Chapter 1:

Review of Derivatives

1.1: The Power and Exponential Rules with the Chain Rule

In PreCalculus, we developed the idea of the Derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the Derivative will be Explored. In this Chapter, we are primarily interested in how to find the Derivative and what it is used for.

Derivative—Def'n: $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

—Means: The function that yields the slope of the tangent line.

Numerical Derivative—Def'n: $f'(a) = \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$

—Means: The numerical value of the slope of the tangent line at x = a.

Symbols for the Derivative

$\dfrac{dy}{dx}$ = "d-y-d-x" $f'(x)$ = "f prime of x" y' = "y prime"

$\dfrac{d}{dx}$ = "d-d-x" D_x = "d sub x"

OBJECTIVES

Use the Power Rule and Exponential Rules to find Derivatives.
Find the Derivative of Composite Functions.

Key Idea from PreCalc: The derivative yields the slope of the tangent line. [But there is more to it than that.]

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in PreCalculus were the Exponential Rules. They look similar to each another, therefore it would be a good idea to view them together.

The Power Rule:	The Exponential Rules:
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}[e^x] = e^x$ $\frac{d}{dx}[a^x] = a^x \cdot \ln a$

The difference between these is where the variable is. The Power Rule applies when the variable is in the base, while the Exponential Rules apply when the variable is in the Exponent. The difference between the two Exponential rules is what the base is. e = 2.718281828459…, while a is any positive number other than 1.

Ex 1 Find a) $\frac{d}{dx}\left[x^5\right]$ and b) $\frac{d}{dx}\left[5^x\right]$

The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\frac{d}{dx}\left[x^5\right] = 5x^4$ b) $\frac{d}{dx}\left[5^x\right] = 5^x \ln 5$

There were a few other basic rules that we need to remember.

> D_x [constant] is always 0
>
> $D_x [cx^n] = (cn)x^{n-1}$
>
> $D_x [f(x)+g(x)] = D_x [f(x)] + D_x [g(x)]$

These rules allow us to easily differentiate a polynomial--term by term.

Ex 2 $y = 3x^2 + 5x + 1$; find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{d}{dx}[3x^2 + 5x + 1]$$
$$= 3(2)x^{2-1} + 5(1)x^{1-1} + 1(0)$$
$$= 6x + 5$$

Ex 3 $f(x) = x^2 + 4x - 3 + e^x$; find $f'(x)$

$$f'(x) = 2x + 4 + e^x$$

Ex 4 $y = \sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$; find $\frac{dy}{dx}$

$$y = \sqrt{x^3} + \frac{4}{\sqrt{x}} - \sqrt[4]{x^3} + e^4$$
$$= x^{3/2} + 4x^{-1/2} - x^{3/4} + e^4$$
$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} - 2x^{-3/2} - \frac{3}{4}x^{-1/4}$$

Note in Ex 4 that e^4 is a constant, therefore, its derivative is 0.

As we have seen, when the variable is in the Exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an Exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right] = \frac{d}{dx}\left[e^{x^2 \ln(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for later.

Ex 5 If $y=\left(x^2+1\right)\left(x^3-4x\right)$, find $\frac{dy}{dx}$.

$$y=\left(x^2+1\right)\left(x^3-4x\right)=x^5-4x^3+x^3-4x=x^5-3x^3-4x$$
$$\frac{dy}{dx}=5x^4-9x^2-4$$

Ex 6 If $y=\frac{x^2-4x+6}{\sqrt[3]{x}}$, find $\frac{dy}{dx}$.

$$y=\frac{x^2-4x+6}{\sqrt[3]{x}}=\frac{x^2-4x+6}{x^{1/3}}=x^{5/3}-4x^{2/3}+6x^{-1/3}$$
$$\frac{dy}{dx}=\frac{5}{3}x^{2/3}-\frac{8}{3}x^{-1/3}-2x^{-4/3}$$

1.1 Free Response Homework

Differentiate.

1. $f(x) = x^2 + 3x - 4$

2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$

3. $y = x^{-2/3}$

4. $y = 5e^x + 3$

5. $v(r) = \frac{4}{3}\pi r^3$

6. $g(x) = x^2 + \frac{1}{x^2}$

7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

9. $z = \frac{A}{y^{10}} + Be^y$

10. $y = e^{x+1} + 1$

11. If $f(x) = 3x^5 - 5x^3 + 3$, find $f'(x)$

12. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$

13. $\frac{d}{dx}\left[x^6 - 3\sqrt[6]{x^7} + 5^x - \frac{1}{\sqrt[3]{x^5}} + \frac{1}{2x}\right]$

14. $\frac{d}{dx}\left[x^4 - 14\sqrt[7]{x^9} + 8^x - \frac{1}{\sqrt[3]{x^7}} + \frac{1}{8x}\right]$

15. $\frac{d}{dx}\left[(x-1)\sqrt{x}\right]$

16. $\frac{d}{dz}\left[\left(z^2-4\right)\sqrt{z^3}\right]$

17. $\frac{d}{dx}\left[\left(x^2-4x+3\right)\sqrt{x^5}\right]$

18. $\frac{d}{dt}\left[\left(4t^2+1\right)\left(3t^3+7\right)\right]$

19. $\frac{d}{dy}\left(\frac{4y^3-2y^2-5y}{\sqrt{y}}\right)$

20. $\frac{d}{dv}\left(\frac{v^2-4v+7}{2\sqrt{v}}\right)$

21. $\frac{d}{dw}\left(\frac{7w^2-4w+1}{5w^3}\right)$

22. $\frac{d}{dw}\left(\frac{5w^2-3w-4}{7w^2}\right)$

1.1 Multiple Choice Homework

1. If $f(x)=x^{3/2}$, then $f'(4)=$

a) –6 b) –3 c) 3 d) 6 e) 8

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2. The derivative of $\sqrt{x}-\frac{1}{x\sqrt[3]{x}}$ is

a) $\frac{1}{2}x^{-1/2}-x^{-4/3}$

b) $\frac{1}{2}x^{-1/2}+\frac{4}{3}x^{-7/3}$

c) $\frac{1}{2}x^{-1/2}-\frac{4}{3}x^{-1/3}$

d) $-\frac{1}{2}x^{-1/2}+\frac{4}{3}x^{-7/3}$

e) $-\frac{1}{2}x^{-1/2}-\frac{4}{3}x^{-1/3}$

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3. Given $f(x)=\frac{1}{2x}+\frac{1}{x^2}$, find $f'(x)$.

a) $-\frac{1}{2x^2}-\frac{2}{x^3}$ b) $-\frac{2}{x^2}-\frac{2}{x^3}$ c) $\frac{2}{x^2}-\frac{2}{x^3}$

d) $-\frac{1}{2x^2}+\frac{2}{x^3}$ e) $\frac{1}{2x^2}-\frac{2}{x^3}$

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4. If $f(x)=e^{5x^2}+x^4$, then $f'(1)=$

a) e^5+1 b) $5e^4+4$ c) $5e^5+1$

d) $10e+4$ e) $10e^5+4$

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5. If h is the function defined by $h(x)=e^{5x}+x+3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

1.2: Composite Functions and the Chain Rule

The Chain Rule is one of the cornerstones of Calculus. It applies to any composite funtion.

Vocab:

Composite Function--A function made of two other functions, one within the other. For Example, $y=\sqrt{16x-x^3}$, $y=\sin x^3$, $y=\cos^3 x$, and $y=(x^2+2x-5)^3$. The general symbol is *f(g(x))*.

Ex 1 Given $f(x)=\cos^{-1}x$, $g(x)=x^2-1$, and $h(x)=\sqrt{1+x^2}$, find a) $f\left(g\left(\sqrt{2}\right)\right)$, b) $h(g(1))$, and c) $f(h(g(1)))$.

a) $g\left(\sqrt{2}\right)=\left(\sqrt{2}\right)^2-1=1$, so $f\left(g\left(\sqrt{2}\right)\right)=f(1)=\cos^{-1}(1)=0$.

b) $g(1)=0$, so $h(g(1))=h(0)=\sqrt{1+0^2}=1$.

c) $g(1)=0$ and $h(g(1))=h(0)=\sqrt{1+0^2}=1$, so $f(h(g(1)))=f(1)=\cos^{-1}(1)=0$

OBJECTIVE
Find the Derivative of Composite Functions.

How do you find the derivative of these composite functions? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. You end up with two open dolls next to each other.

The Chain Rule: $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$

If you think of the inside function as equaling *u*, we could write The Chain Rule like this:

$$\frac{d}{dx}\left[y=f(u)\right]=\frac{dy}{du}\cdot\frac{du}{dx}.$$

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other Rules, as it was in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:

$$\frac{d}{dx}\left[u^n\right]=nu^{n-1}\cdot\frac{du}{dx}$$

The Exponential Rules:

$$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right]=\left(a^u\cdot\ln a\right)\frac{du}{dx}$$

where *u* is a function of *x*.

Ex 2 $\frac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]$

$$\frac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]=10\left(4x^2-2x-1\right)^9(8x-2)$$

$$=20\left(4x^2-2x-1\right)^9(4x-1)$$

Ex 3 $\frac{d}{dx}\left[e^{4x^2}\right]$

$$\frac{d}{dx}\left[e^{4x^2}\right] = e^{4x^2} \cdot 8x = 8xe^{4x^2}$$

Ex 4 If $y = \sqrt{16x - x^3}$, find $\frac{dy}{dx}$.

$$y = \sqrt{16x - x^3} = \left(16x - x^3\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(16x - x^3\right)^{-1/2}\left(16 - 3x^2\right)$$
$$= \frac{\left(16 - 3x^2\right)}{2\left(16x - x^3\right)^{1/2}}$$

In this case, the $\sqrt{\ }$ is the f function and the polynomial $16x - x^3$ is the g. Each derivative is found by the Power Rule, but, as $16x - x^3$ is inside the $\sqrt{\ }$, it is inside the derivative of the $\sqrt{\ }$.

EX 5 $\frac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5 + 7}\right]$

$$\frac{d}{dx}\left[\sqrt{\left(x^2+1\right)^5+7}\right] = \frac{d}{dx}\left[\left(\left(x^2+1\right)^5+7\right)^{1/2}\right]$$
$$= \frac{1}{2}\left(\left(x^2+1\right)^5+7\right)^{-1/2} 5\left(x^2+1\right)^4 (2x)$$
$$= \frac{5x\left(x^2+1\right)^4}{\left(\left(x^2+1\right)^5+7\right)^{1/2}}$$

As opposed to PreCalculus and Algebra, Calculus does not stop with the Algebraic approach to most topics. We will also explore the graphical, numerical and verbal approaches. Here is a numerical view of derivatives:

1.2 Free Response Homework

1. $\frac{d}{dx}\left[x^3 + 4x - \pi\right]^{-7}$

2. $f(x) = \sqrt[4]{1 + 2x + x^3}$, find $f'(x)$

3. $f(x) = \sqrt[5]{\left(\frac{1}{x} + 2x + e^x\right)^3}$, find $f'(x)$

4. If $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$.

5. If $g(2) = 3$ and $g'(2) = -4$, find $f'(2)$ if $f(x) = e^{(g(x))}$.

6. $f(x) = \sqrt{4 - \frac{4}{9}x^2}; find f'\left(\sqrt{5}\right)$

7. $\frac{d}{dx}\left[\sqrt{3x^2 - 4x + 9}\right]$

8. $y = \sqrt[7]{x^3 - 2x}; find \frac{dy}{dx}$

9. If $f(x) = e^{\sqrt{9 - x^2}}$, find $f'(x)$

10. $y = e^{\sqrt{x}}$, find $\frac{dy}{dx}$.

11. $v(t) = \sqrt{\left[\left(\frac{E(t)}{3} + 3t\right)^{3/7} - 4\right]}$, find $v'(t)$.

12. $v(t) = \sqrt[3]{\left(\frac{C(t)}{7} + 4t^2\right)^{5/7} - 1}$, find $v'(t)$.

1.2 Multiple Choice Homework

1. If $f(x) = \cos^2(3 - x)$, then $f'(0) =$

a) $-2\cos 3$ b) $-2\sin 3\cos 3$ c) $6\cos 3$

d) $2\sin 3\cos 3$ e) $6\sin 3\cos 3$

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2. If $y = \left(x^4 + 4\right)^2$, then $\frac{dy}{dx} =$

a) $2\left(x^4 + 4\right)$ b) $\left(4x^3\right)^2$ c) $2\left(4x^3 + 4\right)$

d) $4x^3\left(x^4 + 4\right)$ e) $8x^3\left(x^4 + 4\right)$

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3. If $h(x) = [f(x)]^2 g(x)$ and $g(x) = 3$, then $h'(x) =$

a) $2f'(x)g'(x)$

b) $6f'(x)f(x)$

c) $g'(x)[f(x)]^2 + 2f(x)f'(x)g(x)$

d) $2f'(x)g(x) + g'(x)[f(x)]^2$

e) 0

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4. Which of the following statements must be true?

I. $\frac{d}{dx}\sqrt{e^{x}+3}=\frac{e^{x}}{2\sqrt{e^{x}+3}}$ II. $\frac{d}{dx}\left(5^{3x^{2}}\right)=6x\ln 5\left(5^{3x^{2}}\right)$

III. $\frac{d}{dx}\left(6x^{3}-\pi+\sqrt[3]{x^{8}}-\frac{2}{x^{3}}\right)=18x^{2}+\frac{8}{3}\sqrt[3]{x^{5}}+\frac{6}{x^{4}}$

a) I only b) II only c) III only

d) I and III only e) I, II, and III

1.3: Trig, Trig Inverse, and Log Rules

Trigonometric--Defn: "A function (sin, cos, etc.) whose independent variable represents an angle measure."

Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic--Defn: "The inverse of an exponential function."

Means: there is a Log or Ln in the equation.

$$\frac{d}{dx}\left[\sin u\right]=(\cos u)\ \frac{du}{dx} \qquad \frac{d}{dx}\left[\csc u\right]=(-\csc u\cot u)\ \frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos u\right]=(-\sin u)\ \frac{du}{dx} \qquad \frac{d}{dx}\left[\sec u\right]=(\sec u\tan u)\ \frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan u\right]=\left(\sec^2 u\right)\ \frac{du}{dx} \qquad \frac{d}{dx}\left[\cot u\right]=\left(-\csc^2 u\right)\ \frac{du}{dx}$$

$$\frac{d}{dx}\left[\ln u\right]=\left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\log_a u\right]=\left(\frac{1}{u\cdot\ln a}\right)\frac{du}{dx}$$

Note that all these Rules are Expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives involving Trig, Trig Inverse, and Logarithmic Functions.

