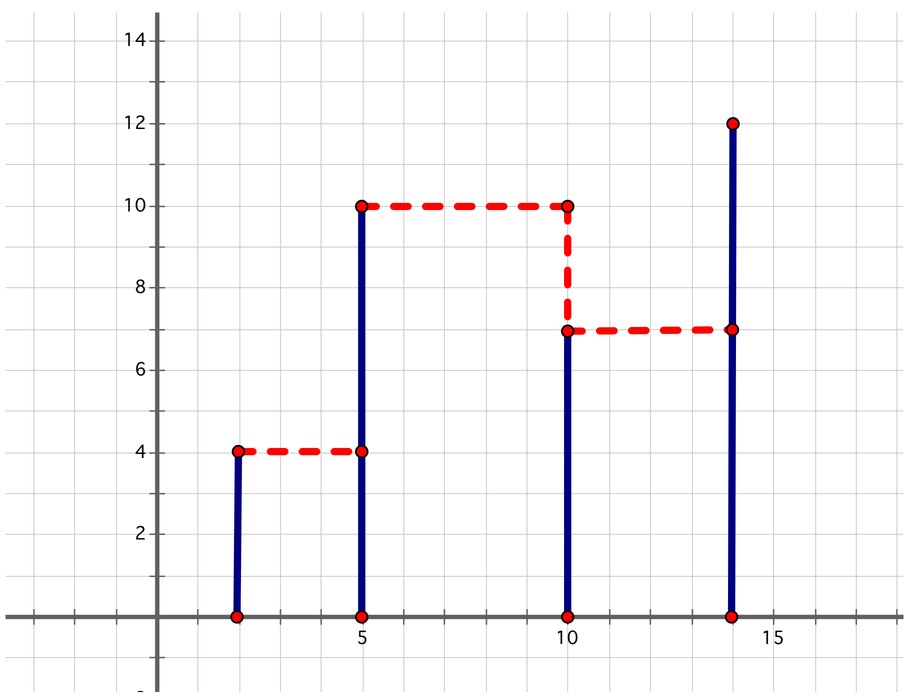
Using the sub-intervals defined by the table values, use the right-hand Riemann sum to approximate $\int_2^{14} f(x)\,dx$.





$$\int_2^{14} f(x)\,dx \approx 3(10)+5(7)+4(12)=113$$

If the question had asked for left hand rectangles, the picture and solution would look like this:



$$\int_2^{14} f(x)dx \approx 3(4)+5(10)+4(7)=90$$

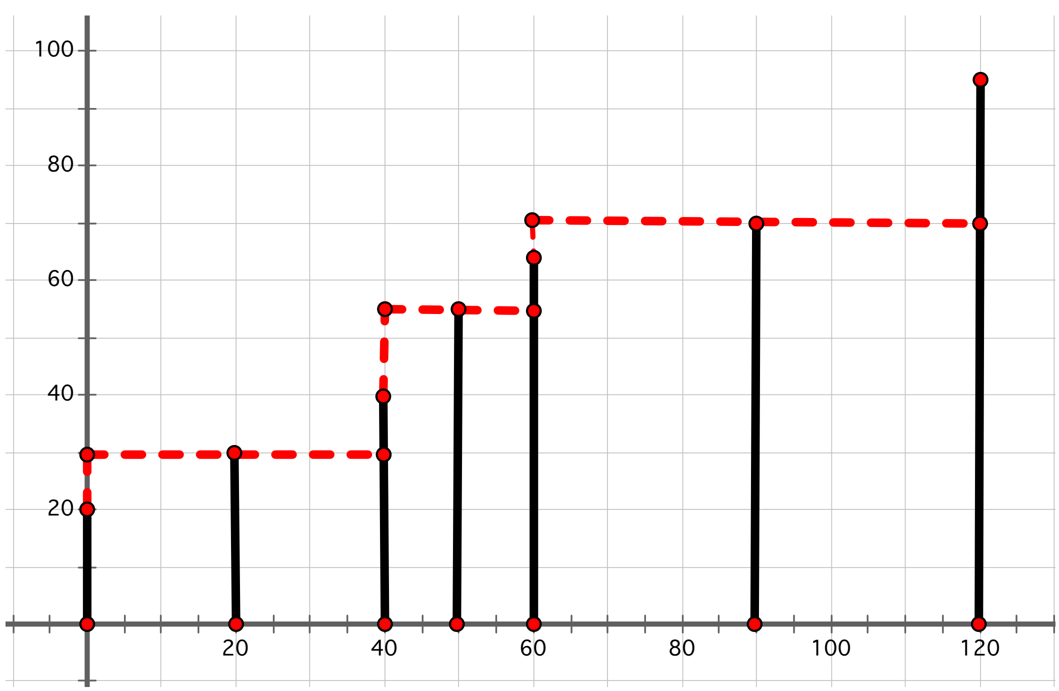
The third kind of Riemann rectangles is where the height comes from the midpoints of the rectangles. Two conditions are needed for Midpoint Rectangles;

1. There need to be an odd number of values.
2. The 2nd, 4th, 6th, etc. x-values must be midpoints between the odd values.

Ex 3 The rate of consumption, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values of $R(t)$ for the time interval $0 \le t \le 90$ is shown below.

t minutes	0	20	40	50	60	90	120
R(t) (Gallons per minute)	20	30	40	55	65	70	95

Use the Riemann sum with Midpoint Rectangles and the subintervals given by the table to approximate the value of $\int_0^{120} R(t)dt$.



$$\int_0^{120} R(t)dt \approx 40(30)+20(55)+60(70)=6500$$

Notice that we have half as many rectangles, but each is double width.

Ex 4 Use the midpoint rule and the given data to approximate the value of $\int_0^{2.6} f(x)dx$.

x	$f(x)$	x	$f(x)$
0	3.5	1.6	4.7
0.4	2.3	2.1	5.9
0.8	3.2	2.6	4.1
1.2	4.3		

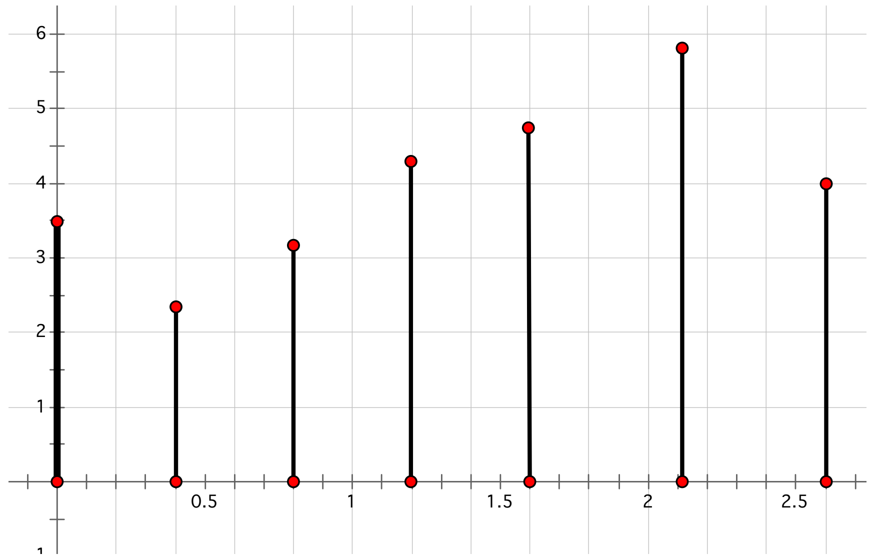
We do not really need the picture to set up the problem:

$$\int_0^{2.6} f(x)dx \approx 0.8(2.3)+0.8(4.3)+1.0(5.9)=11.18$$

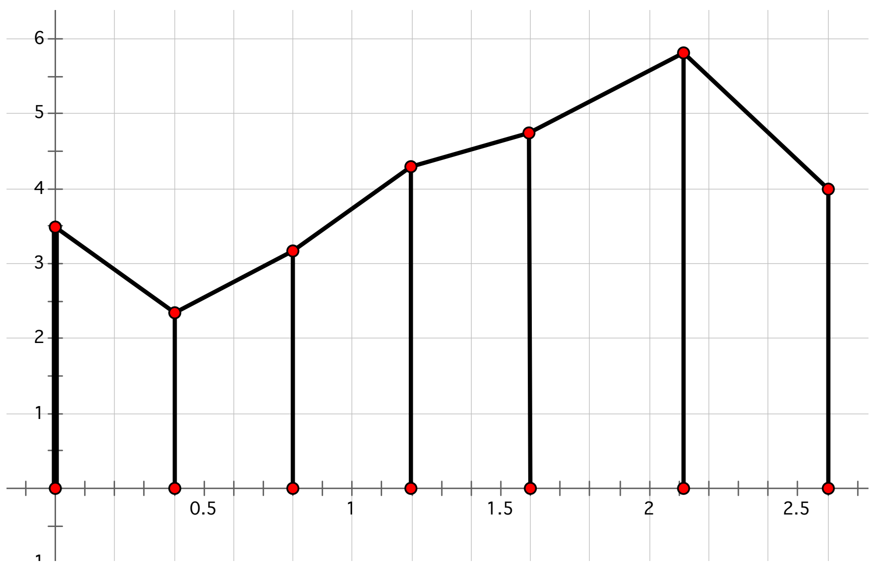
A geometric alternative to rectangles would be to use trapezoids.

Ex 5 Using the table of data from Ex 4, approximate $\int_0^{2.6} f(x)dx$ with 6 trapezoids.

We can start by drawing the heights, just as we did with rectangles.



Then we can connect the tops of the heights, making trapezoids:



Remember the area formula for a trapezoid from Geometry:

$$A_{\text{Trap}} = \left(\frac{b_1 + b_2}{2}\right) \cdot h,$$

In this case, b_1 and b_2 are the verticals (i.e., the parallel sides of the trapezoid), and h is the width.

$$\int_0^{2.6} f(x)\,dx \approx (0.4)\left(\frac{3.5+2.3}{2}\right)+(0.4)\left(\frac{2.3+3.2}{2}\right)+(0.4)\left(\frac{3.2+4.3}{2}\right)$$
$$+(0.4)\left(\frac{4.3+4.7}{2}\right)+(0.5)\left(\frac{4.7+5.9}{2}\right)+(0.5)\left(\frac{5.9+4.1}{2}\right)$$

$$=8.15$$

If the table had had equal width subintervals, we could set up the trapezoid area formula and factor out the $\frac{1}{2}$ and the widths. The widths would be $\frac{b-a}{n}$**,** where n is the number of trapezoids. Left behind would be the sum of the heights where each height, *except for the first and last*, would appear twice. The result would look like this:

<u>The Trapezoidal Rule*</u>

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+...+2f(x_{n-1})+f(x_n)\right]$$

*The Trapezoidal Rule requires ***equal*** sub-intervals

Ex 6 The following table gives values of a continuous function. Approximate the average value of the function using the Trapezoidal Rule.

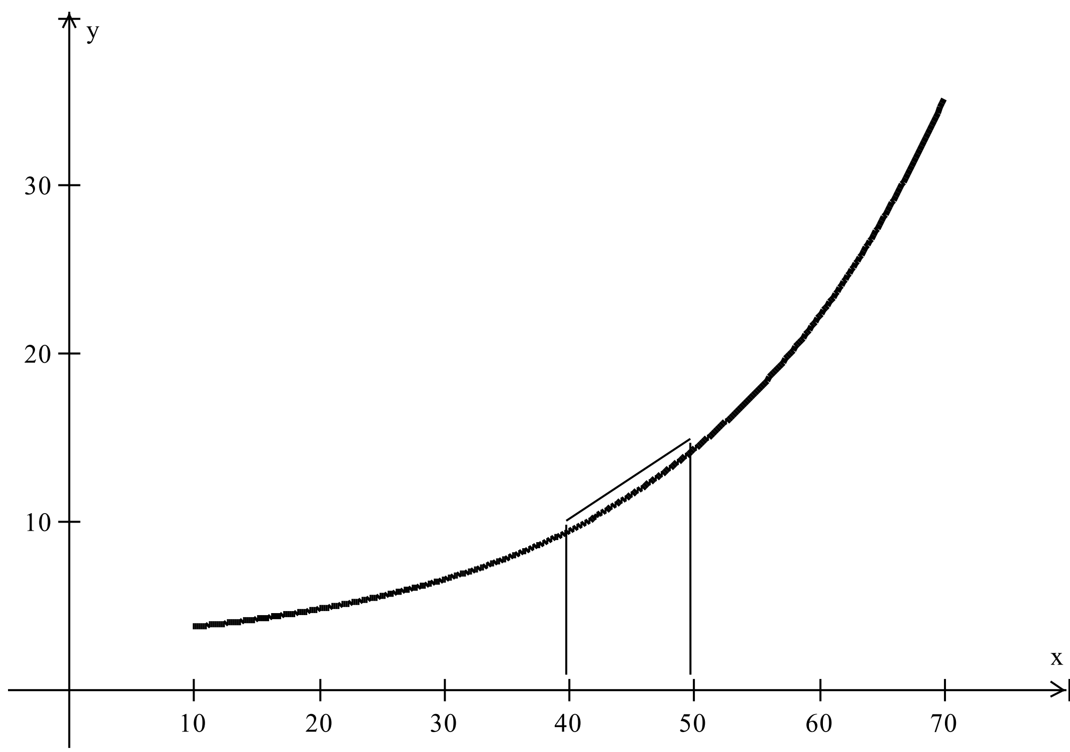
x	10	20	30	40	50	60	70
$f(x)$	3.649	4.718	6.482	9.389	14.182	22.086	35.115

$$
\begin{aligned}
f_{avg} &= \frac{1}{70-10}\int_{10}^{70} f(x)\,dx \\
&\approx \frac{1}{70-10}\left[\frac{70-10}{12}\begin{pmatrix}3.649+2(4.718)+2(6.482)+ \\ 2(9.389)+2(14.182)+2(22.086)+35.115\end{pmatrix}\right] \\
&= 12.707
\end{aligned}
$$

Ex 7 Let us assume that the function that determined the values in the chart above is $f(x)=2+e^{.05x}$. Calculate the average value of the function and compare it to your approximations.

$$f_{avg} = \frac{1}{70-10}\int_{10}^{70}\left(2+e^{.05x}\right)dx = 12.489$$

If we want to figure out if our approximations are overestimates or underestimates, we have to look at the graph of the function.

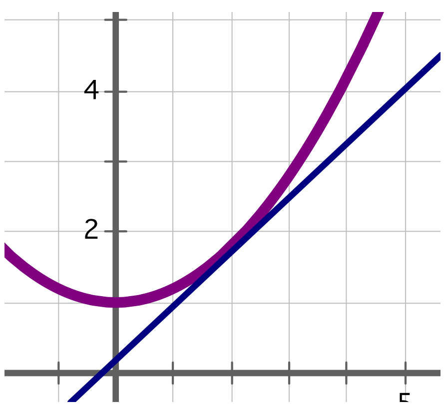
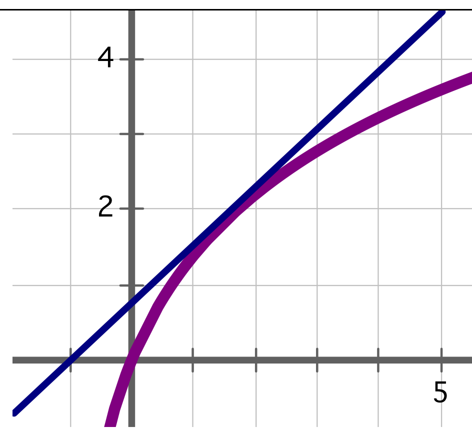


We can see that, rather than the function being increasing or decreasing, over- and under-estimation with trapezoids (as well as with midpoint rectangles) is determined by the concavity of the curve.