Ex 1 $\frac{d}{dx}\left(\sin^3 x\right)$

$$\frac{d}{dx}\left(\sin^3 x\right) = 3\sin^2 x\cos x$$

Ex 2 $\frac{d}{dx}\left[\sin\left(x^3\right)\right]$

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \cos x^3\left(3x^2\right)$$

$$= 3x^2\cos x^3$$

Ex 3 $\frac{d}{dx}\left[\ln 4x^3\right]$

$$\frac{d}{dx}\left[\ln 4x^3\right] = \frac{1}{4x^3}\cdot 12x^2$$

$$= \frac{3}{x}$$

We could have also simplified algebraically before taking the derivative:

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 4 $\frac{d}{dx}\left(\sec^5 3x^4\right)$

$$\frac{d}{dx}\left(\sec^5 3x^4\right) = 5\sec^4 3x^4\left(\sec 3x^4\tan 3x^4\right)\left(12x^3\right)$$

$$= 60x^3\sec^5 3x^4\tan 3x^4$$

Ex 5 $\frac{d}{dx}\ln\left(\cos\sqrt{x}\right)$

$$\frac{d}{dx}\ln\left(\cos\sqrt{x}\right) = \frac{1}{\cos x^{1/2}} \cdot \left(-\sin x^{1/2}\right)\left(\frac{1}{2}x^{-1/2}\right)$$
$$= -\tan x^{1/2}\left(\frac{1}{2}x^{-1/2}\right)$$
$$= \frac{-\tan x^{1/2}}{2x^{1/2}}$$

General inverses are not all that interesting. We are more interested in particular TRANSCENDENTAL inverse functions, like the Ln. Another particular kind of inverse function that bears more study is the Trig Inverse Function. Interestingly, as with the Log functions, the derivatives of these Transcendental Functions become Algebraic Functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1}u\right] = \frac{1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1}u\right] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1}u\right] = \frac{-1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1}u\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1}u\right] = \frac{1}{u^2+1} \cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1}u\right] = \frac{-1}{u^2+1} \cdot D_u$$

Ex 6 $\frac{d}{dx}\left[\tan^{-1}3x^4\right]$

$$\frac{d}{dx}\left[\tan^{-1}3x^4\right] = \frac{1}{\left(3x^4\right)^2+1} \cdot \left(12x^3\right)$$
$$= \frac{12x^3}{9x^8+1}$$

Ex 7 $\frac{d}{dx}\left[\sec^{-1}x^2\right]$

$$\frac{d}{dx}\left[\sec^{-1}x^2\right]=\frac{1}{\left|x^2\right|\sqrt{\left(x^2\right)^2-1}}\cdot 2x$$
$$=\frac{2x}{\left(x^2\right)\sqrt{\left(x^2\right)^2-1}}$$
$$=\frac{2}{x\sqrt{x^4-1}}$$

General Inverse Derivative

$$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'\left[f^{-1}(x)\right]}$$

Ex 8 If $f(x)=x^2+2x+3$, $g(x)=f^{-1}(x)$, and $g(1)=2$; find $g'(1)$.

$f'(x)=2x+2\rightarrow f'(g(x))=2[g(x)]+2$

$\frac{d}{dx}\left[f^{-1}(x)\right]=\frac{1}{f'\left[f^{-1}(x)\right]}=\frac{1}{f'[g(x)]}=g'(x)$

$g'(1)=\frac{1}{f'\left[g(1)\right]}=2\left[g(1)\right]+2=6$

1.3 Free Response Homework

Find the derivatives of the given functions. Simplify where possible.

1. $y = \sin 4x$

2. $y = 4\sec x^5$

3. $f(t) = \sqrt[3]{1 + \tan t}$

4. $f(\theta) = \ln(\cos\theta)$

5. $y = a^3 + \cos^3 x$

6. $y = \cos\left(a^3 + x^3\right)$

7. $f(x) = \cos(\ln x)$

8. $f(x) = \sqrt[5]{\ln x}$

9. $f(x) = \log_{10}(2 + \sin x)$

10. $f(x) = \log_2(1 - 3x)$

11. $y = \sin^{-1}(e^x)$

12. $\frac{d}{dx}\left(\sin^{-1}\left(e^{3x}\right)\right)$

13. $y = \tan^{-1}\left(\sqrt{x}\right)$

14. $\frac{d}{dx}\left(\cot^{-1}\left(e^{2x}\right)\right)$

15. $y = \tan^{-1} x^2$, find y'

16. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x$

17. $\frac{d}{dx}\left(3e^{x^2 + 2x}\right)$

18. $\frac{d}{dx}\left(3\cos\left(x^2 + 2x\right)\right)$

19. $\frac{d}{dx}\left(\sqrt[3]{16 + x^3}\right)$

20. $\frac{d}{dx}\left(\sec^{-1}\left(2x^2\right)\right)$

21. If $g(x) = \ln\left(x^2 + 16\right)$, find $g'(x)$

22. If $f(x) = \left(x^2 + 1\right)^{3/2}$, find $f'(x)$.

23. $\frac{d}{dx}\left(\ln(\sec x)\right)$

24. $y = \cos x^2$, find y'

25. $f(x) = \ln(x^2 + 3)$, find $f'(x)$

26. $g(x) = \ln(x^2 - 4x + 4)$, find $g'(x)$

27. $h(x) = \sqrt{x^2 + 5}$, find $h'(x)$

28. $F(x) = \sqrt[3]{3x^2 - 6x + 1}$, find $F'(x)$

29. $y = \sin^{-1}(\cos x)$, find y'

30. $y = \sin(\cos^{-1} x)$, find y'

31. $\frac{d}{dx}\left(5e^{\tan(7x)}\right)$

32. $\frac{d}{dx}\left(\sqrt{\cos(1 - x^2)}\right)$

33. $\frac{d}{dx}\left(\ln^3(x^2 + 1)\right)$

34. $\frac{d}{dx}\left(\ln(\sin x^3)\right)$

35. $y = \tan^2(3\theta)$, find y'

36. $y = \cot^7(\sin\theta)$, find y'

The following table shows some values of $g(x)$, $g'(x)$, and $h(x)$, where $h(x) = g^{-1}(x)$.

x	$g(x)$	$h(x)$	$g'(x)$	$h'(x)$
1	2	3	$1/2$	$1/3$
2	5	-1	4	$1/5$
3	1	2	-2	$1/2$

37. Find $h'(1)$.

38. Find $g'(1)$

1.3 Multiple Choice Homework

1. If $y = \sin^{-1} e^{3\theta}$, then $\dfrac{dy}{d\theta} =$

a) $\dfrac{1}{\sqrt{1 - e^{3\theta}}}$ b) $\dfrac{e^{3\theta}}{\sqrt{1 - e^{6\theta}}}$ c) $\dfrac{e^{3\theta}}{\sqrt{1 - e^{9\theta^2}}}$

d) $-3e^{3\theta}\cos^{-1} e^{3\theta}$ e) $\dfrac{3e^{3\theta}}{\sqrt{1 - e^{6\theta}}}$

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2. If $f(x) = \tan^{-1}(\cos x)$, then $f'(x) =$

a) $\sec^{-2}(\cos x)$ b) $-\sin x \sec^{-2}(\cos x)$ c) $-\csc x$

d) $\dfrac{-\cos x}{1 - \sin^2 x}$ e) $\dfrac{-\sin x}{\cos(x^2) + 1}$

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3. If $h(x) = \ln(x^2)\tan^{-1}(x)$, then $h'(1) =$

a) $\dfrac{\pi}{4}$ b) $\dfrac{\pi}{4} + 1$ c) $\dfrac{\pi}{2}$ d) $\dfrac{\pi}{2} + 1$ e) $\dfrac{\pi}{2} + 2$

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4. If $f(t) = t\sqrt{1 - t^2} + \cos^{-1} t$, then $f'(t) =$

a) $\dfrac{t - 2}{2\sqrt{t^2 - 1}}$ b) $\dfrac{-2t^2}{\sqrt{1 - t^2}}$ c) $\dfrac{-2t^2 + 2}{\sqrt{1 - t^2}}$

d) $\dfrac{-1 - t^2}{\sqrt{1 - t^2}}$ e) $\dfrac{1 - t^2}{\sqrt{1 - t^2}}$

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5. If h is the function defined by $h(x) = e^{5x} + x + 3$, then $h'(0)$ is

a) 2 b) 4 c) 5 d) 6 e) 8

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6. Given that $f(x) = 8\sin^2(5x)$, find $f''\left(\frac{\pi}{30}\right)$

a) $40\sqrt{3}$ b) $40\sqrt{2}$ c) 40 d) 200 e) 0

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7. If $g(x) = \cos^2(2x)$, then $g'(x)$ is

a) $2\cos 2x \sin 2x$ b) $-4\cos 2x \sin 2x$

c) $2\cos(2x)$ d) $-2\cos(2x)$ e) $4\cos(2x)$

---

8. If $f(x) = \sin^2(3 - x)$, then $f'(0) =$

a) $-2\cos 3$ b) $-2\sin 3\cos 3$ c) $6\cos 3$

d) $2\sin 3\cos 3$ e) $6\sin 3\cos 3$

---

9. The function $f(x) = \tan(3^x)$ has one zero in the interval $[0,\ 1.4]$. The derivative at this point is

a) 0.411 b) 1.042 c) 3.451 d) 3.763 e) undefined

---

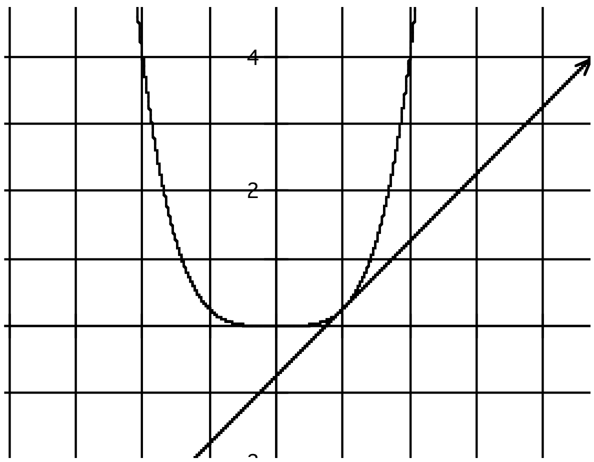
10. If $f(x) = \cos^2(3 - x)$, then $f'(0) =$

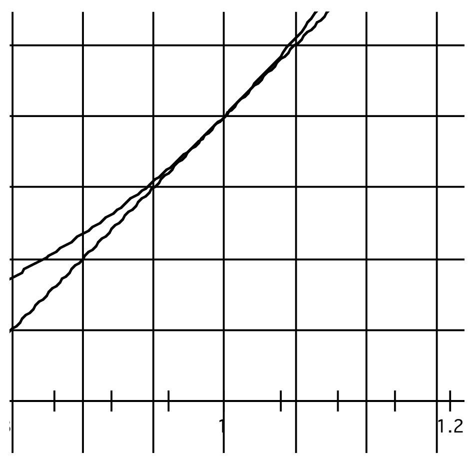
a) $-2\cos 3$ b) $-2\sin 3\cos 3$ c) $6\cos 3$

d) $2\sin 3\cos 3$ e) $6\sin 3\cos 3$

1.4: Derivatives in Context: Local Linearity and Rectilinear Motion

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, *y*-values on the line are very close to *y*-values on the function. This idea is called **local linearity**—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency. Consider the graphs of $y = .25x^4$ and its tangent line at $x = 1$, $y = x + .75$.





The closer you zoom in, the more the line and the curve become one. The *y*-values on the line are good approximations of the *y*-values on the curve. For a good animation of this concept, see

http://www.ima.umn.edu/~arnold/tangent/tangent.mpg

Since it is easier to find the *y*-value of a line arithmetically than for other functions—especially transcendental functions—the tangent line approximation is useful if you have no calculator.

OBJECTIVES
Use the equation of a tangent line to approximate function values.

Remember from PreCalculus:

The equation of a line is

$$y - y_1 = m(x - x_1)$$

This equation requires that, for a specific line, numbers are needed for the ordered pair (x_1, y_1) and the slope *m*.

For a tangent line, $f(x_1) = y_1$ and $m = f'(x_1)$

For a normal line, $f(x_1) = y_1$ and $m = \dfrac{-1}{f'(x_1)}$

Ex 1 Find the equations of the lines tangent and normal to $f(x) = x^4 - x^3 - 2x^2 + 1$ at x = -1

The slope of the tangent line will be f '(-1)

$$f'(x) = 4x^3 - 3x^2 - 4x$$

$$f'(-1) = -3$$

[Note that we could have gotten this more easily with the nDeriv function on our calculator.]

f(-1) = 1, so the tangent line will be

$$y - k = m(x - h)$$

$$y - 1 = -3(x + 1)$$

or

$$y = -3x - 2$$

The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope = 1/3. The normal line is

$$y - 1 = \frac{1}{3}(x + 1)$$

One of the uses of the tangent line is based on the idea of Local Linearity. This means that in small areas, algebraic curves act like lines—namely their tangent lines. Therefore, one can get an approximate y-value for points near the point of tangency by plugging x-values into the equation of the tangent line.

Ex 2 Use the tangent line equation found in Example 1 above to get an approximate value of $f(-0.9)$.

While we can find the Exact value of f(-0.9) with a calculator, we can get a quick approximation from the tangent line. If x=-.9 on the tangent line, then

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = .7$$

This example is somewhat trite in that we could have just plugged -0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but it is doable. Consider the next example, though.

Ex 3 Find the tangent line equation to $f(x) = e^{2x}$ at $x = 0$ and use it to approximate value of $e^{.2}$.

Without a calculator, we could not find the exact value of $e^{.2}$. In fact, even the calculator only gives an approximate value.

$$f'(x) = 2e^{2x} \text{ and } f'(0) = 2e^{2x} = 2$$
$$f(0) = e^0 = 1$$

So the tangent line equation is $y - 1 = 2(x - 0)$ or $y = 2x + 1$

$$e^{.2} \approx 2(.2) + 1 = 1.2$$

Note that the value that you get from a calculator for $e^{0.2}$ is 1.221403…
Our approximation of 1.2 seems very reasonable.

Though not as practically useful (in 2 dimensions) as the tangent lines, another context for the derivative is in finding the equation of the normal line.

Ex 4   Use the tangent line equation to $f(x) = \sqrt[3]{x}$ at $x = 8$ to approximate value of $\sqrt[3]{7}$.

$$f(8) = \sqrt[3]{8} = 2$$

$$f'(x) = \frac{1}{3}x^{-2/3} \text{ and } f'(8) = \frac{1}{3}(8)^{-2/3} = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$$

$$\text{So, } \; y - 2 = \frac{1}{12}(x - 8)$$

$$\text{Therefore, } \; \sqrt[3]{7} \approx y(7) = \frac{1}{12}(7 - 8) + 2 = \frac{23}{12}$$

Summary of Tangent Line Approximations

NB. a. Use $y - y_1 = m(x - x_1)$ as the general equation of the tangent line.

b. There are three unknowns— $x_1, y_1,$ *and* m. One of them will have a given value.

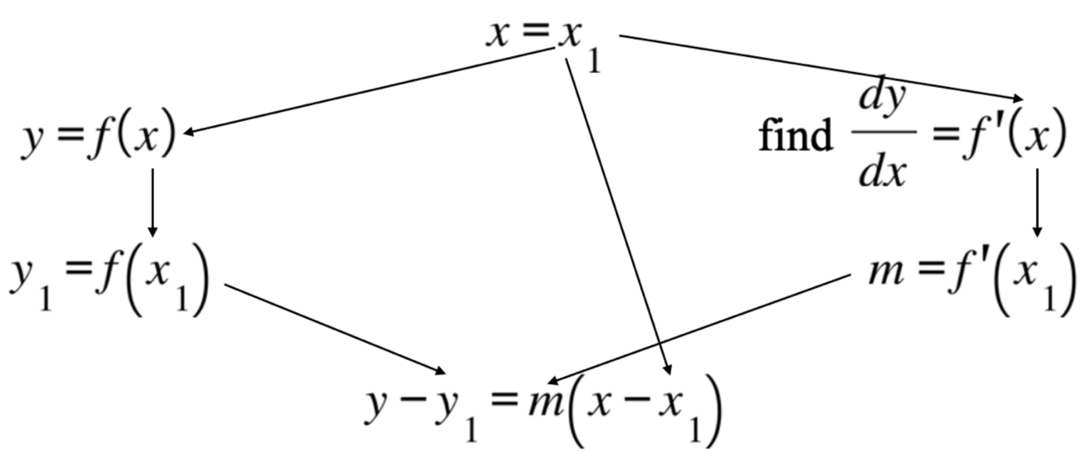
Steps

1. Find values for the other two unknowns.

2. Set up the specific tangent line equation.

3. Approximate $f(a)$ by substituting *a* for *x* in the tangent line equation and simplifying.

Summary of Tangent Line Approximations



$$f(a) \approx y = m(a - x_1) + y_1$$

Rectilinear Motion

A key application of the derivative as a rate of change is the application to motion. Typically, we refer to horizontal position in terms of $x(t)$ and vertical position in terms of $y(t)$. Since the derivative is a rate of change of a function and the rate of change of position is velocity, it should be pretty obvious that the derivative of position of velocity. Likewise, the derivative of velocity is acceleration since the rate of change of velocity is acceleration.

Vocabulary:

1. ***Rectilinear Motion –*** movement that occurs in a straight line
2. ***Parametric Motion –*** movement that occurs in two dimensions.
3. ***Velocity*** – Defn: directed speed
   Means: how fast something it is going and whether it is moving right or left, up or down
4. ***Average Velocity*** – Defn: distance traveled divided by time or $\dfrac{x_2 - x_1}{t_2 - t_1}$
   Means: the average rate, as used in algebra
5. ***Instantaneous Velocity*** – Defn: velocity at a particular time t
   Means: $\dfrac{ds}{dt}$, $\dfrac{dx}{dt}$, or $\dfrac{dy}{dt}$, or the rate at any given instant
6. ***Acceleration*** – the rate of change of the velocity or $\dfrac{dv}{dt}$

The basics about derivatives and motion were explored in PreCalculus.

Remember:

Position = $x(t) \quad or \quad y(t)$

$\int x'(t)\, dt \quad or \quad \int y'(t)\, dt$

Velocity = $x'(t) \quad or\ y'(t)$

$\int x''(t)\, dt \quad or \quad \int y''(t)\, dt$

Acceleration = $x''(t) \quad or \quad y''(t)$

Therefore:

Position =

Velocity =

Summary of Key Phases

When = solve for *t*
Where = solve for position
Which direction = is the velocity positive or negative
Speeding up or slowing down = are the velocity and acceleration in the same direction or opposite (do they have the same sign or not)

Things to remember from PreCalculus

1. The sign of the velocity determines the direction of the movement:

   Velocity > 0 means the movement is to the right (or up)
   Velocity < 0 means the movement is to the left (or down)
   Velocity = 0 means the movement is stopped.

2. Speeding up and slowing down is not determined by the sign of the acceleration.
   An object is speeding up when $v(t)$ and $a(t)$ have the same sign.
   An object is slowing when $v(t)$ and $a(t)$ have opposite signs.

Here is an example of a motion problem from PreCalculus which only uses derivatives.

EX 1 The position of a particle is described by $x(t) = t^3 - 3t^2 - 24t + 3$.

a) Where is the particle at $t = 3$?

b) When the particle is stopped?

c) Which direction it is moving at $t = 3$ seconds?

d) Find $a(3)$.

e) Is the particle speeding up or slowing down at $t = 3$ seconds?

a) "Where it is at $t = 3$" means find $x(3)$

$$x(3) = 3^3 - 3(3)^2 - 24(3) + 3 = -69$$

b) "When the particle is stopped" means "at what time is the velocity zero?"

$$v(t) = x'(t) = 3t^2 - 6t - 24 = 0$$
$$\frac{3t^2 - 6t - 24 = 0}{3}$$
$$t^2 - 2t - 8 = 0$$
$$(t + 2)(t - 4) = 0$$
$$T = -2 \text{ or } 4$$

c) "Which direction it is moving at $t = 3$ seconds" means what is the sign of the velocity?"

$$v(3) = 3(3)^2 - 6(3) - 24 = -15$$

The particle is moving left because the velocity is negative.

d) $a(3)$ means plug 3 into the acceleration equation.

$$a(t) = v'(t) = 6t - 6$$
$$a(3) = v'(t) = 6(3) - 6 = 12$$

e) From parts c) and d) above, $v(3) = -15$ and $a(3) = 12$. Since these have opposite signs, the particle is slowing down.

Ex 2 The position of a particle is described by $y(t) = t^3 - 6t^2 + 12$.

a) Find the position and velocity when the acceleration is zero.

$a(t) = 6t - 12 = 0 \rightarrow t = 2$
$y(2) = -4$
$v(2) = -12$

b) Find the position and acceleration when the velocity is zero.

$v(t) = y'(t) = 3t^2 - 12t = 3t(t-4) = 0 \rightarrow t = 0,\ 4$
$y(0) = 12$
$y(4) = -20$

$a(t) = v'(t) = 6t - 12$
$a(0) = -12$
$a(4) = 12$

1.4 Free Response Homework

1. Use the tangent line equation to $g(x) = x^3 - x^2 + 4x - 4$ at $x = -3$ to approximate the value of $g(-2.9)$

2. Use the equation of the tangent line to $f(x) = \frac{1}{2}x^4 + \frac{1}{3}x^3 + 2x^2 + 2x - 7$ at $x = -1$ to approximate the value of $f(-0.9)$.

3. Use the equation of the tangent line to $f(x) = \sqrt{6 - x}$ at $x = 2$ to approximate $\sqrt{4.1}$.

4. Use the equation of the tangent line to $f(x) = \sqrt{x + 5}$ at $x = 4$ to approximate $f(3.9)$.

5. Find the equation of the tangent line at $x = 2$ for $h(x) = \ln\left(9 - x^3\right)$. Use this to approximate $h(2.1)$

6. Find the equation of the tangent line at $x = 2$ for $g(x) = \ln\left(x^2 - 3\right)$. Use this to approximate $g(2.1)$

7. Find an equation of the line tangent to the curve $y = x^4 + 2e^x$ at the point $(0, 2)$.

8. Find the approximate value, using the tangent line at $x = 0$, of $f(0.08)$ if $f(x) = \sqrt{3 + e^x}$.

9. Use the equation of the tangent line to $f(x) = 2x + \cos(x - 2)$ at $x = 2$ to approximate $f(1.9)$.

10. Find the equation of the line tangent to $g(x) = 5 + 2x + \tan\left(x^2 - 1\right)$ when $x = 1$. Use this tangent line to find an approximation for $g(1.1)$.

11. Find the equation of the tangent line to $y = x + \cos x$ at the point $(0, 1)$.

12. Find the equation of the tangent line to $y = \sec x - 2\cos x$ at the point $\left(\pi/3, 1\right)$.

13. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

14. Find the equation of the line tangent to $y = \sec(2x) + \cot(2x)$ through the point $\left(\frac{\pi}{8}, 1 + \sqrt{2}\right)$. Use exact values in your answers.

15. At what point on the graph of $y = x^2 - 3x - 4$ is the tangent parallel to the line $5x - y = 3$?

16. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent parallel to the line $2x - 4y = 3$?

17. Find the equation of the line tangent to $f(x) = 2x^3 - 9x^2 - 12x$ where $f'(x) = 12$.

18. Find the equation of the line tangent to $f(x) = \frac{1}{5}x^5 + \frac{2}{3}x^3 - 8x$ where $f'(x) = 1$.

19. Find all points on the graph of $y = 2\sin x + \sin^2 x$ where the tangent line is horizontal.

20. Find the equation of the line tangent to $y = x + \ln x$ and parallel to $3x - y = 7$.

The position of a particle is described by the following distance equations. For each, find:

a) when the particle is stopped,
b) which direction it is moving at $t = 3$ seconds,
c) <u>where</u> it is at $t = 3$
d) $a(3)$
e) whether the particle is slowing down or speeding up at $t = 3$.

21. $x(t) = 2t^3 - 21t^2 + 60t + 4$

22. $x(t) = t^3 - 6t^2 + 12t + 5$

23. $y(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$

24. $y(t) = 12t^5 - 15t^4 - 220t^3 + 270t^2 + 1080t$

25. $x(t) = t^2 - 5t + 4$; find $x(t)$ when $v = 0$.

26. $x(t) = t^3 - 6t^2 - 63t + 4$; find $x(t)$ when $v = 0$.

27. $x(t) = 2t^3 - 21t^2 + 60t + 4$; find $x(t)$ and $v(t)$ when $a(t) = 0$.

28. $x(t) = 9t^4 - 4t^3 - 240t^2 + 576t - 48$; find $x(t)$ and $v(t)$ when $a(t) = 0$.

29. The motion of a particle is described by $y = 2t^3 - t^2 - 4t + 2$
a) Find the position and acceleration when $v(t) = 0$.
b) Find the position and velocity when $a(t) = 0$.

30. The motion of a particle is described by $y(t) = t^4 - 2t^2 - 8$
b) Find the position and acceleration when $v(t) = 0$.
b) Find the position and velocity when $a(t) = 0$.

1.4 Multiple Choice Homework

1. Let *f* be the function given by $f(x) = 2e^{4x^2}$. For what value of *x* is the slope of the line tangent to the graph of *f* at $(x, f(x))$ equal to 3?

a) 0.168 b) 0.274 c) 0.318 d) 0.342 e) 0.551

---

2. Line *l* is tangent to the graph of $y = e^x$ at the point (k, e^k). What is the positive value of *k* for which the *y*-intercept of *l* is $\frac{1}{2}$?

a) 0.405 b) 0.768 c) 1.500

d) 1.560 e) There is no value of *k*

---

3. If $f(x)$ is a differentiable function where $f(2)=1$ and the tangent line approximation at $x=2$ for $f(2.1)$ is 0.7, what is $f'(2)$?

a) 0.7 b) –3 c) 0.3 d) 7 e) –2

---

4. Using the line tangent to $y=\sqrt[4]{3x}$ at $x=27$, approximate $\sqrt[4]{90}$.

a) 3.070 b) 3.078 c) 3.080 d) 3.083 e) 3.105

---

5. Let $f(x)$ be the function with $f(1)=2$ and $f'(x)=\sqrt{x^2+3}$. Using the tangent line approximation to the graph of $f(x)$ at $x=1$, estimate $f(0.98)$.

a) 1.99 b) 1.98 c) 1.97 d) 1.96 e) 1.95

---

6. Let $f(x)$ be the function with $f(2)=4$ and $f'(x)=\sqrt{x^3+1}$. Using the tangent line approximation to the graph of $f(x)$ at $x=2$, estimate $f(2.2)$.

a. 4.0 b. 4.2 c. 4.4 d. 4.6 e. 4.8

---

7. If h is the function defined by $h(x)=x^2-5x+3$, what is the equation of the tangent line to the function when $h'(x)=-1$?

a) $y=-x-1$ b) $y=-x+24$ c) $y=-x-5$

d) $y=-7x+2$ e) $y=-7x$

---

8. Let f be the function defined by $f(x) = 4x^2 - 5x + 3$. Which of the following is an equation of the line tangent to the graph of f at the point where $x = -2$

a) $y = -21x - 13$ b) $y = -21x + 29$ c) $y = -21x - 71$

d) $y = -11x + 7$ e) $y = -11x - 51$

---

9. Which of the following is an equation of the line tangent to the graph of $f(x) = x^6 + x^5 + x^2$ at the point where $f'(x) = -1$?

a) $y = -3x - 2$

b) $y = -3x + 4$

c) $y = -x + 0.905$

d) $y = -x + 0.271$

e) $y = -x - 0.271$

---

10. At what point on the graph of $y = \frac{1}{2}x^2$ is the tangent parallel to the line $2x - 4y = 3$

a) $\left(\frac{1}{2}, -\frac{1}{2}\right)$ b) $\left(\frac{1}{2}, \frac{1}{8}\right)$ c) $\left(\frac{1}{2}, -\frac{1}{4}\right)$

d) $\left(1, -\frac{1}{2}\right)$ e) $(2, 2)$

---

11. An equation of the normal line to the curve $y = \sqrt[3]{x^2 - 1}$ at the point where $x = 3$ is

a) $y + 12x = 38$
b) $y - 4x = 10$
c) $y + 2x = 4$
d) $y + 2x = 8$
e) $y - 2x = -4$

---

12. A particle moves along a straight line with equation of motion $s = t^3 + t^2$. Find the value of t at which the acceleration is zero.

a) $-\frac{2}{3}$ b) $-\frac{1}{3}$ c) $\frac{2}{3}$ d) $\frac{1}{3}$ e) $-\frac{1}{2}$

---

13. A particle moves on the x-axis with velocity given by $v(t) = 3t^4 - 11t^2 + 9t - 2$ for $-3 \le t \le 3$. How many times does the particle change direction as t increases from –3 to 3?

a) Zero b) One c) Two

d) Three e) Four

---

14. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the velocity of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

15. A particle moves on the x-axis so that its position is given by $x(t) = t^2 - 6t + 5$. For what value of t is the acceleration of the particle zero?

a) 1 b) 2 c) 3 d) 4 e) No such value of t

---

16. Find the acceleration at time $t = 9$ seconds if the position (in cm.) of a particle moving along a line is $s(t) = 6t^3 - 7t^2 - 9t + 2$.

a) 310cm/sec^2 b) 310cm/sec c) 1323cm/sec^2

d) 1323cm/sec e) -1323cm/sec

1.5: The Product Rule

Remember:

The Product Rule: $\frac{d}{dx}\left[u \cdot v\right] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

OBJECTIVE
Find the derivative of the product of two functions.