Remember from Chapter 1:

<u>Tangent Line Approximations</u>

- Tangent line approximations are an overestimate if the curve is concave down (since your "tangent lines" will be above the curve).
- Tangent line will be an underestimate if the curve is concave up (since your "tangent lines" will be below the curve).

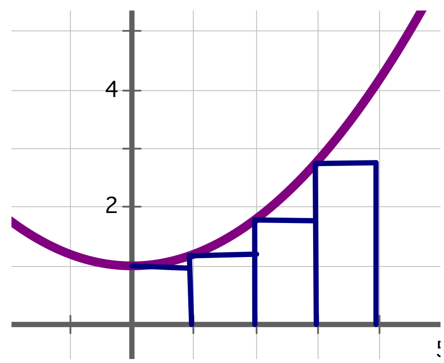
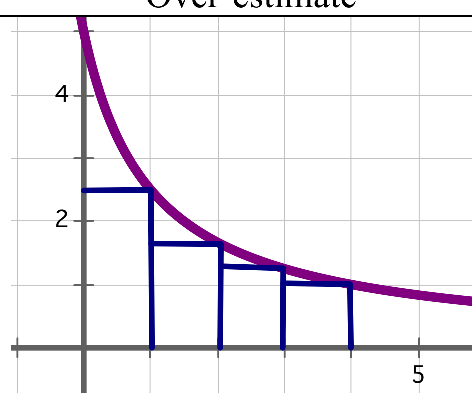
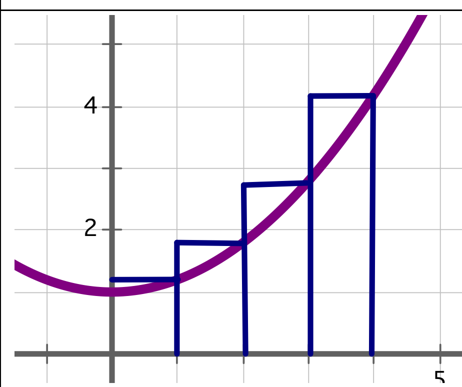
Concave Up	**Concave Down**
 Under-estimate	 Over-estimate

Add to that:

Integral Approximations

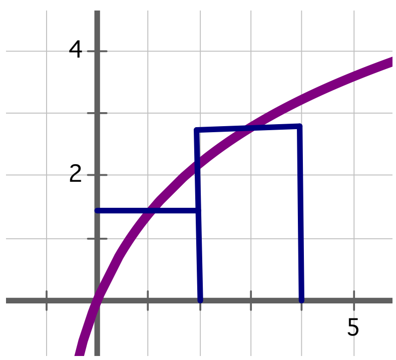
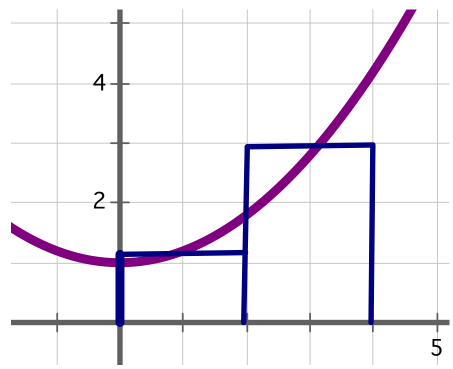
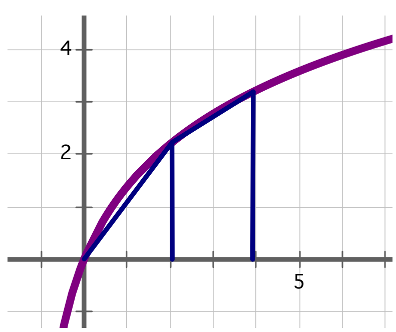
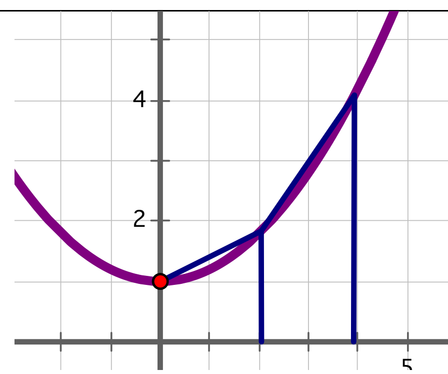
- Left Hand Rectangles are an overestimate if the curve is decreasing and an underestimate if the curve is decreasing.
- Right Hand Rectangles are an underestimate if the curve is decreasing and an overestimate if the curve is decreasing.

These facts might be better understood visually:

	Decreasing	**Increasing**
Left Hand	 Over-estimate	 Under-estimate
Right Hand	 Under-estimate	 Over-estimate

<u>Integral Approximations</u>

- Midpoint Rectangles are an overestimate if the curve is concave down and an underestimate if the curve is concave up.
- Midpoint Rectangles and Trapezoids are an underestimate if the curve is concave down and an overestimate if the curve is concave up.

	Concave Down	**Concave up**
Midpoint	 Under-estimate	 Over-estimate
Trapezoidal	 Under-estimate	 Over-estimate

3.5 Free Response Homework

1. The following table gives values of a continuous function.

x	0	1	2	3	4	5	6	7	8
$F(x)$	10	15	17	12	3	-5	8	-2	10

Estimate the average value of the function on $x \in [0, 8]$ using
(a) Right-Hand Riemann rectangles,
(b) Left-Hand Riemann rectangles, and
(c) Midpoint Riemann rectangles.

2. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

$t(s)$	v (mi/h)	$t(s)$	v (mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

3. Below is a chart showing the rate of a rocket flying according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	10	20	30	40	50	60
$v(t)$ (in km/min)	30	28	32	18	52	48	28

a) Find an approximation for $\int_0^{60} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{30} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

c) Find an approximation for $\int_{30}^{60} v(t)\,dt$ using left rectangles. Make sure you express your answer in correct units.

d) Find an approximation for $\int_0^{40} v(t)\,dt$ using right rectangles. Make sure you express your answer in correct units.

4. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	8	16	24	32	40	48
$V(t)$ (in m^3/min)	26	32	43	24	19	24	26

a) Find an approximation for $\int_0^{48} V(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{16} V(t)\,dt$ using right Riemann rectangles. Make sure you express your answer in correct units.

5. Below is a chart of your speed driving to school in meters/second. Use the information below to find the values in a) and b) below.

t (in seconds)	0	30	90	120	220	300	360
$v(t)$ (in m/sec)	0	21	43	38	30	24	0

a) Find an approximation for $\int_0^{360} v(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{220} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

6. Below is a chart showing the velocity of the Flash as he runs across the country. Use this information to answer each of the following.

t (in seconds)	0	4	8	12	16	20	24
$W(t)$ (in km/second)	10	12	15	19	24	18	7

a) Find an approximation for $\int_0^{24} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units. Describe what this integral means.

b) Find an approximation for $\int_0^{16} v(t)\,dt$ using Trapezoids. Make sure you express your answer in correct units.

7. Below is a chart showing the rate of sewage flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	4	6	10	13	15	20
$V(t)$ (in gallons/min)	83	68	82	40	38	30	68

a) Find an approximation for $\int_0^{20} V(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{20} V(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

8. Star Formation Rate (SFR) observations of red-shift allow scientists to track the total mass gained in a galaxy by the making of new stars. Below is a table of such data:

t	0	1	2	3	4	5	6	7	8
SFR	0.0029	0.0051	0.0055	0.0049	0.0042	0.0035	0.0029	0.0025	0.0021

SFR is measured in solar masses per cubic parsec per gigayear (millions of years) and t is measured in gigayears.

a) Use midpoint rectangles to approximate the total mass of stars formed from $t=0$ to $t=8$.

b) Use right-hand rectangles to approximate the average solar masses per cubic parsec per gigayear.

9. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the $\int_0^2 \sqrt[4]{1+x^2}\,dx$ with the specified value of $n=8$.

10. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the $\int_1^2 \frac{\ln x}{1+x}\,dx$ with $n=10$.

11. AP Handout: AB 1998 #3, AB 2001 #2, BC2007 #5

3.5 Multiple Choice Homework

1. The graph of the function *f* is shown below for $0 \le x \le 3$.

Of the following, which has the smallest value?

a) $\int_1^3 f(x)dx$

b) Left Riemann sum approximation of $\int_1^3 f(x)dx$ with 6 equal sub intervals.

c) Right Riemann sum approximation of $\int_1^3 f(x)dx$ with 6 equal sub intervals.

d) Midpoint Riemann sum approximation of $\int_1^3 f(x)dx$ with 6 equal sub intervals.

e) Trapezoidal sum approximation of $\int_1^3 f(x)dx$ with 6 equal sub intervals.

Time (sec)	0	10	25	37	46	60
Rate (gal/sec)	500	400	350	280	200	180

2. The table above gives the values for the rate (in gal/sec) at which water flowed into a lake, with readings taken at specific times. A right Riemann sum, with the five subintervals indicated by the data in the table, is used to estimate the total amount of water that flowed into the lake during the time period $0 \le t \le 60$. What is this estimate?

a) 1,910 gal b) 14,100 gal c) 16,930 gal

d) 18,725 gal e) 20,520 gal

---

3. A car is traveling on a straight road such that selected measures of the velocity have values given on the table below.

t	10	20	40	70	80
$v(t)$	90	88	100	90	85

Using four Left Hand Reimann rectangles based on the table, the estimated distance traveled by the car between $t = 10$ and $t = 80$ seconds is

a) 6125 b) 6380 c) 6430 d) 6495 e) 6560

---

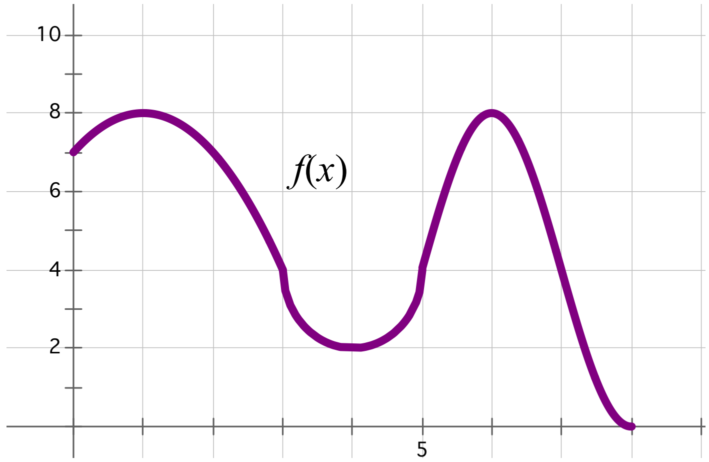
x	3	6	9	12
$f(x)$	12	18	7	5

4. The function f is continuous and differentiable on the closed interval [3,12], what is the right Riemann approximation of $\int_3^{12} f(x)dx$?

a) 69 b) 90 c) 111 d) 126 e) 201

---

5. Consider the function f whose graph is shown below:

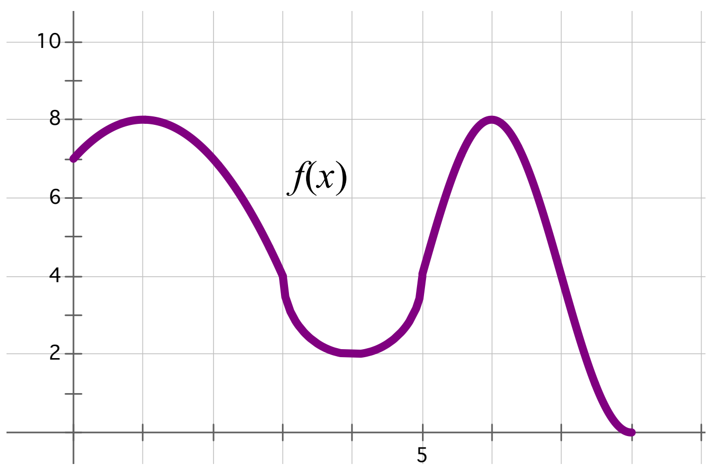


The approximate value of $\int_0^8 f(x)\,dx$, using 4 midpoint rectangles with equal width, is

a) 20 b) 37 c) 40 d) 40.5 e) 44

---

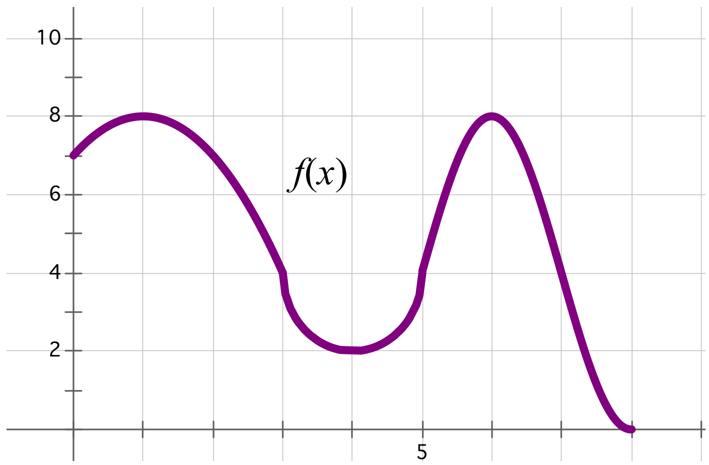
6. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight right-hand rectangles with equal width, is

a) 18.5 b) 37 c) 40 d) 40.5 e) 44

7. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight left-hand rectangles with equal width, is

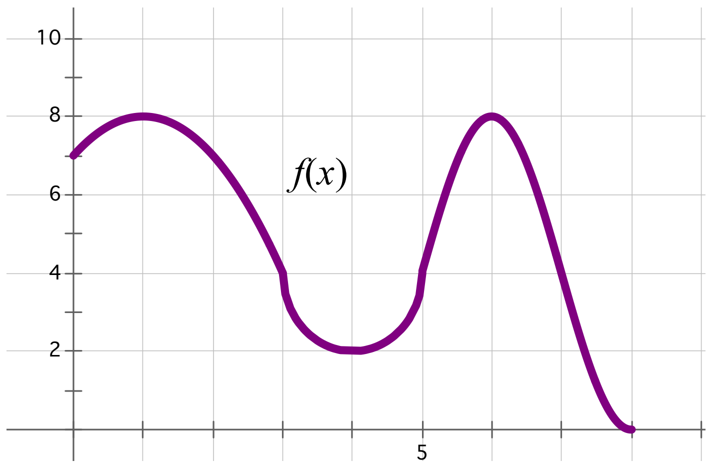
a) 23 b) 37 c) 40 d) 40.5 e) 44

x	2	5	10	14
$f(x)$	12	28	34	30

8. Let f be a differentiable function on the closed interval [2, 14] and which has values as shown on the table above. Using the sub-intervals defined by the table values and using right hand Riemann sums, $\int_2^{14} f(x)dx =$

a) 296 b) 312 c) 343 d) 374 e) 390

9. Consider the function f whose graph is shown below:



The approximate value of $\int_0^8 f(x)\,dx$, using eight trapezoids with equal width, is

a) 37 b) 40 c) 40.5 d) 44 e) 48

---

10. The following table lists the known values of a function $f(x)$.

x	1	2	3	4	5
$f(x)$	0	1.1	1.4	1.2	1.5

If the Trapezoidal Rule is used to approximate $\int_1^5 f(x)\,dx$ the result is

a) 4.1 b) 4.3 c) 4.5 d) 4.7 e) 4.9

---

t	0	1	2	3	4
$H(t)$	0	1.3	1.5	2.1	2.6

11. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of each day for four days. The data, where $H(t)$ is measured in inches per day and t is measured in days, are listed above. Using the trapezoidal rule, which of the following represents an estimate of of the average rate of growth of the plant over the four-day period?