Ex 1 $\frac{d}{dx}\left(x^2 \sin x\right)$

$$\frac{d}{dx}\left(x^2 \sin x\right) = x^2 \cos x + (\sin x)(2x)$$

$$= x^2 \cos x + 2x \sin x$$

Ex 2 $\frac{d}{dx}(5^x \cos x)$

$$\frac{d}{dx}(5^x \cos x) = 5^x(-\sin x) + \cos x(5^x \ln 5)$$

$$= 5^x\left((\ln 5)\cos x - \sin x\right)$$

The Product Rule is pretty straight forward. The tricky part is simplifying the Algebra.

Ex 3 If $f(x) = x^2 e^{-x/2}$, find $f'(x)$.

$$u = x^2 \qquad v = e^{-x/2}$$

$$\frac{du}{dx} = 2x \qquad \frac{dv}{dx} = e^{-x/2}\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-x/2}$$

$$f'(x) = x^2\left(-\frac{1}{2}e^{-x/2}\right) + e^{-x/2}(2x)$$

$$= xe^{-x/2}\left(-\frac{1}{2}x + 2\right)$$

Ex 4 $\frac{d}{dx}\left[x\sqrt{1-x^2}\right]$

$u = x$

$D_u = 1$

$v = \sqrt{1-x^2} = \left(1-x^2\right)^{1/2}$

$D_v = \frac{1}{2}\left(1-x^2\right)^{-1/2}(-2x) = \frac{-x}{\left(1-x^2\right)^{1/2}}$

$$\frac{d}{dx}\left[x\sqrt{1-x^2}\right] = (x)\frac{-x}{\left(1-x^2\right)^{1/2}} + \left(1-x^2\right)^{1/2}(1)$$

$$= \frac{-x^2 + \left(1-x^2\right)}{\left(1-x^2\right)^{1/2}}$$

$$= \frac{1-2x^2}{\left(1-x^2\right)^{1/2}}$$

<u>Steps to Simplifying a Product (or Quotient) Rule:</u>

1. Identify there is a product (or quotient) and identify *u* and *v*.
2. Using scratch work on the side, find *du* and *dv*.
3. Simplify the scratch work.
4. Plug *u, v, du* and *dv* into the formula.
5. Simplify the formula.
    - Factor
    - Common denominators
    - Reduce

Ex 5 $\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]$

$$u=(2x-3)^8 \qquad\qquad v=\left(3x^2-1\right)^7$$

$$\frac{du}{dx}=8(2x-3)^7(2) \qquad\qquad \frac{dv}{dx}=7\left(3x^2-1\right)^6(6x)$$

$$=16(2x-3)^7 \qquad\qquad =42x\left(3x^2-1\right)^6$$

$$\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]=(2x-3)^8 42x\left(3x^2-1\right)^8+\left(3x^2-1\right)^7 16(2x-3)^7$$

This, then, is factorable:

$$\begin{aligned}\frac{d}{dx}\left[(2x-3)^8\left(3x^2-1\right)^7\right]&=42x(2x-3)^8\left(3x^2-1\right)^6+16\left(3x^2-1\right)^7(2x-3)^7\\&=2(2x-3)^7\left(3x^2-1\right)^6\left[21x(2x-3)+8\left(3x^2-1\right)\right]\\&=2(2x-3)^7\left(3x^2-1\right)^6\left[42x^2-63x+24x^2-8\right]\\&=2(2x-3)^7\left(3x^2-1\right)^6\left(66x^2-63x-8\right)\end{aligned}$$

Remember that, in section 1.1, we said we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the Exponent. We can now address that situation.

Ex 6 $\frac{d}{dx}\left[(\cos x)^{x^2}\right]$

$$\begin{aligned}\frac{d}{dx}\left[(\cos x)^{x^2}\right]&=\frac{d}{dx}\left(e^{x^2\ln\cos x}\right)\\&=e^{x^2\ln\cos x}\left(x^2\frac{1}{\cos x}(-\sin x)+(\ln\cos x)2x\right)\\&=(\cos x)^{x^2}\left(2x\ln\cos x-x^2\tan x\right)\end{aligned}$$

1.5 Free Response Homework

Find the derivative of the following functions.

1. $y = t^3 \cos t$

2. $y = xe^{-x^2}$

3. $\frac{d}{dx}\left[xe^{-x}\right]$

4. $\frac{d}{dx}\left[xe^{2x}\right]$

5. $y = e^{-5x}\cos 3x$

6. $f(x) = x\sqrt{\ln x}$

7. $\frac{d}{dx}\left(x^3 \sec x\right)$

8. $D_x\left(x^2 \csc x\right)$

9. $D_x\left(x^2 \sin x + 2x\cos x\right)$

10. If $y = x\tan^2 x$, find $\frac{dy}{dx}$

11. If $f(x) = 2\sin^2 x\cos^2 x$, find $f'(x)$

12. $D_x\left(x^3 \sec x + x^2 \tan x\right)$

13. $f(x) = \sec x \tan x; \text{find} f'\left(\frac{\pi}{4}\right)$

14. $f(x) = x\cos x + x\sin x; \text{find} f'\left(\frac{\pi}{4}\right)$

15. $\frac{d}{dx}\left[\left(x^2 - 2x - 8\right)e^x\right]$

16. $\frac{d}{dx}\left[\left(4 - x^2\right)e^{2x}\right]$

17. $\frac{d}{dx}\left[\left(x^2 - 1\right)e^{-1/2x}\right]$

18. $\frac{d}{dx}\left[\left(x - x^3\right)e^x\right]$

19. $\frac{d}{dx}\left[x^2 e^{-4x}\right]$

20. $\frac{d}{dx}\left[\left(2x^2 + 5x + 2\right)e^{-3x}\right]$

21. $\frac{d}{dx}\left[e^x\sqrt{7 - x}\right]$

22. $\frac{d}{dx}\left[e^{-x}\sqrt{x + 4}\right]$

23. $\frac{d}{dx}\left[x\sqrt{4-x^2}\right]$

24. $\frac{d}{dx}\left[-x\sqrt{9-x^2}\right]$

25. $\frac{d}{dx}\left[(x^2)\sqrt{9-x^2}\right]$

26. $\frac{d}{dx}\left[(-x^2)\sqrt{1-x^2}\right]$

27. $y=(4x^5-3)^7(7x^2+1)^5$

28. $y=(2x-5)^4(8x^2-5)^{-3}$

29. $y=(3x^2-4)^3(6x^2+7)^2$

30. $g(x)=(1+4x)^5(3+x-x^2)^8$

31. $y=e^{x\cos x}$

32. $y=\sin^x x$

33. Find the equation of the line tangent to $y=x^2e^{-x}$ at the point $\left(1, 1/e\right)$.

34. $y=x^2\sqrt{5-x^2}$, find $y'(1)$

35. Write the equation of the line tangent to $f(x)=x\cdot\sqrt[4]{7+x^2}$ at $x=3$.

36. Write the equation of the line tangent to $f(x)=x\cdot\sqrt[3]{1-x^2}$ at $x=3$.

37. Find the equation of the lines tangent and normal to $y=x\sin\left(\frac{\pi}{2}\ln x\right)$ when $x=e$

38. Find the equation of the line tangent to $y=e^{x\sin(4x)}+2$ when $x=0$

39. Find the equation of the lines tangent and normal to $y=x\left(\frac{1}{x}\right)$ when $x=\frac{4}{\pi}$.

40. $H(x)=(1+x^2)\tan^{-1}(x)$

41. $y=x\cos^{-1}x-\sqrt{1-x^2}$

42. If $f(x)=e^x-x^2\arctan x$, find $f'(x)$.

43. $y = \cos^{-1}x + x\sqrt{1 - x^2}$

44. $y = \ln\left(x^2 + 4\right) - x\tan^{-1}\left(\frac{x}{2}\right)$

45. $y = 3\sin^{-1}\left(\frac{x}{3}\right) + \sqrt{9 - x^2}$

1.5 Multiple Choice Homework

1. If $y = x^2\cos 2x$, then $\frac{dy}{dx} =$

a) $-2x\sin 2x$ b) $-4x\sin 2x$ c) $2x(\cos 2x - \sin 2x)$

d) $2x(\cos 2x - x\sin 2x)$ e) $2x(\cos 2x + \sin 2x)$

---

2. If $x(t) = 2t\cos t^2$, find $x'(t)$.

a) $x(t) = -4t^2\sin t^2$ b) $x(t) = -4t^2\sin t^2 + 2\cos t^2$

c) $x(t) = \sin t^2 + 3$ d) $x(t) = -\sin t^2 + 4$

e) $x(t) = \sin t^2 + 2$

---

3. If $f(x) = x\tan x$, then $f'\left(\frac{\pi}{4}\right) =$

a) $1 - \frac{\pi}{2}$ b) $1 + \frac{\pi}{2}$ c) $1 + \frac{\pi}{4}$

d) $1 - \frac{\pi}{4}$ e) $\frac{\pi}{2} - 1$

---

4. If $y = x^2 e^{2x}$, then $\frac{dy}{dx} =$

a) $2xe^{2x}$ b) $4xe^{2x}$ c) $xe^{2x}(x+1)$

d) $2xe^{2x}(x+1)$ e) $xe^{2x}(x+2)$

---

5. The equation of the line **normal** to $y = 3x\sqrt{x^2+6} - 3$ at $(0, -3)$ is

a) $x - 3\sqrt{6}y = 9\sqrt{6}$
b) $x + 3\sqrt{6}y = -9\sqrt{6}$
c) $3\sqrt{6}x + y = -3$
d) $3\sqrt{6}x - y = -3$
e) $x + 3\sqrt{6}y = -3$

---

6. Which of the following statements must be true?

I. $\frac{d}{dx}(x\tan x) = x\tan x + x\sec^2 x$ II. $\frac{d}{dx}(x\ln x) = 1 + \ln x$

III. $\frac{d}{dx}\sqrt{1-x} = \frac{1}{2\sqrt{1-x}}$

a) I only b) II only c) III only

d) I and II only e) I, II, and III

---

7. Let f and g be differentiable functions with the following properties:

i. $g(x) < 0$ for all x

ii. $f(2) = 3$

If $h(x) = f(x)g(x)$, and $h'(x) = f(x)g'(x)$, then $f'(x) =$

a) $f'(x)$ b) $g(x)$ c) e^{-x}

d) 0 e) 3

1.6: The Quotient Rule

Remember:

The Quotient Rule: $\frac{d}{dx}\left[\frac{u}{v}\right]=\frac{v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^2}$

OBJECTIVE
Find the derivative of the quotient of two functions.

Ex 1 $\frac{d}{dx}\left(\frac{6x}{x^2+4}\right)$

$$u=6x,\text{ so}\frac{du}{dx}=6$$

$$v=x^2+4,\text{ so}\frac{dv}{dx}=2x$$

$$f'(x)=\frac{v\cdot\frac{du}{dx}-u\cdot\frac{dv}{dx}}{v^2}=\frac{\left(x^2+4\right)(6)-(6x)(2x)}{\left(x^2+4\right)^2}$$

$$=\frac{6x^2+24-12x^2}{\left(x^2+4\right)^2}$$

$$=\frac{24-6x^2}{\left(x^2+4\right)^2}$$

Ex 2 $\dfrac{d}{dx}\left(\dfrac{x^2+2x-3}{x-4}\right)$

$u = x^2+2x-3$, so $\dfrac{du}{dx} = 2x+2$

$v = x-4$, so $\dfrac{dv}{dx} = 1$

$$f'(x) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2} = \frac{(x-4)\cdot(2x+2) - (x^2+2x-3)\cdot 1}{(x-4)^2}$$

$$= \frac{2x^2-6x-8-x^2-2x+3}{(x-4)^2}$$

$$= \frac{x^2-8x-5}{(x-4)^2}$$

Ex 3 $\dfrac{d}{dx}\left(\dfrac{x^2-4x+3}{2x^2-5x-3}\right)$

Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right) = \frac{d}{dx}\left(\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right)$$

$$= \frac{d}{dx}\left(\frac{x-1}{2x+1}\right)$$

$$= \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)^2}$$

$$= \frac{3}{(2x+1)^2}$$

Ex 4 $\frac{d}{dx}\left(\frac{\cot 3x}{x^2+1}\right)$

$$\frac{d}{dx}\left(\frac{\cot 3x}{x^2+1}\right)=\frac{\left(x^2+1\right)\left(-\csc^2 3x\right)(3)-(\cot 3x)\,2x}{\left(x^2+1\right)^2}$$

$$=\frac{-3x^2\csc^2 3x-3\csc^2 3x-2x\cot 3x}{\left(x^2+1\right)^2}$$

As with the Product Rule, the difficulty with the Quotient Rule arises from the Algebra needed to simplify our answer.

Ex 5 If $y=\frac{4x}{\sqrt{x^2+4}}$, find $\frac{dy}{dx}$.

$$u=4x \qquad\qquad v=\left(x^2+4\right)^{1/2}$$

$$\frac{du}{dx}=4 \qquad\qquad \frac{dv}{dx}=\frac{1}{2}\left(x^2+4\right)^{-1/2}(2x)=\frac{x}{\left(x^2+4\right)^{1/2}}$$

$$\frac{dy}{dx}=\frac{\left(x^2+4\right)^{1/2}(4)-(4x)\left(\frac{x}{\left(x^2+4\right)^{1/2}}\right)}{\left(x^2+4\right)^1}$$

$$=\frac{\frac{4\left(x^2+4\right)}{\left(x^2+4\right)^{1/2}}-(4x)\left(\frac{x}{\left(x^2+4\right)^{1/2}}\right)}{\left(x^2+4\right)^1}$$

$$=\frac{4x^2+16-4x^2}{\left(x^2+4\right)^{3/2}}$$

$$=\frac{16}{\left(x^2+4\right)^{3/2}}$$

Ex 6 Find the equation of the tangent to $f(x) = \dfrac{x}{\sqrt{x^2+9}}$ at $x = -\sqrt{7}$.

As we recall, for the equation of a line, we need a point and a slope:

The point: $f\left(-\sqrt{7}\right) = \dfrac{-\sqrt{7}}{\sqrt{7+9}} = -\dfrac{\sqrt{7}}{4}$

The slope is the derivative at the given *x*-value:

$$\frac{dy}{dx} = \frac{\left(x^2+9\right)^{1/2}(1) - (x)\left(\dfrac{x}{\left(x^2+9\right)^{1/2}}\right)}{\left(x^2+9\right)^{1}}$$

Rather than simpifying the algebra, find the slope by substituting $x = -\sqrt{7}$
:

$$m_{\tan} = \frac{(7+9)^{1/2}(1) - \left(-\sqrt{7}\right)\left(\dfrac{-\sqrt{7}}{(7+9)^{1/2}}\right)}{(7+9)} = \frac{4-\frac{7}{4}}{16} = \frac{9}{64}$$

The tangent line equation:

$$y + \frac{\sqrt{7}}{4} = \frac{9}{64}\left(x + \sqrt{7}\right)$$

1.6 Free Response Homework

Find the derivative of the following functions.