a) $\frac{1}{4}(0+1.3+1.5+2.1+2.6)$

b) $\frac{1}{4}\left[\frac{1}{2}(0+1.3+1.5+2.1+2.6)\right]$

c) $\frac{1}{4}\left[\frac{1}{2}(0+2(1.3)+2(1.5)+2(2.1)+(2.6))\right]$

d) $\frac{1}{4}\left[\frac{1}{2}(0+2(1.3)+2(1.5)+2(2.1)+2(2.6))\right]$

e) $\frac{1}{4}\left[\frac{1}{4}(0+2(1.3)+2(1.5)+2(2.1)+2.6)\right]$

3.6 Table Problems

Reasoning from Tabular Data (aka Table Problems) has been one of the most commonly recurring topics on the AP Exam. It has been the first question on the Free Response part of the Exam for the past six years (2014 - 2019).

There are two main kinds of table problems:

1. Multiple choice questions often have tables of values to plug into the Chain, Product, or Quotient Rule.
    - The trick here is that much of the data are distractors.

2. The most common is a word problem involving an unknown function but with specific data points.
    - These are very much in line with the Accumulation of Rates and Rectilinear Motion problems.

It is this second kind which we will investigate here.

Remember:

Instantaneous rate of change: $\frac{dx}{dt}$

Average rate of change: $\frac{f(b)-f(a)}{b-a}$

Average value: $f_{avg} = \frac{1}{b-a}\int_a^b f(x)dx$

Total change: $\int_a^b R(t)dt$ or $\int_a^t [incoming\ rate - outgoing\ rate]\ dx$

Total rate of change: $incoming\ rate - outgoing\ rate$

Total Amount: $Total(t) = initial\ value + \int_a^t [incoming\ rate - outgoing\ rate]\ dx$

Amount Increasing (or decreasing): Total rate of change is positive (or negative)

Rate of Change Increasing (or decreasing): $\frac{d}{dt}(Rate\ of\ Change)$ is positive (or negative)

Amount Increasing at an increasing rate:

Total rate of change is positive AND $\frac{d}{dt}(Rate\ of\ Change)$ is positive

Objective

Analyze the relationship between rates of change and integrals in light of data given in a tabular format.

Let us consider a problem similar to the ones in the Accumulation of Rates Section, but with a table of data for part of the problem.

Ex 1 At 6am at the *Popular Potatoes* potato chip factory, there are already 5 tons of potatoes in the factory. More potatoes are delivered from 6am ($t = 6$) until noon ($t = 12$) at a rate modeled by

$$P(t) = 9 - \frac{9\sin(x-2)}{x-2} \text{ tons of potatoes per hour.}$$

Workers arrive at 6am and begin to process the potatoes to turn them into potato chips. Their supervisor measures their rate of output every hour and records her findings in the chart below.

t = time after midnight in hours	6	9	13	14	16
$C(t)$ = Rate of potatoes processed in tons/hour	7.9	6.5	3.9	3.1	1.3

The supervisor determines that the workers' rate of processing is a decreasing function throughout the day.

(a) How many tons of potatoes arrive at the *Popular Potatoes* factory between 6am and noon?
(b) Use a Left Riemann sum with subintervals indicated by the table to approximate $\int_6^{16} C(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.
(c) Is your approximation in part (b) an under- or over-approximation? Explain.
(d) The workers end their shift at 4pm. At that time, are there still potatoes in the factory left to process? Explain your reasoning.

(a) How many tons of potatoes arrive at the *Popular Potatoes* factory between 6am and noon?

$$\int_6^{12} P(t)\,dt = 54.899 \ \ tons\, of \ \ potatos$$

(b) Use a Left Riemann sum with subintervals indicated by the table to approximate $\int_6^{16} C(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

$$\int_6^{16} C(t)\,dt \approx 3(7.9)+4(6.5)+1(3.9)+2(3.1)=59.8$$

Approximately 59.8 tons of potatoes were processed into chips between 6 a.m. and 4 p.m.

(c) Is your approximation in part (b) an under- or over-approximation? Explain.

The approximation is an over-estimate because the data on the table show a decreasing function and left-hand Riemann rectangles over-estimate a decreasing function.

(d) The workers end their shift at 4pm. At that time, are there still potatoes in the factory left to process? Explain your reasoning.

Yes, there were still potatoes left at the end of the day because, while 59.8 tons of potatoes were processed, there were 59.899 tons on site—5 tons at the beginning of the day and 54.899 tons which were delivered.

Ex 2 Below is a chart showing specific values of the rate of sewage flowing through a pipeline according to time in minutes.

t (in minutes)	0	4	6	10	13	15	20

$V(t)$ (in gallons/min)	83	68	83	48	38	30	38

Assume $V(t)$ is a continuous and differentiable function.

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{20} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(c) Find the value of $\int_0^{20} V'(t)\,dt$ and explain the meaning of this value in the context of the problem.

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

$$V'(7) = \frac{V(10)-V(6)}{10-6} = \frac{48-83}{4} = 8.75\,{}^{gal}\!/_{\min^2}$$

(b) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{20} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

$$\int_0^{20} V(t)\,dt \approx 4\left(\frac{83+68}{2}\right) + 2\left(\frac{68+83}{2}\right) + 4\left(\frac{83+48}{2}\right) + 3\left(\frac{48+38}{2}\right)$$
$$+2\left(\frac{38+30}{2}\right) + 5\left(\frac{30+38}{2}\right)$$
$$= 1082$$

Approximately 1082 gallons of sewage flowed through the pipeline between $t = 0$ and $t = 20$ minutes.

(c) Find the value of $\int_0^{20} V'(t)\,dt$ and explain the meaning of this value in the context of the problem.

$$\int_0^{20} V'(t)\,dt = V(20) - V(0) = 38 - 83 = -45 \,{}^{gal}\!/_{\min}$$

The total change of the rate of flow between $t=0$ and $t=20$ minutes is -45 gallons per minute.

3.6 Free Response Homework

1. Below is a chart showing the rate of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	8	16	24	32	40	48
$V(t)$ (in m^3/min)	26	32	43	24	19	24	26

(a) Estimate $V'(7)$. Show the work that leads to your answer. Indicate the units.

(b) Find $\int_8^{40} V'(t)\,dt$

(c) Use a trapezoidal sum with subintervals indicated by the table to approximate $\int_0^{48} V(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{48}\int_0^{48} V(t)\,dt$ in the context of the problem.

t days	0	1	2	3	4
$H(t)$ in mm per day	0	1.3	1.5	2.1	2.6

2. A small plant is purchased from a nursery and the change in height of the plant is measured at the end of each day for four days. The data, where $H(t)$ is measured in millimeters per day and t is measured in days, are listed above.

(a) Estimate $H'(3)$. Show the work that leads to your answer. Indicate the units.

(b) Explain how one would know that the plant's growth is not increasing at a decreasing rate

(c) Use right-hand rectangles with subintervals indicated by the table to approximate $\int_0^4 H(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

(d) Using correct units, explain the meaning of $\frac{1}{4}\int_0^4 H(t)\,dt$ in the context of the problem.

---

3. The rate of consumption of fuel, in gallons per minute, recorded during an airplane flight is given by a twice differentiable and strictly increasing function $R(t)$. A table of selected values of $R(t)$ for the time interval $0 \le t \le 90$ is shown below.

t minutes	0	20	40	50	60	90
$R(t)$ (gallons per minute)	20	30	40	55	65	70

(a) Estimate $R'(30)$. Show the work that leads to your answer. Indicate the units.

(b) Use right hand Riemann rectangles to approximate $\int_0^{90} R(t)dt$ and indicate units of measure. Explain the meaning of $\int_0^{90} R(t)dt$ in terms of the fuel consumption.

(c) Use left hand rectangles to find $\frac{1}{70}\int_{20}^{90} R(t)dt$. Using the correct units, explain the meaning of $\frac{1}{70}\int_{20}^{90} R(t)dt$ in terms of the fuel consumption.

---

4. A diabetic patient tests his blood glucose level every morning. After being put on insulin, the data below show the glucose levels in milligrams per deciliter (mg/dL) over one week.

t days	1	2	3	4	5	6	7
$G(t)$ (mg/dL)	233	198	185	168	147	130	147

Let $G(t)$ represent the glucose level where t is measured in days.

(a) Estimate $G'(3.7)$. Using the correct units, explain the meaning of the result.

(b) Use midpoint Riemann rectangles to approximate $\int_1^7 G(t)dt$. Using the correct units, explain the meaning of $\frac{1}{7}\int_1^7 G(t)dt$ in terms of the patient's glucose levels.

(c) Ignoring the last data point, $M(t) = 237.6e^{-0.082t}$ is a model of $G(t)$. Find $M'(3.7)$. Is $M(t)$ decreasing at an increasing rate? Show the work that leads to your conclusion.

---

5. Diabetic patients take a test called an A1c every three months which measures the three-month average percentage of glycated hemoglobin (that is, hemoglobin covered in glucose).

Month	0	3	6	9	12	15	18	21
$A(t)$ (as a %)	10.2	10.0	10.5	9.1	8.0	8.9	8.3	8.6

Let $A(t)$ represent the A1c score, measured as a percentage.

(a) Find $\int_0^{21} A'(t)dt$. Show the work that leads to your answer. Explain the meaning of $\int_0^{21} A'(t)dt$ in terms of A1c scores.

(b) Use right-hand Riemann rectangles to approximate $\int_0^{21} A(t)dt$. Using the correct units, explain the meaning of $\frac{1}{21}\int_0^{21} A(t)dt$ in terms of the patient's A1c score.

(c) $B(t) = -.305x + 10.571 + .1\sin \pi x$ is a model of $A(t)$. Find $\frac{1}{21}\int_0^{21} B(t)dt$.

---

t months	0	1	2	3	4	5	6
$P(t)$	160.3	192.8	345.7	746.1	944.2	873.0	1128.6

t months	7	8	9	10	11	12
$P(t)$	928.3	851.3	751.3	535.5	216.4	150.7

6. A family leases solar panels on their house. At the end of the year, they receive a report, including the tables above, which shows the monthly production $P(t)$ of electricity, in kilowatts per month (kW/month), from the panels.

(a) Use right-hand Riemann rectangles to approximate $\int_0^{12} P(t)dt$. Indicate the units.

(b) $k(t) = 660 - 489\cos\frac{\pi}{6}t$ is a model of $P(t)$. Find $\int_0^{12} k(t)dt$.

(c) Using the model $k(t) = 660 - 489\cos\frac{\pi}{6}t$, show that the production is decreasing at $t = 9$. Is the production decreasing at an increasig rate?

---

t months	0	1	2	3	4	5	6
$C_e(t)$	390.7	660	667.1	538.4	420.5	412.1	347.8
$C_g(t)$	87.6	84.6	109	116	79.8	53.9	42.9

t months	7	8	9	10	11	12
$C_e(t)$	287.5	303.1	322.4	342.5	390.3	384.2
$C_g(t)$	24.9	25.6	18	20.3	48.9	91.8

7. Dr. Quattrin analyzes his PG&E bill to track his consumption of both electricity ($C_e(t)$) and gas ($C_g(t)$) over the course of a year. The tables above are the result. $C_e(t)$ is measures in kilowatts (kW) and $C_g(t)$ is measured in therms (thm).

(a) Approximate $C_e'(3.4)$ and $C_g'(3.4)$. Use correct units, explain the meaning of these estimations in terms of increasing and/or decreasing consumption of each commodity at $t = 3.4$.

(b) Use right hand Riemann rectangles to approximate $\int_0^{12} C_e(t)dt$. Indicate the units.

(c) Use midpoint Riemann rectangles to approximate $\int_0^{12} C_g(t)dt$. Indicate the units.

(d) Using the correct units, explain the meaning of $\frac{1}{12}\int_0^{12} C_g(t)dt$.

8. Dr. Quattrin decides to lease solar panels from Sunrun Solar. After a year, he reanalyzes his PG&E bill to track both his consumption of electricity ($C_e(t)$) and his production of electricity ($P_e(t)$) over the course of a year. The tables below show the consumption of electricity, measured in kilowatts (kWs).

t months	0	1	2	3	4	5	6
$C_e(t)$	326.5	660.0	667.1	538.4	420.5	412.1	347.8

t months	7	8	9	10	11	12
$C_e(t)$	287.5	303.1	322.4	342.5	390.3	384.2

$P_e(t) = 407 - 374.2\cos\frac{\pi}{6}t$ models the production in kW per month that PG&E buys back.

(a) How much power does PG&E buy back from the Quattrins over the course of the year? Indicate the units.
(b) Using the trapezoidal sum, approximate the amount of power the Quattrins consume over the course of the year. Based on the estimates to (a) and (b), does Dr. Quattrin owe PG&E for electricity at the end of the year or does PG&E owe Dr. Quattrin a refund?
(c) Electricity costs $0.28 per kW. Write an expression for amount due on the PG&E bill at time *t* months.

9. Dr. Quattrin's paternal grandmother's family originated in the Alpine town of Sauris, Italy, where the temperature in January changes at a rate of $W(t)$ degrees Celsius per hour. $W(t)$ is a twice-differentiable, increasing and concave up function with selected values in the table below. At midnight ($t=0$), the temperature in Sauris is $-8°C$.

t (in hours after midnight)	0	1	3	6	8
$W(t)$ (in degrees Celsius per hour)	-2.6	-3.1	-1.2	1.9	2.5

a) At approximately what rate is the rate of change of the temperature changing at 2am ($t=2$)? Include units.

b) Use a right Riemann sum with subdivisions indicated by the table to approximate $\int_0^8 W(t)\,dt$. Using correct units, explain the meaning of this value in the context of this problem.

c) Set up, but do not solve, an integral equation which would determine the temperature in Sauris at 1pm.

10. Handout of AP Questions:

Definite Integral Chapter Practice Test

1. Find $\int_1^4 \frac{6}{\sqrt{x}}\,dx$

a) 12 b) 8 c) 47 d) 24 e) 48

---

2. $\int_0^5 \frac{dx}{\sqrt{3x+1}}$

a) $\frac{1}{2}$ b) $\frac{2}{3}$ c) 1 d) 2 e) 6

---

3. $\int_1^2 \frac{1}{\sqrt{1-\frac{1}{4}t^2}}\,dt$

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{6}$ d) π e) $\frac{2\pi}{3}$

---

4. For $t \geq 0$ hours, H is a differentiable function of t that gives the change in temperature, in degrees Celsius per hour, at an Arctic weather station. Which of the following is the best interpretation of $\frac{1}{t}\int_0^t H(x)\,dx$?

a) The change in temperature during the first t hours.
b) The change in temperature during the first day.
c) The average change the temperature during the first t hours.
d) The rate at which the temperature is changing during the first day.
e) The rate at which the temperature is changing at the end of the 24^{th} day.