1. $y = \left(\dfrac{x^2 - 3}{x^2 - 4}\right)$; find $\dfrac{dy}{dx}$

2. $D_x\left(\dfrac{3x^2 + 4x - 3}{x^2 - 9}\right)$

3. $f(x) = \dfrac{x^2 + 2x - 8}{x^2 - x - 3}$; find $f'(x)$

4. $\dfrac{d}{dx}\left(\dfrac{x^3 - 2x^2 - 5x + 6}{x + 2}\right)$

5. $\dfrac{d}{dx}\left(\dfrac{3x + 3}{x^3 + 1}\right)$

6. $y = \dfrac{x^2 + 2x - 3}{x - 4}$; find y'

7. $\dfrac{d}{dx}\left[\dfrac{x^5 - 12x^3 - 19x}{3x^3}\right]$

8. $D_x\left[\dfrac{x - 4}{x^2 - 9x + 20}\right]$

9. $\dfrac{d}{dx}\left(\dfrac{\tan x + 5}{\sin x}\right)$

10. $\dfrac{d}{dx}\left(\dfrac{\sin x}{1 - \cos x}\right)$

11. If $y = \dfrac{\tan x}{\cos x - 3}$, find $\dfrac{dy}{dx}$

12. $\dfrac{d}{dx}\left(\dfrac{x^2}{\cos x}\right)$

13. $y = \dfrac{\tan x - 1}{\sec x}$

14. $y = \dfrac{\sec^{-1} x}{x}$

15. $f(x) = \dfrac{\tan x}{\tan x + 1}$; find $f'\left(\dfrac{\pi}{4}\right)$

16. $y = \dfrac{\sin x}{x^2}$

17. $y = \dfrac{r}{\sqrt{r^2 + 1}}$

18. If $f(x) = \dfrac{x}{\ln x}$, find $f'(e)$.

19. Find the equation of the lines tangent and normal to $y = \frac{-2x}{x^2 + 16}$ at $x = -1$.

20. Find the equation of the line tangent to $y = \frac{x^2 - 3}{x^2 - 4}$ at $x = 1$.

21. Find the equation of the lines tangent and normal $y = \frac{-3x}{x^2 + 1}$ at $x = 1$.

22. Find the equation of the lines tangent and normal $y = \frac{x^2 - 4x + 3}{2x^2 - 5x - 3}$ at $x = 2$

1.6 Multiple Choice Homework

1. If f is a function that is differentiable throughout its domain and is defined by $f(x) = \frac{1 + e^x}{\sin(x^2)}$, then the value of $f'(0) =$

a) -1 b) 0 c) 1 d) e e) nonexistent

---

2. If $y = \frac{5x - 4}{4x - 5}$, then $\frac{dy}{dx} =$

a) $\frac{-9}{(4x - 5)^2}$ b) $\frac{9}{(4x - 5)^2}$ c) $\frac{40x - 41}{(4x - 5)^2}$

d) $\frac{40x + 41}{(4x - 5)^2}$ e) $\frac{5}{4}$

---

3. If $y = \frac{3 - 2x}{3x + 2}$, then $\frac{dy}{dx} =$

a) $\frac{12x + 2}{(3x + 2)^2}$ b) $\frac{12x - 2}{(3x + 2)^2}$ c) $\frac{13}{(3x + 2)^2}$

d) $\dfrac{-13}{(3x+2)^2}$ e) $-\dfrac{2}{3}$

---

4. If $y=\dfrac{3}{4+x^2}$, then $\dfrac{dy}{dx}=$

a) $\dfrac{-6x}{\left(4+x^2\right)^2}$

b) $\dfrac{3x}{\left(4+x^2\right)^2}$

c) $\dfrac{6x}{\left(4+x^2\right)^2}$

d) $\dfrac{-3x}{\left(4+x^2\right)^2}$

e) $\dfrac{3}{2x}$

---

5. An equation of the line normal to the graph of $y=\dfrac{3x+4}{4x-3}$ at $(1,\ 7)$ is

a) $25x+y=32$

b) $25x-y=18$

c) $7x-y=0$

d) $x-25y=-174$

e) $x+25y=176$

1.7: Higher Order Derivatives

What we have been calling the Derivative is actually the First Derivative. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. In this section, we will Explore the process of finding the higher order derivatives.

Second Derivative--Defn: The derivative of the derivative.

Just as with the First Derivative, there are several symbols for the 2nd Derivative:

Higher Order Derivative Symbols

Liebnitz: $\frac{d^2y}{dx^2}$= d squared y, d x squared; $\frac{d^3y}{dx^3}$;… $\frac{d^ny}{dx^n}$

Function: $f''(x)$ = f double prime of x; $f'''(x)$ **;** $f^{IV}(x)$;… $f^{n}(x)$

Combination: y" = y double prime

OBJECTIVE

Find higher order derivatives.

Ex 1 $\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]$

$$\frac{d^2}{dx^2}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right] = \frac{d}{dx}\left[\frac{d}{dx}\left[x^4 - 7x^3 - 3x^2 + 2x - 5\right]\right]$$

$$= \frac{d}{dx}\left[4x^3 - 21x^2 - 6x + 2\right]$$

$$= 12x^2 - 42x - 6$$

Ex 2 Find $\frac{d^3y}{dx^3}$ if $y = \sin 3x$

$$y = \sin 3x$$

$$\frac{dy}{dx} = \cos 3x \cdot 3 = 3\cos 3x$$

$$\frac{d^2y}{dx^2} = 3(-\sin 3x) \cdot 3 = -9\sin 3x$$

$$\frac{d^3y}{dx^3} = -9\cos 3x \cdot 3 = -27\cos 3x$$

More complicated functions, in particular Composite Functions, have a complicated process. When the Chain Rule is applied, the answer becomes a product or quotient. Therefore, the 2nd Derivative will require the Product or Quotient Rules as well as, possibly, the Chain Rule again.

Ex 3 $y = e^{3x^2}$, find y".

$$\frac{dy}{dx} = e^{3x^2} \cdot 6x = 6xe^{3x^2}$$

$$\frac{d^2y}{dx^2} = 6x\left(e^{3x^2} \cdot 6x\right) + e^{3x^2} \cdot 6$$

$$= 36x^2e^{3x^2} + 6e^{3x^2}$$

$$= 6e^{3x^2}\left(6x^2 + 1\right)$$

Ex 4 $y = \sin^3 x$, find y"

$$y' = 3\sin^2 x \cdot \cos x$$

$$y'' = 3\sin^2 x(-\sin x) + \cos x(6\sin x \cdot \cos x)$$

$$= 3\sin x\left(2\cos^2 x - \sin^2 x\right)$$

Ex 5 $f(x) = \ln\left(x^2 + 3x - 1\right)$, find f''(x).

$$f'(x) = \frac{1}{x^2 + 3x - 1}(2x + 3) = \frac{2x + 3}{x^2 + 3x - 1}$$

$$f''(x) = \frac{\left(x^2 + 3x - 1\right)(2) - (2x + 3)(2x + 3)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \frac{\left(2x^2 + 6x - 2\right) - \left(4x^2 + 12x + 9\right)}{\left(x^2 + 3x - 1\right)^2}$$

$$= \frac{-2x^2 - 6x - 11}{\left(x^2 + 3x - 1\right)^2}$$

Ex 6 $g(x) = \sqrt{4x^2 + 1}$, find g''(x).

$$g'(x) = \frac{1}{2}\left(4x^2 + 1\right)^{-1/2}(8x) = \frac{4x}{\left(4x^2 + 1\right)^{1/2}}$$

$$g''(x) = \frac{\left(4x^2 + 1\right)^{1/2}(4) - (4x)\left[\frac{1}{2}\left(4x^2 + 1\right)^{-1/2}(8x)\right]}{\left[\left(4x^2 + 1\right)^{1/2}\right]^2}$$

$$= \frac{\left(4x^2 + 1\right)^{1/2}(4) - \frac{16x^2}{\left(4x^2 + 1\right)^{1/2}}}{\left(4x^2 + 1\right)}$$

$$= \frac{\left(4x^2 + 1\right)(4) - 16x^2}{\left(4x^2 + 1\right)^{3/2}}$$

$$= \frac{4}{\left(4x^2 + 1\right)^{3/2}}$$

1.7 Free Response Homework

In #1-8, find the second derivative of the function.

1. $f(x) = x^5 + 6x^2 - 7x$

2. $h(x) = \sqrt{x^2 + 1}$

3. $y = \left(x^3 + 1\right)^{2/3}$

4. $H(t) = \tan 3t$

5. $g(t) = t^3 e^{5t}$

6. $y = e^{3x^2}$

7. $y = \sin^3 x$

8. $f(t) = t\cos t$

9. $\frac{d^2}{dx^2}\left[5x^4 + 9x^3 - 4x^2 + x - 8\right]$

10. $\frac{d^2}{dx^2}\left[4x^7 - 3x^5 + 3x^3 + 6x - 1\right]$

11. $y = \cos x^2$, find y''

12. $y = \tan^2 x$, find y''

13. $y = \sec 3x$, find $\frac{d^2y}{dx^2}$

14. $y = xe^{2x}$, find $\frac{d^2y}{dx^2}$

15. $f(x) = \ln\left(x^2 + 3\right)$, find $f''(x)$

16. $g(x) = \ln\left(x^2 - 4x + 4\right)$, find $g''(x)$

17. $h(x) = \sqrt{x^2 + 5}$, find $h''(x)$

18. $F(x) = \sqrt{3x^2 - 2x + 1}$, find $F''(x)$

19. $y = \frac{x^2 - 3}{x^2 - 10}$, find $\frac{d^2y}{dx^2}$

20. $y = \frac{3x + 3}{x^3 + 1}$, find $\frac{d^2y}{dx^2}$

21. $y = x^3 + x^2 - 7x - 15$

22. $y = 3x^4 - 20x^3 + 42x^2 - 36x + 16$

23. $y = \frac{-4x}{x^2 + 4}$

24. $y = \frac{x^2 - 1}{x^2 - 4}$

25. $y = x\sqrt{8 - x^2}$

26. $y = \frac{1}{2}x + \sin x$

27. $y = xe^{-x}$

28. $y = e^{-x^2}$

29. $y = \dfrac{x}{x^2 - 9}$

30. $y = 2x - x^{2/3}$

1.7 Multiple Choice Homework

1. If f and g are twice differentiable and if $h(x) = g(f(x))$, then $h''(x) =$

a) $g''(f(x))$

b) $g''(f(x))f''(x)$

c) $g''(f(x))\left[f'(x)\right]^2$

d) $g'(f(x))\left[f'(x)\right]^2 + f'(x)(f''(x))$

e) $g'(f(x))f''(x) + [f'(x)]^2 g''(f(x))$

---

2. Find $\dfrac{d^2y}{dx^2}$ if $y = \dfrac{x+2}{x-3}$

a) $\dfrac{-2}{(x-3)^2}$ b) 0 c) $\dfrac{10}{(x-3)^3}$ d) $\dfrac{2}{(x-3)^2}$ e). None of these

---

3. If $y = \ln(\cos x)$ and $0 \leq x \leq \pi$, then $\dfrac{d^2y}{dx^2}$ is

a) $-\tan x$ b) $-\sec^2 x$ c) $\tan x$

d) $\sec^2 x$ e) $\sec x \tan x$

---

4. If $y = \ln\left(x^2 + 4\right)$, then $\dfrac{d^2y}{dx^2}$ is

a) $\dfrac{1}{x^2 + 4}$ b) $\dfrac{2x}{x^2 + 4}$ c) $\dfrac{-2x^2 + 8}{x^2 + 4}$

d) $\dfrac{2x}{\left(x^2 + 4\right)^2}$ e) $\dfrac{-2x^2 + 8}{\left(x^2 + 4\right)^2}$

---

5. If $y = e^{x^2}$, then $\dfrac{d^2y}{dx^2} =$

a) e^{x^2} b) $2e^{x^2}\left(2x^2 + 1\right)$ c) $2xe^{x^2}$

d) $4x^2e^{x^2}$ e) $2e^{x^2}\left(2x^2 - 1\right)$

1.8: Intro to AP: Basic Derivatives Numerically and Graphically

Traditionally, Calculus was an algebraically heavy subject. One of the philosophical changes that the CollegeBoard made in the 1990s was to emphasize that the Calculus should be understood in a variety of modes. As they state in their enduring understanding:

- Students should be able to work with functions represented in a variety of ways: Graphical, numerical, verbal, and analytic (algebraic). They should understand the connections between these representations.

Later, they added that students should be able to verbalize their understanding and be able to communicate that understanding through proper writing. We will consider this later as we consider more context-oriented problems.

OBJECTIVE
Determine derivative values from numerical or graphical data.

Ex 1 Given this table of values, find $\frac{d}{dx}\left[f(g(x))\right]$ and $\frac{d}{dx}\left[g(f(x))\right]$ at $x=1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

These are two different, but similar problems, so let us consider them individually:

$\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x)$ at $x=1$,

$$\frac{d}{dx}\left[f(g(x))\right]=f'(g(1))g'(1)$$
$$=f'(2)(6)$$
$$=(5)(6)$$
$$=30$$

$$\frac{d}{dx}\left[g(f(x))\right] = g'(f(x))f'(x) \text{ at } x = 1,$$

$$\frac{d}{dx}\left[g(f(x))\right] = g'(f(1))f'(1)$$

$$= g'(3)(4)$$

$$= (9)(4)$$

$$= 36$$

Ex 2 If $g(2) = -5$ and $g'(2) = 4$, find $f'(2)$ if $f(x) = e^{g(x)} + g\left(x^3 - 6\right) + (g(x))^3$

Notice that while we do not actually know the function that *g* represents, we still can take its derivative, because we know the derivative of *g* is *g*'. Of course, the Chain Rule is still essential in this process.

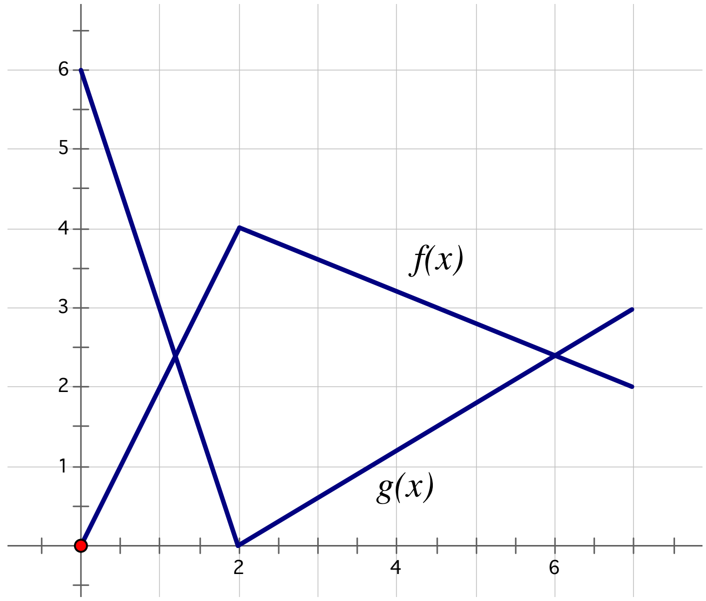
$$f(x) = e^{g(x)} + g\left(x^3 - 6\right) + (g(x))^3$$

$$f'(x) = e^{g(x)} \cdot g'(x) + g'\left(x^3 - 6\right) \cdot \left(3x^2\right) + 3(g(x))^2 \cdot g'(x)$$

$$f'(2) = e^{g(2)} \cdot g'(2) + g'\left(2^3 - 6\right) \cdot \left(3(2)^2\right) + 3(g(2))^2 \cdot g'(2)$$

$$f'(2) = e^{-5} \cdot 4 + g'(2) \cdot (12) + 3(-5)^2 \cdot 4$$

$$f'(2) = \frac{4}{e^5} + 4 \cdot (12) + 300 = \frac{4}{e^5} + 348$$



Ex 3 Given the graph above, find

(a) $u'(1)$ if $u = f(g(x))$
(b) $v'(1)$ if $v = g(f(x))$
(c) $w'(1)$ if $w = f(x)g(x)$

(a) $u'(1)$ if $u = f(g(x))$

$\Rightarrow u'(x) = f'(g(x)) \cdot g'(x)$

$\Rightarrow u'(1) = f'(g(1)) \cdot g'(1)$

$\Rightarrow u'(1) = f'(3) \cdot (-3)$

$\Rightarrow u'(1) = \left(-2/5\right) \cdot (-3) = 6/5$

(b) $v'(1)$ if $v = g(f(x))$

$\Rightarrow v'(x) = g'(f(x)) \cdot f'(x)$

$\Rightarrow v'(1) = g'(f(1)) \cdot f'(1)$

$\Rightarrow v'(1) = g'(2) \cdot 2$

$\Rightarrow$ dne

Note that $g'(2)$ does not exist. The slope cannot be determined at $x = 1$ because the slopes to the left and right of $x = 1$are different. The is called a corner point or a cusp point and will be explored further in a later chapter.

(c) $w'(1)$ if $w = f(x)g(x)$

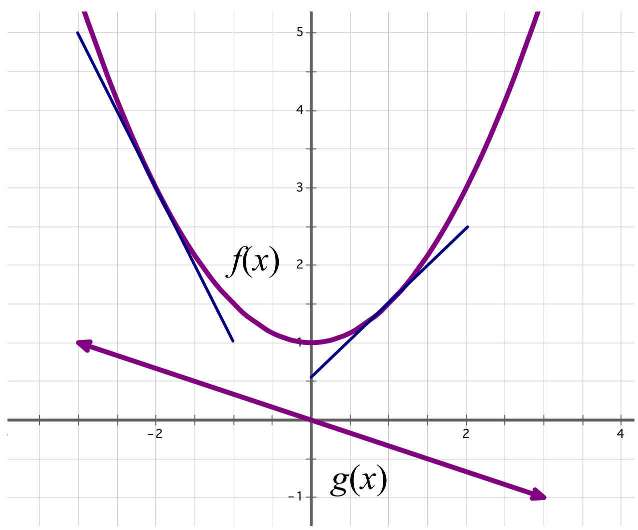
$$\Rightarrow w'(x) = f'(x)(g(x)) + g'(x)f(x)$$

$$\Rightarrow w'(1) = f'(1)(g(1)) + g'(1)f(1)$$

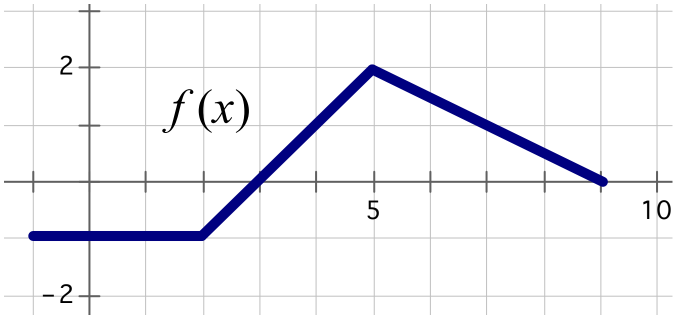
$$\Rightarrow w'(1) = (2)(3) + \left(-\frac{1}{3}\right)(2)$$

$$\Rightarrow w'(1) = 6 + \left(-2/3\right) = 16/3$$

Ex 4 The figure below shows the graph of the functions *f* and *g*. The graphs of the lines tangent to the graph of *f* at $x = -2$ and $x = 1$ are also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $-\frac{5}{6}$ b) $-\frac{1}{2}$ c) $\frac{1}{6}$ d) $\frac{1}{3}$ e) $\frac{1}{2}$



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

Ex 5 Let $f(x)$ be the function whose graph is given above and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above. Furthermore, let h be the function defined by $h(x)=\ln(x^2+4)$.

(a) Find the equation of the line tangent to $f(x)$ at $x=4$.

(b) Let K be the function defined by $K(x)=h(f(x))$. Find $K'(3)$.

(c) Let M be the function defined by $M(x)=g(x)\cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x)=\dfrac{g(x)}{h\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

(a) $f(4)=1$ and $f'(4)=1$. The tangent line equation is $y-1=1(x-4)$.

(b) $h(x)=\ln(x^2+4)\rightarrow h'(x)=\dfrac{2x}{x^2+4}$

$K(x)=h(f(x))\rightarrow K'(x)=h'(f(x))\cdot f'(x)$

$K'(3)=h'(f(3))\cdot f'(3)=h'(0)\cdot f'(3)=(0)\cdot(1)=0$

(c) $M(x)=g(x)\cdot f(x)\rightarrow M'(x)=g(x)\cdot f'(x)+g'(x)\cdot f(x)$

$M'(6)=g(6)\cdot f'(6)+g'(6)\cdot f(6)=(6)\cdot\left(\frac{1}{2}\right)+(12)\cdot\left(\frac{3}{2}\right)=15$

(d) $J(x)=\dfrac{g(x)}{h\left(\frac{1}{2}x\right)}\rightarrow J'(x)=\dfrac{h\left(\frac{1}{2}x\right)g'(x)-g(x)\cdot h'\left(\frac{1}{2}x\right)\left(\frac{1}{2}\right)}{\left[h\left(\frac{1}{2}x\right)\right]^2}$

$$J'(8) = \frac{h(4)g'(8) - g(8) \cdot h'(4)\left(\frac{1}{2}\right)}{\left[h(4)\right]^2} = \frac{8\ln 8 - \frac{4}{5}}{\ln^2 8}$$

1.8 Free Response Homework

1. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a. $g'(2)$, where $g(x)=[f(x)]^3$ b. $h'(2)$, where $h(x)=f(x^3)$

For problems 14 – 19, use the values of $f(x)$ and $g(x)$ given on the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	2	8	4	3
8	8	-12	2	4

2. If $h(x)=f(g(x))$, find $h'(8)$

3. If $h(x)=f(g(x))$, find $h'(2)$

4. If $k(x)=g(f(x))$, find $k'(2)$

5. If $m(x)=f(f(x))$, find $\mathrm{m}'(4)$

6. $P_1(x)=f(x)g(x)$, find $P_1'(2)$

7. $P_1(x)=f(x)g(x)$, find $P_1'(8)$

8. $P_2(x)=f(2x)g(x)$, find $P_2'(2)$

9. $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3{}'(4)$

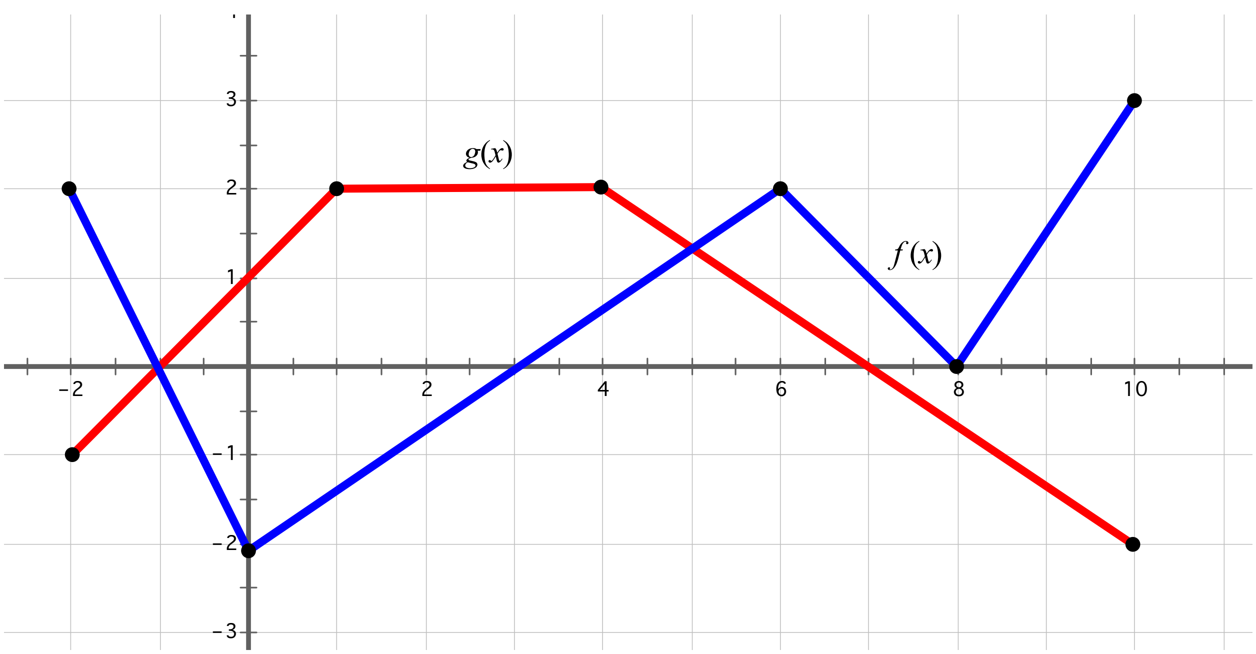
10. $Q_1(x) = \frac{f(x)}{g(x)}$, find $Q_1{}'(2)$

11. $Q_2(x) = \frac{g(x)}{f(x)}$, find $Q_2{}'(8)$

12. $Q_3(x) = \frac{f(2x)}{g(x)}$, find $Q_3{}'(4)$

13. $Q_2(x) = \frac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4{}'(4)$

For problems 14 – 25, the graphs of $f(x)$ and $g(x)$ are given below.



14. $u'(2)$ if $u = f(g(x))$

15. $v'(4)$ if $v = g(f(x))$

16. $w'(6)$ if $w = g(g(x))$

17. $t'(8)$ if $t = f(f(x))$

18. $P_1(x) = f(x)g(x)$, find $P_1'(2)$

19. $P_1(x) = f(x)g(x)$, find $P_1'(8)$

20. $P_2(x) = f(2x)g(x)$, find $P_2'(2)$

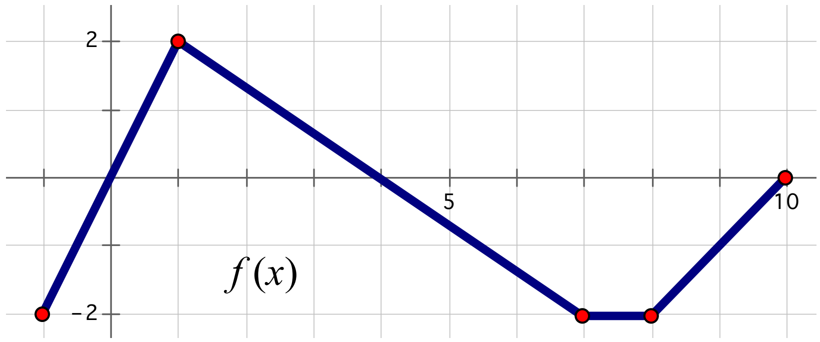
21. $P_3(x) = f(x)g\left(\frac{1}{2}x\right)$, find $P_3'(2)$

22. $Q_1(x) = \dfrac{f(x)}{g(x)}$, find $Q_1'(2)$

23. $Q_2(x) = \dfrac{g(x)}{f(x)}$, find $Q_2'(8)$

24. $Q_3(x) = \dfrac{f(2x)}{g(x)}$, find $Q_3'(4)$

25. $Q_2(x) = \dfrac{g\left(\frac{1}{2}x\right)}{f(2x)}$, find $Q_4'(4)$



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

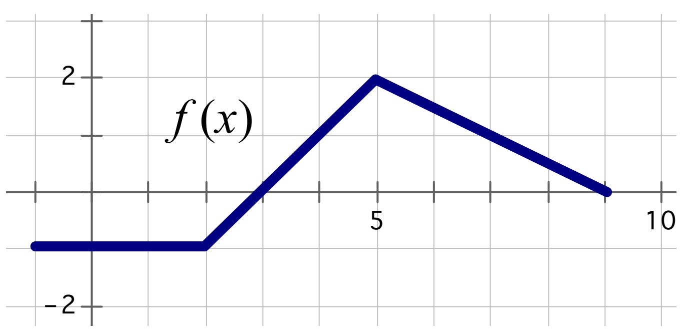
26. Let $f(x)$ be the function whose graph is given above and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above.

a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

b) Let K be the function defined by $K(x) = g(f(x))$. Find $K'(1)$.

c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

d) Let J be the function defined by $J(x) = \dfrac{f(2x)}{g(x)}$. Find $J'(2)$.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

27. Let $f(x)$ be the function whose graph is given above and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above.

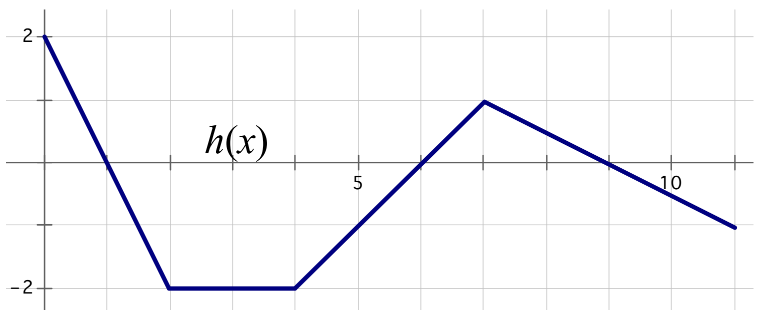
a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

b) Let K be the function defined by $K(x) = g(g(x))$. Find $K'(8)$.

c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(4)$.

d) Let J be the function defined by $J(x) = \dfrac{g(2x)}{f(x)}$. Find $J'(1)$.

$$f(x) = 4x - x^3$$



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

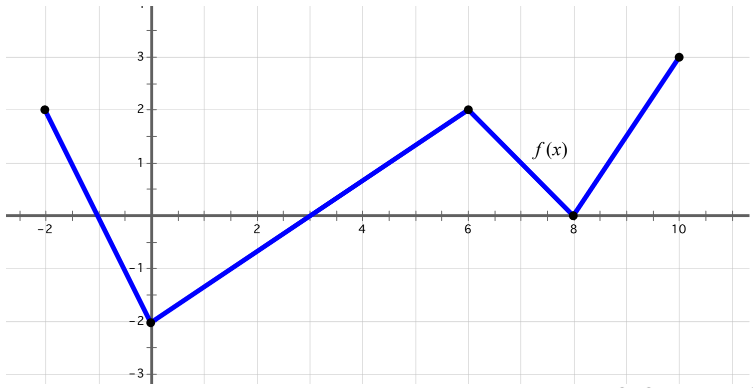
28. Let $f(x)$ be the function defined by the equation above, let $h(x)$ be the function whose graph is given above, and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above.