---

5. Which of the following is equal to $\int_0^{\pi} \cos x \, dx$?

a) $\int_0^{\pi} \sin x \, dx$ b) $\int_{-\pi/2}^{\pi/2} \cos x \, dx$ c) $\int_{-\pi/2}^{\pi/2} \sin x \, dx$

d) $\int_{\pi}^{2\pi} \sin x \, dx$ e) $\int_{\pi/2}^{3\pi/2} \cos x \, dx$

---

6. The following table lists the known values of a function $f(x)$.

x	1	2	3	4	5
$f(x)$	0	1.1	1.4	1.2	1.5

If the Right-Hand Riemann Sum is used to approximate $\int_1^5 f(x) \, dx$, the result is

a) 3.7 b) 4.5 c) 4.6 d) 5.2

e) none of these

---

7. Find the average rate of change of $y = x^2 + 5x + 14$ on $x \in [-1, 2]$

a) 3 b) 6 c) 9 d) $\frac{65}{6}$ e) 18

---

1. $\int_0^{\pi/9} \left(\tan 3x + \frac{x}{4+x^2} \right) dx$. Show the anti-derivatives.

---

2. Find the average value of $y = \frac{4}{x} \ln^3 x$ on $x \in [1, e]$. Show the anti-derivative.

---

3. Find the area between $y = xe^{3x^2}$ and the x-axis on $x \in [-2, 1]$. Show the anti-derivative.

4. Find the area on $x \in [0, 2]$ under $f(x) = \frac{1}{x^2+9} + \sin 4x$. Show the anti-derivative.

5. When a certain grocery store opens, it has 50 pounds of bananas on a display table. Customers remove bananas from the display table at a rate modeled by

$$f(t) = 10 + .8t\sin\left(\frac{t^3}{100}\right) \text{ for } 0 < t \le 12,$$

where $f(t)$ is measured in pounds per hour and t is the number of hours after the store opened. After the store has been open for three hours, store employees add bananas to the display table at a rate modeled by

$$g(t) = 3 + 2.4\ln\left(t^2 + 2t\right) \text{ for } 3 < t \le 12,$$

where $g(t)$ is measured in pounds per hour and t is the number of hours after the store opened.

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a) How many pounds of bananas are removed from the display table during the first 2 hours the store is open?

---

b) Find $f'(7)$. Using correct units, explain the meaning of $f'(7)$ in the context of the problem.

---

c) Is the number of pounds of bananas on the display table increasing or decreasing at time $t = 5$? Give a reason for your answer.

.

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d) How many pounds of bananas are on the display table at time $t = 8$?

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6. The C&A Smelter opened in Douglas, Arizona in 1904 to process copper ore from the Copper Queen Mine in Bisbee and the Nacozari Mine in Sonora. By 1907, the Smelter was processing 10,000 tons of ore per month. During a 10-hour shift on a given day, copper ore arrived at Douglas at a rate modeled by

$$E(t) = 18 + 71\cos^2 \frac{2}{3}t \text{ tons of ore per hour.}$$

Workers arrive at 4am and begin to process the ore. At that time, there were 17 tons of unprocessed ore left from the last shift.

The shift supervisor measures their rate of output every few hours and records the findings in the chart below.

t = time after midnight in hours	4	6	8	11	14
$P(t)$= Rate of ore processed in tons/hour	90	82.6	72.7	46.7	34.9

The supervisor notices that the workers' rate of processing decreased throughout the day.

---

a) How many tons of copper ore arrive at the C&A Smelter between 4am and 2pm?

b) Use a Right Riemann sum with subintervals indicated by the table to approximate $\int_{4}^{14} P(t)\,dt$. Using correct units, explain the meaning of this value in the context of the problem.

---

c) Is your approximation in part (b) an under- or over-approximation? Explain.

---

d) The workers end their shift at 2pm. Base on the estimates in (a) and (b) above, is there still unprocessed ore at the Smelter? Explain your reasoning.

Chapter 3 Answer Key

3.1 Free Response Answers

1.	$\frac{364}{3}$	2.	107.5	3.	0	4.	280.915	5.	$\frac{7}{8}$
6.	0	7.	1	8.	$3\ln 3$	9.	2	10.	$\frac{1}{\sqrt{2}}-1$
11.	$84\frac{2}{3}$	12.	2	13.	$\frac{1}{\sqrt{2}}$	14.	3	15.	$7\ln 2-1$
16.	$\frac{48844}{33}$	17.	-5	18.	-5	19.	-11	20.	12
21.	1	22.	7	23.	4	24.	27		
25.	-22	26.	$\frac{53}{2}$	27.	15	28.	$\frac{37}{10}$		
29.	$y^2 \sin y$	30.	$\sqrt{1+2x}$						
31.	$-\cos(x^2)$	32.	$-\frac{1}{x^2}\arctan \frac{1}{x}$	33.	$\frac{\cos\sqrt{x}}{2x}$	34.	$\sqrt{257}$		
35.	$2x\ln\left(x^4+1\right)$	36.	$e^{5\sqrt{x}}\left(\frac{1}{2}\sqrt{x}\right)$	37.	$4x^3\ln x$				
38.	$m\cos^2 m$	39.	$\frac{y}{\ln^4 y}$	40.	$-\left(e^{3x}+e^{x}+1\right)$				

3.1 Multiple Choice Answers:

1.	D	2.	B	3.	D	4.	B	5.	E	6.	B
7.	B	8.	C								

3.2 Free Response Answers:

1. $\frac{182}{9}$ 2. $\frac{671}{16}$ 3. 0 4. $\frac{3}{2}$

5. $\frac{20}{3}\left(19^{3/4}-1\right)$ 6. $\sqrt{11}-\sqrt{6}$ 7. $\frac{5}{2}\ln 5$ 8. $\frac{1}{2}\ln 3$

9. $\frac{1}{2}(e-1)$ 10. $-\frac{1}{3}\ln\frac{7}{8}$ 11. 2 12. 1

13. 2 $\frac{1}{2}$ 14. 4 15. $\frac{107}{10}$ 16. $\ln\left(\frac{\ln 4}{\ln 2}\right)$

17. $\tan^{-1}2-\frac{\pi}{4}$ 18. $\frac{9}{128}$ 19. $\frac{1}{2}$ 20. 1

21. 0 22. ln3 23. $\frac{1-\sqrt{2}}{2\sqrt{2}}$ 24. $\frac{1}{4}$

25. $-\frac{1}{10}$ 26. $-\frac{2}{7}$ 27. -2 28. $\frac{243}{5}$

29. 2 30. 3 31. $\frac{16}{3}$ 32. $\frac{2\sqrt{3}}{3}$

33. $\frac{4}{\pi}$ 34. $\frac{1}{2}\ln 3$ 35. $30\frac{4}{9}$ 36. 46.8

37. 0 38 $-\frac{1}{8}\left(e^{-25}-e^{-1}\right)$ 39. $\frac{3}{25}$ 40. $\frac{2}{17}$

41. $225.753°F$ 42. 8.401 hours 43. $63.045°F$

3.2 Multiple Choice Answers:

1.	B	2.	E	3.	C	4.	E	5.	A	6.	A
7.	C	8.	E	9.	B	10.	E	11.	C	12.	B
13.	E										

3.3 Free Response Answers:

1a. 4 1b. $\frac{17}{4}$ 2a. 4 2b. $\frac{23}{4}$

3a. 2 3b. 4 4a. $\frac{4}{3}$ 4b. $\frac{23}{12}$

5. 11.833 6. 4.917 7. 35.5 8. 3

9. $\ln 2$ 10. 2.704 11. 1.627 12. 9.519

13. 2.707 14. 9.804 15. (a) $-\frac{3}{2}m$ and (b) $\frac{41}{6}m$

16. (a) $-\frac{10}{3}m$ and (b) $\frac{98}{3}m$

17. $\int_0^a \left(e^{-x^2} - x\right)dx - \int_a^2 \left(e^{-x^2} - x\right)dx = 3.080$, where $a = 0.65291864$

18. $\int_0^b \left(e^{-x^2} - 2x\right)dx - \int_b^2 \left(e^{-x^2} - 2x\right)dx = 5.305$, where $b = 0.41936482$

19. 2.235

20. 13.030

3.3 Multiple Choice Answers:

1.	B	2.	C	3.	B	4.	C	5.	B

3.4 Free Response Answers:

1. $\int_5^{10} w'(t)\, dt$ represents the change of the child's weight in pounds from ages 5 to 10 years

2. $\int_0^{120} r(t)\, dt$ represents how much oil, in gallons, has leaked from the tank in the first 120 minutes.