(a) Find the equation of the line tangent to $g(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(1)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{f(x)}$. Find $J'(4)$.



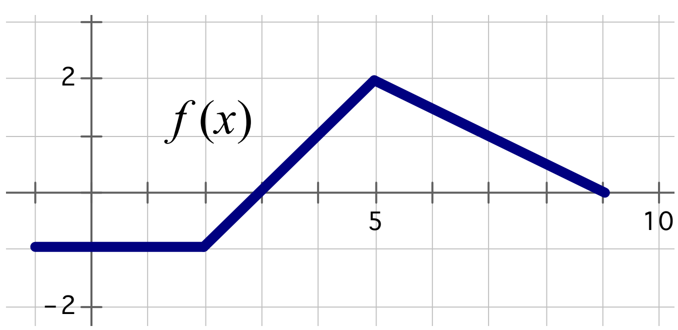
x	$G(x)$	$G'(x)$
-4	3	2
-2	5	-1
0	7	0
2	5	-1
4	3	2

29. Let h be the function defined by $h(x) = \sin(x) + e^{\cos 3x}$. Let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g(x)$ given on the table above. And let $f(x)$ be the function whose graph is given above.

(a) Find the equation of the line tangent to $h(x)$ at $x = \dfrac{\pi}{2}$.

(b) Let K be the function defined by $K(x) = f(h(x))$. Find $K'\left(\dfrac{\pi}{2}\right)$.

(d) Let J be the function defined by $J(x) = g(2x) \cdot f(x)$. Find $J'(2)$.



x	$g(x)$	$g'(x)$
0	-1	1
2	1	3
4	3	6
6	6	12
8	4	8

30. Let $f(x)$ be the function whose graph is given above and let $g(x)$ be a differentiable function with selected values for $g(x)$ and $g'(x)$ given on the table above. Furthermore, let h be the function defined by $h(x) = \ln(x^2 + 4)$.

(a) Find the equation of the line tangent to $f(x)$ at $x = 4$.

(b) Let K be the function defined by $K(x) = h(f(x))$. Find $K'(3)$.

(c) Let M be the function defined by $M(x) = g(x) \cdot f(x)$. Find $M'(6)$.

(d) Let J be the function defined by $J(x) = \dfrac{g(x)}{h\left(\frac{1}{2}x\right)}$. Find $J'(8)$.

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31. 2017 AP Calculus AB #6

1.8 Multiple Choice Homework

1. Let the function f be differentiable on the interval $[0, 2.5]$ and define g by $g(x) = f(f(x))$. Use the table to find $g'(1.5)$.

x	0.0	0.5	1.0	1.5	2.0	2.5
$f(x)$	0.5	1.5	2.0	2.5	1.0	0.0
$f'(x)$	0.1	0.3	0.6	1.1	2.0	2.2

a) 0.0 b) 1.24 c) 1.65 d) 2.08 e) 2.42

---

2. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given on the table below, find $h'(2)$, given that $h(x) = g(x) \cdot f(x)$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

a) -12 b) -2 c) 0 d) 30 e) 64

---

3. Let $f(x)$ and $g(x)$ be differentiable functions. The table below gives the values of $f(x)$ and $g(x)$, and their derivatives, at several values of x.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	-6
2	1	8	-5	7
3	7	-2	7	9

If $h(x) = \dfrac{f(x)}{g(x)}$, what is the value of $h'(2)$?

a) -4 b) -63 c) 51 d) $-\dfrac{47}{64}$ e) $-\dfrac{33}{64}$

---

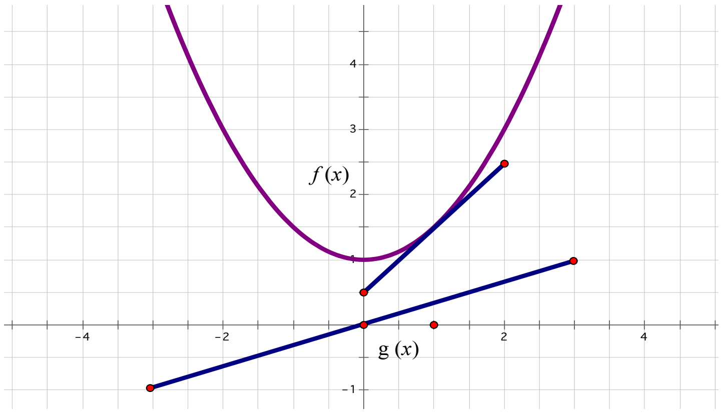
x	1	2	4	8
$f(x)$	-3	4	9	-1
$g(x)$	0	6	2	1
$f'(x)$	9	-4	3	2
$g'(x)$	10	1	3	5

4. Let $h(x) = g(x) \cdot f(x^3)$. What is the value of $h'(2)$?

(A) -6 (B) 2 (C) 11 (D) 24 (E) 143

---

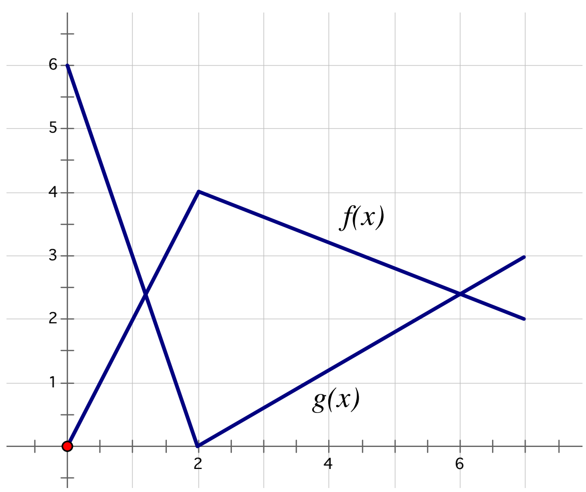
5. The figure below shows the graph of the functions f and g. The graphs of the lines tangent to the graph of f at $x = -2$ and $x = 1$ are also shown. If $B(x) = f(x) \cdot g(x)$, what is $B'(1)$?



a) $\frac{5}{6}$ b) $-\frac{1}{2}$ c) $-\frac{1}{6}$ d) $\frac{1}{3}$ e) $\frac{7}{6}$

---

6. The figure below shows the graph of the functions f and g. If $B(x) = \dfrac{g(x)}{f(x)}$, what is $B'(1)$?



a) $-\frac{5}{3}$ b) $\frac{20}{27}$ c) $\frac{4}{3}$ d) $\frac{16}{27}$ e) -3

---

7. The figure below shows the graph of the functions *f* and *g*. The graphs of the lines tangent to the graph of *g* at $x = -3$ and $x = 1$ are also shown. If $B(x) = f(g(x))$, what is $B'(1)$?

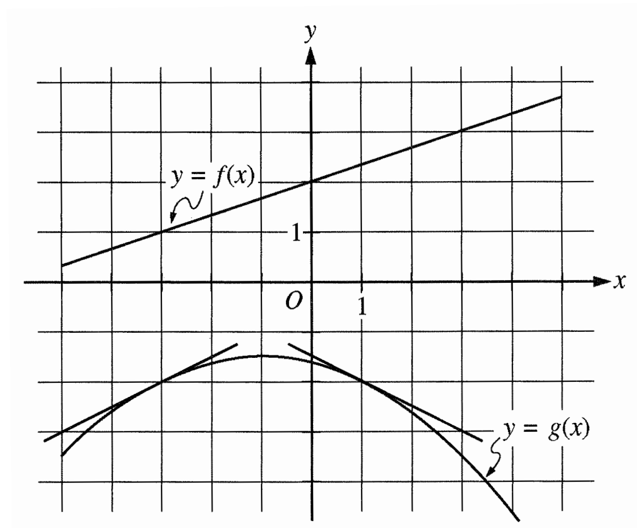
a) $-\frac{1}{2}$

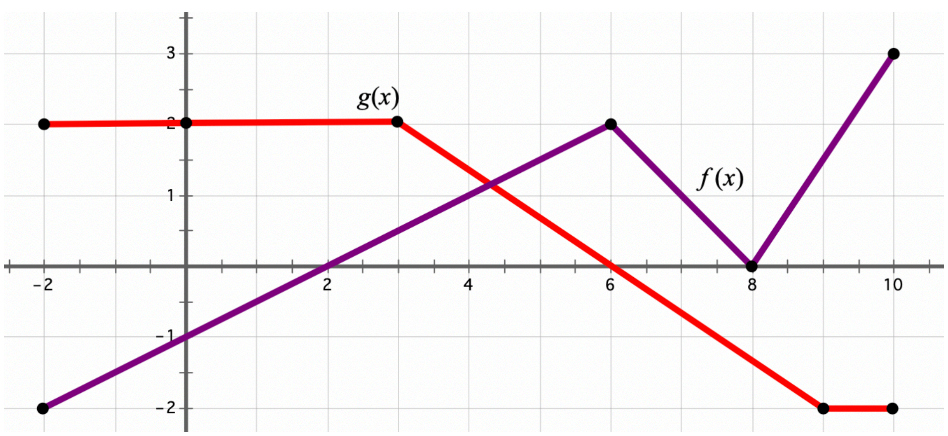
b) $-\frac{1}{6}$

c) $\frac{1}{6}$

d) $\frac{1}{3}$

e) $\frac{1}{2}$





8. Given the graphs of the two functions above and the fact that $B(x) = f(g(x))$, $B'(4) =$

a) 0 b) 1 c) 1/2 d) -1/3 e) dne

Derivative Review Practice Test

1. If $y = \ln(\sin x)$ and $0 \leq x \leq \pi$, then $\frac{dy}{dx}$ is

a) $-\tan x$ b) $-\cot x$ c) $\tan x$

d) $\cot x$ e) $\csc x$

---

2. If $y = \sin^{-1}\left(e^{2x}\right)$, then $\frac{dy}{dx}$ is

a) $\frac{2e^{2x}}{\sqrt{1-e^{4x}}}$ b) $\frac{e^{2x}}{\sqrt{1-e^{4x}}}$ c) $\frac{2e^{2x}}{\sqrt{1+e^{4x}}}$

d) $\frac{e^{2x}}{1+e^{4x}}$ e) $\frac{2e^{2x}}{\sqrt{e^{4x}-1}}$

---

3. Given the functions $f(x)$ and $g(x)$ that are both continuous and differentiable, and that have values given on the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	-2	8	1
4	10	8	4	3
8	6	-12	2	4

Given that $h(x) = g(g(x))$, $h'(8) =$

a) 1 b) 2 c) 3 d) 4 e) 8

---

4. If $g(x) = \tan^2(e^x)$, then $g'(x)$ is

a) $2\tan(e^x)\sec^2(e^x)$ b) $2e^x\tan(e^x)\sec^2(e^x)$

c) $2\tan^2(e^x)\sec(e^x)$ d) $e^x\sec^2(e^x)$ e) $2e^x\tan(e^x)$

---

5. Let $f(x)$ be the function with $f(2) = 4$ and $f'(x) = \sqrt{x^3 + 1}$. Using the tangent line approximation to the graph of $f(x)$ at x=2, estimate $f(2.2)$.

a) 4.0 b) 4.2 c) 4.4 d) 4.6 e) 4.8

---

6. Which of the following statements must be true?

I. $\frac{d}{dx}\sqrt{e^x + 3} = \frac{e^x}{2\sqrt{e^x + 3}}$ II. $\frac{d}{dx}(\ln\cos x) = \tan x$

III. $\frac{d}{dx}\left(6x^3 - \pi + \sqrt[3]{x^8} - \frac{2}{x^3}\right) = 18x^2 + \frac{8}{3}\sqrt[3]{x^5} + \frac{6}{x^4}$

a) I only b) II only c) III only

d) I and III only e) I, II, and III

---

7. The value of the derivative of $y = \frac{(x^2 - 3)^3}{(5x - 9)^2}$ at $x = 2$ is

a) -4 b) -2 c) 0 d) 2 e) 4

---

8. $\frac{d}{dx}\left[x^7 - 4\sqrt[8]{x^7} + 7^x - \frac{1}{\sqrt[7]{x^4}} + \frac{1}{5x}\right]$

9. $D_x\left[e^{3x^2}\cos 4x\right]$

10. $f(x) = e^{\sin 4x}$; find the exact value of $f''\left(\frac{\pi}{4}\right)$.