3. Newtons-meters, or joules

4a. 11,977.905 *gallons* 4b. falling 4c. 3538.181 *gallons*

5a. 971 or 972 passengers 5b. decreasing

5c. 1921 or 1922 people

6a. 133.111 yottatons 6b. yes

6c. $t = 28.000,\ 50.376,\ 77.948,\ 91.465$

7a. 76.844 cg 7b. 4.143 cg

7c. The rate at which the liver is cleansing the blood is decreasing at a rate of $-1.213\ {}^{cg}\!/_{hr^2}$ when $t = 6$.

7d. $\int_0^t \left(8 - \frac{e^{0.47t}}{x+6}\right) - \left(7 - .46x\cos(x)\right) dx = 0$

8a. 1382 *cars* 8b) 2880.737 8c) Yes, between $t = 13$ and $t = 15$

9a) $P'(2) = 0.398$ hundreds of letters per hour per hour. The rate at which the 100s of letters per hour are coming intot he post office is increasing by approximately 40 letters per hr per hr.

9b) $\int_0^3 P(t)\,dt = 24.604\ \textit{hundreds of letters}$ or 2460 letters.

9c) Write an expression for $L(t) = 3 + \int_0^t \left[P(x) - 5\right]dx$, the total number of letters in the post office at time *t*.

10a) No, the water is NOT increasing. 10b) $A = 1310$

10c) $\int_0^k R(t)\ dt = 1310$

11a) $A = 70.571$ 11b) Falling 11c) $A = 122.026\ \textit{gallons}$

11d) No, the tank never overflows.

12. See AP Central

3.4 Multiple Choice Answers:

1.	E	2.	A	3.	D	4.	D	5.	B	6.	E
7.	C	8.	C	9.	E	10.	A	11.	D	12.	D

3.5 Free Response Solutions

1a. 7.25 2b. 7.25 2c. 30

2. 4940 miles

3a. 1880 km 3b. 840 km 3c. 1880 km 3d. 1300 km

4a) $1280\ {m^2}/{\min}$ 4b) $600\ {m^2}/{\min}$

5a) $10{,}190m$ 5b) $6850\ m$

6a. 392 6b. This would mean that he ran 392 km in these 24 seconds.

6c. $252\,km$

7a. 1,130 *gallons* 7b. 1,134 *gallons*

8a. 0.032 solar masses per cubic parsecs

8b. 0.0022 solar masses per gigayears per cubic parsecs

9a. 2.414 9b. 2.411

10a. .147 10b. .147

11. See AP Central

3.5 Multiple Choice Solutions

1.	C	2.	C	3.	E	4.	B	5.	C	6.	B
7.	E	8.	D	9.	C	10.	C	11.	C		

3.6 Free Response Answers:

1a) $\frac{3}{4}\, m^3/\min^2$ 1b) -8

1c) $1344\ \text{m}^3$ of water flows through the pipeline between $t=0$ and $t=48$ minutes.

1d) $\frac{1}{48}\int_0^{48} V(t)\,dt$ represents the average rate, in m^3/min, of the flow of water through the pipeline between $t=0$ and $t=48$ minutes.

2a) $0.55\, cm/day^2$.

2b) The plant's growth is increasing at an increasing rate rather than a decreasing because, while $H(3)$ being positive shows that the growth is increasing, $H'(3)$ being positive means the growth is occurring at an increasing rate.
2c) Over the four days, the plant grew 9.9 mm.
2d) $\frac{1}{4}\int_0^4 H(t)\,dt$ would be the average number of millimeters per day that the plan grew.

3a) $R'(30) \approx \frac{1}{2} \, {}^{gallons}\!/_{\min^2}$
3b) $\int_0^{90} R(t)\,dt \approx 4700\, gallons$
$\int_0^{90} R(t)\,dt$ it the total consumption of gallons of fuel between $t=0$ and $t=90$ minutes.
3c) $\frac{1}{70}\int_{20}^{90} R(t)\,dt \approx 50 \, {}^{gal}\!/_{\min}$
The average rate of consumption of fuel, in gallons per minute, between $t=0$ and $t=90$ minutes is 50.

4a) $-17 \, {}^{mg/dL}\!/_{day}$. Using the correct units, explain the meaning of the result.
4b) 992. $\frac{1}{7}\int_1^7 G(t)\,dt$ is the patient's average morning blood glucose level in mg/dL.

4c) -14.348. Yes, $M'(3.7)<0$ and $M''(3.7)>0$

5a) The patient's A1c score hase dropped 1.7 over the span from $t=0$ to $t=21$ months.
5b) $\int_0^{21} A(t)\,dt \approx 189.3$. $\frac{1}{21}\int_0^{21} A(t)\,dt \approx 9.01$ is the patient's average A1c score per month from $t=0$ to $t=21$ months.
5c) $=7.372$

6a) 766.3 kW
6b) 7920 kW
6c) The production is decreasing at a constant rate.

7a) $C_e{}'(3.4) = -117.9\, {}^{kW}\!/_{month}$; $C_g{}'(3.4) = -36.2\, {}^{therms}\!/_{month}$
Consumption of both commodities were decreasing
7b) 5075.3 kW
7c) 692.6 therms
7d) $\frac{1}{12}\int_0^{12} C_g(t)dt$ is the average gas consumption in therms per month during these 12 months.

8a) 4884 kW
8b) 5099.55 kW ; Dr Quattrin owes PG&E
8c) $E(t) = 0.28\int_0^t C_e(x)dx - 0.28\int_0^t P_e(x)\,dx$

9a) $W(2) \approx 0.45\, {}^{\circ C}\!/_{hr^2}$

9b) $\int_0^8 W(t)\,dt \approx 1(-3.1) + 2(-1.2) + 3(1.9) + 2(2.5) = 2.1°C$. The temperature in Sauris on this night has risen approximately $2.1°C$ between midnight and 8:00am.

9c) $T(1pm) = T(13) = -8 + \int_0^{13} W(t)\,dt$.

10. See AP Central

Integral Practice Test Answers

1. A 2. D 3. E 4. C 5. A 6. D

7. B

1. 0.246 2. $\frac{1}{e-1} = 0.582$ 3. 27128.812

4. $A = 1.336$ where $\int f(x)dx = \frac{1}{3}\tan^{-1}\frac{x}{3} - \frac{1}{4}\cos 4x + c$

5a. 20.051 *lbs*

5b. $f'(7) = -8.120\ lbs/hr/hr$

5c. Decreasing

5d. 23.347 *lbs*

6a. $\int_4^{14} E(t)\,dt = 551.813\ tons$

6b. Approximately 562.6 tons of copper were processed between 4 a.m. and 2 p.m.

6c. The approximation is an under-estimate because the data on the table show a decreasing function and right-hand Riemann rectangles under-estimate a decreasing function.

6d. Yes, there was still ore left at the end of the day because, while 555.4 tons of ore were processed, there were 568.813 tons on site—17 tons at the beginning of the day and 551.813 tons which were delivered.