11. A fourth differentiable function is defined for all real numbers and satisfies each of the following:

$$g(2) = 5,\ g'(2) = -2, \text{ and } g''(2) = 3$$

and the function f is given by $f(x) = e^{k(x-1)} + g(2x)$, where k is a constant.

a. Find $f(1)$, $f'(1)$, $f''(1)$

b. Show that the fourth derivative of f is $k^4e^{k(x-1)} + 16g^{IV}(2x)$

Chapter 1 Answer Key

1.1 Free Response Answers

1. $f'(x) = 2x + 3$

2. $f'(t) = t^3$

3. $y' = -2/3 x^{-5/3}$

4. $y' = 5e^x$

5. $v'(r) = 4\pi r^2$

6. $g'(x) = 2x - \dfrac{2}{x^3}$

7. $\dfrac{dy}{dx} = \dfrac{3}{2}x^{1/2} + 2x^{-1/2} - \dfrac{3}{2}x^{-3/2}$

8. $\dfrac{du}{dt} = \dfrac{2}{3}t^{-1/3} + 3t^{1/2}$

9. $\dfrac{dz}{dy} = \dfrac{-10A}{y^{11}} + Be^y$

10. $\dfrac{dy}{dx} = e^{x+1}$

11. $f'(x) = 15x^4 - 15x^2.$

12. $7x^6 - \dfrac{7}{2}x^{-1/8} + 7^x\ln 7 + \dfrac{4}{7\sqrt[7]{x^4}}x^{-11/7} - \dfrac{1}{5}x^{-2}$

13. $6x^5 - \dfrac{7}{2}x^{1/6} + 5^x\ln 5 + \dfrac{5}{3}x^{-8/3} - \dfrac{1}{2}x^{-2}$

14. $4x^3 - 18x^{2/7} + 8^x\ln 8 + \dfrac{7}{3}x^{-10/3} + \dfrac{1}{8}x^{-2}$

15. $\dfrac{3}{2}x^{1/2} - \dfrac{1}{2}x^{-1/2}$

16. $\dfrac{7}{2}z^{5/2} - 6z^{1/2}$

17. $\dfrac{9}{2}x^{7/2} - 14x^{5/2} + \dfrac{15}{2}x^{3/2}$

18. $60t^4 + 9t^2 + 56t$

19. $= 10y^{3/2} - 3y^{1/2} - \frac{5}{2}y^{-1/2}$

20. $\frac{3}{4}v^{1/2} - v^{-1/2} - \frac{7}{4}v^{-3/2}$

21. $-\frac{7}{5}w^{-2} + \frac{8}{5}w^{-3} - \frac{3}{5}w^{-4}$

22. $\frac{3}{7}w^{-2} + \frac{8}{7}w^{-3}$

1.1 Multiple Choice Answers

1. C 2. B 3. A 4. E 5. D

1.2 Free Response Answers

1. $-7(x^3 + 4x - \pi)^{-8}(3x^2 + 4)$

2. $f'(x) = \dfrac{2 + 3x^2}{4(1 + 2x + x^3)^{3/4}}$

3. $f'(x) = \dfrac{-3x^{-2} + 6 + 3e^x}{5(x^{-1} + 2x + e^x)^{2/5}}$

4. $f'(x) = 37(3x^2 + 2) \cdot (x^3 + 2x)^{36}$

5. $f'(2) = -4e^3$

6. $f'(x) = \dfrac{-4x}{9\sqrt{4 - \frac{4}{9}x^2}}$

7. $\dfrac{3x-2}{\sqrt{3x^2-4x+9}}$

8. $\dfrac{3x^2-2}{7\left(x^3-2x\right)^{6/7}}$

9. $\dfrac{-x}{\sqrt{9-x^2}}e^{\sqrt{9-x^2}}$

10. $\dfrac{1}{2\sqrt{x}}e^{\sqrt{x}}$

11. $v'(t)=\dfrac{1}{2}\left[\left(\dfrac{E(t)}{3}+3t\right)^{3/7}-4\right]^{-1/2}\left[\dfrac{3}{7}\left(\dfrac{E(t)}{3}+3t\right)^{-4/7}\right]\left(\dfrac{1}{3}E'(t)+3\right)$

12. $v'(t)=\dfrac{1}{3}\left(\left(\dfrac{C(t)}{7}+4t^2\right)^{5/7}-1\right)^{-2/3}\left[\dfrac{5}{7}\left(\dfrac{C(t)}{7}+4t^2\right)^{-2/7}\right]\left(\dfrac{1}{7}C'(t)+8t\right)$

1.2 Multiple Choice Answers

1. D 2. E 3. B 4. E

1.3 Free Response Answers

1. $\dfrac{dy}{dx}=4\cos 4x$

2. $y'=20x^4\sec x^5\tan x^5$

3. $f'(t)=\dfrac{\sec^2 t}{3(1+\tan t)^{2/3}}$

4. $f'(\theta)=-\tan\theta$

5. $\dfrac{dy}{dx}=-3\cos^2 x\sin x$

6. $y=-3x^2\sin\left(a^3+x^3\right)$

7. $f'(x) = -\dfrac{\sin(\ln x)}{x}$

8. $f'(x) = \dfrac{1}{5x(\ln x)^{4/5}}$

9. $f'(x) = \dfrac{\cos x}{\ln 10(2+\sin x)}$

10. $f'(x) = \dfrac{-3}{\ln 2(1-3x)}$

11. $y' = \dfrac{e^x}{\sqrt{1-e^{2x}}}$

12. $\dfrac{3e^{3x}}{\sqrt{1-e^{6x}}}$

13. $y' = \dfrac{1}{2\left(x^{1/2}+x^{3/2}\right)}$

14. $\dfrac{-2e^{2x}}{1+e^{4x}}$

15. $y' = \dfrac{2x}{1+x^4}$

16. 0

17. $6(x+1)e^{x^2+2x}$

18. $-6(x+1)\sin\left(x^2+2x\right)$

19. $\dfrac{x^2}{\left(16+x^3\right)^{2/3}}$

20. $\dfrac{2}{x\sqrt{4x^4-1}}$

21. $g'(x) = \dfrac{2x}{x^2+16}$

22. $f'(x) = 3x\left(x^2+1\right)^{1/2}$.

23. $\tan x$

24. $y' = -2x\sin x^2$

25. $f'(x) = \dfrac{2x}{x^2+3}$

26. $g'(x) = \dfrac{2}{x-2}$

27. $h'(x) = \dfrac{x}{\left(x^2+5\right)^{1/2}}$

28. $F'(x) = \dfrac{2(x-1)}{\left(3x^2-6x+1\right)^{2/3}}$

29. $y' = -1$

30. $\dfrac{-x}{\sqrt{1-x^2}}$

31. $35e^{\tan(7x)}\left(\sec^2 7x\right)$

32. $\dfrac{-x\sin\left(1-x^2\right)}{\cos^{1/2}\left(1-x^2\right)}$

33. $\dfrac{6x\ln^2\left(x^2+1\right)}{x^2+1}$

34. $3x^2\tan x^3$

35. $y' = 6\tan(3\theta)\cdot\sec^2(3\theta)$

36. $y' = -7\cos\theta\cot^6(\sin\theta)\csc^2(\sin\theta)\cdot\cos\theta$

37. 5

38. 2

1.3 Multiple Choice Answers

1.	E	2.	E	3.	C	4.	B	5.	D	6.	D
7.	B	8.	B	9.	C						

1.4 Free Response Answers

1. $g(-2.9) \approx 5.5$

2. $f(-0.9) \approx \dfrac{199}{6}$

3. $\sqrt{4.1} = f(1.9) \approx 2.25$

4. $f(3.9) \approx \dfrac{17}{6}$

5. $y-1=1(x-0)$

6. $f(2.1) \approx 0.4$

7. $y-2=2(x-0)$

8. 0.002

9. 4.8

10. $g(1.1) \approx 7.4$

11. $y-1=1(x-0)$

12. $y-1=3\sqrt{3}\left(x-\pi/3\right)$

13. $y-2=\frac{2}{\pi}\left(x-\frac{\pi}{2}\right)$

14. $y-1+2\sqrt{2}=\left(2\sqrt{2}-4\right)\left(x-\frac{\pi}{8}\right)$

15. (4, 0)

16. $\left(\frac{1}{2},\ \frac{1}{8}\right)$

17. $y-1=12(x+1)$ and $y+64=12(x-4)$

18. $y-6.8=-4(x+1)$ and $y+6.8=-4(x-1)$

19. $\left(\frac{\pi}{2}\pm 2\pi n,\ 3\right),\ \left(-\frac{\pi}{2}\pm 2\pi n,\ 1\right)$

20. $y-3=3\left(x-\frac{1}{2}\right)$

21a. $t=2$ and $t=5$ b. left c. 49 units right of the origin
1d. –6 e. Speeding up

22a. $t=2$ b. right c. 14 units right of the origin
d. 6 e. Speeding up

23a. $t=-4, t=\frac{4}{3}$, and $t=3$ b. neither; it is stopped
c. 141 units above of the origin d. 420 e. Neither

24a. $t=-3, t=-1, t=2$, and $t=3$ b. neither; it is stopped
c. 1431 units above of the origin d. 1440 e. Neither

25. $x(2.5)=-2.25$

26. $x(-3)=112; x(7)=-388$

27. $x(3.5)=42.5; v(3.5)=-13.5$

28. $x(-2)=-1984; v(-2)=1200,\ x\left(\frac{20}{9}\right)=222.398;\quad v\left(\frac{20}{9}\right)=-154.864$

29a. $x(1) = -1,\ a(1) = 10,\ x\left(-\frac{2}{3}\right) = 3.630,\ a\left(-\frac{2}{3}\right) = -10$

29b. $v\left(\frac{1}{6}\right) = 1.315;\ x\left(\frac{1}{6}\right) = -4.167$

30a. $x(\pm 1) = -9,\ a(\pm 1) = 8,\ x(0) = -8,\ a(0) = -4$

30b. $y\left(\frac{1}{\sqrt{3}}\right) = -\frac{77}{9}; v\left(\frac{1}{\sqrt{3}}\right) = -\frac{8}{3\sqrt{3}}; y\left(-\frac{1}{\sqrt{3}}\right) = -\frac{77}{9}; v\left(-\frac{1}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}$

1.4 Multiple Choice Answers

1.	A	2.	B	3.	C	4.	C	5.	D	6.	D
7.	A	8.	A	9.	E	10.	B	11.	E	12.	B
13.	C	14.	C	15.	E	16.	A				

1.5 Free Response Answers

1. $t^2(3\cos t - t\sin t)$

2. $e^{-x^2}\left[1 - 2x^2\right]$

3. $e^{-x}(1 - x)$

4. $e^{2x}(2x + 1)$

5. $-e^{-5x}\left[5\cos 3x + 3\sin 3x\right]$

6. $\dfrac{1 + 2\ln x}{2(\ln x)^{1/2}}$

7. $x^2 \sec x(x \tan x + 3)$

8. $x \csc x(2 - x \cot x)$

9. $\cos x(x^2 + 2)$

10. $\tan x(2\sec^2 x + \tan x)$

11. $\sin 4x$

12. $x(x^2 \sec x \tan x + 3x \sec x + x \sec^2 x + 2\tan x)$

13. $3\sqrt{2}$

14. $\dfrac{\pi + 4}{2\sqrt{2}}$

15. $e^x(x^2 - 10)$

16. $2e^{2x}[4 - x^2 - x]$

17. $-\dfrac{1}{2}e^{-\frac{1}{2}x}(x^2 - 4x - 1)$

18. $e^x(-x^3 - 3x^2 + x - 1)$

19. $-2xe^{-4x}(2x - 1)$

20. $-2e^{-3x}(x^2 + 5x + 3)$

21. $e^x\left(\dfrac{13 - 2x}{2(7 - x)^{1/2}}\right)$

22. $\dfrac{4 - 2x^2}{(4 - x^2)^{1/2}}$

23. $\dfrac{4-2x^2}{(4-x^2)^{1/2}}$

24. $\dfrac{9+2x^2}{(9-x^2)^{1/2}}$

25. $\dfrac{18x-3x^3}{(9-x^2)^{1/2}}$

26. $\dfrac{-x^3+2x}{(1-x^2)^{1/2}}$

27. $35x(4x^5-3)^6(7x^2+1)^4[36x^5+4x^3-6]$

28. $8(2x-5)^3\cdot(8x^2-5)^{-4}\cdot[-4x^2+30x-5]$

29. $30x(3x^2-4)^2(6x^2+7)[6x^2+1]$

30. $4(1+4x)^4(3+x-x^2)^7[17+9x-21x^2]$

31. $e^{x\cos x}[\cos x - x\sin x]$

32. $\sin^x x[x\cot x+\ln\sin x]$

33. $y-\dfrac{1}{e}=\dfrac{1}{e}(x-1)$

34. $y'(1)=7/2$

35. $y+6=\frac{13}{2}(x-3)$

36. $y+6=\frac{13}{2}(x-3)$

37. Tangent line: $y-e=1(x-e)$ Normal line: $y-e=-1(x-e)$

38. $y=3$

39. Tangent line: $y-2\sqrt{2}\pi=\frac{\sqrt{2}(-\pi+4)}{8}\left(x-4/\pi\right)$

Normal line: $y-2\sqrt{2}\pi=\frac{4\sqrt{2}}{\pi-4}\left(x-4/\pi\right)$

40. $1+2x\tan^{-1}x$

41. $\cos^{-1}x$

42. $e^{x}-\frac{x^2}{1+x^2}-2x\arctan x$

43. $\frac{-2x^2}{\sqrt{1-x^2}}$

44. $-\tan^{-1}\left(\frac{x}{2}\right)$

45. $\dfrac{3-x}{\left(9-x^2\right)^{1/2}}$

1.5 Multiple Choice Answers

1.	D	2.	B	3.	B	4.	D	5.	A	6.	B
7.	B										

1.6 Free Response Answers

1. $\dfrac{-2x}{\left(x^2-4\right)^2}$

2. $\dfrac{-4x^2-48x-36}{\left(x^2-9\right)^2}$

3. $\dfrac{-3x^2+10x-14}{\left(x^2-x-3\right)^2}$

4. $\dfrac{2x^3+4x^2-8x-4}{(x+2)^2}$

5. $\dfrac{-6x+3}{\left(x^2-x+1\right)}$

6. $\dfrac{3x^2-8x-5}{(x-4)^2}$

7. $\dfrac{2x^4+38}{3x^3}$

8. $-(x-5)^{-2}$

9. $\tan^2 x - 5\cot x$

10. $\dfrac{-1}{1-\cos x}$

11. $\dfrac{\sec x - 3\sec^2 x + \sin x \tan x}{(\cos x - 3)^2}$

12. $2x\sec x + x^2 \tan x \sec x$

13. $y' = \dfrac{1+\tan x}{\sec x}$

14. $\dfrac{1}{x^2\sqrt{x^2-1}} - \dfrac{\sec^{-1} x}{x^2}$

15. $\dfrac{3}{2}$

16. $\dfrac{x\cos x - 2\sin x}{x^3}$

17. $\dfrac{1}{\left(r^2+1\right)^{3/2}}$

18. 0

19. Tangent: $y - \dfrac{2}{17} = -\dfrac{38}{289}(x+1)$; Normal: $y - \dfrac{2}{17} = \dfrac{289}{38}(x+1)$

20. Tangent: $y - \frac{2}{3} = \frac{2}{9}(x-1)$; Normal: $y - \frac{2}{3} = -\frac{9}{2}(x-1)$

21. Tangent: $y = -\frac{3}{2}$; Normal: $x = 1$

22. Tangent: $y + \frac{1}{5} = \frac{3}{25}(x-2)$; Normal: $y + \frac{1}{5} = -\frac{25}{3}(x-2)$

1.6 Multiple Choice Answers

1. E 2. A 3. D 4. A 5. D

1.7 Free Response Answers

1. $f'(x) = 5x^4 + 12x - 7$ *and* $f''(x) = 20x^3 + 12$

2. $h'(x) = \frac{x}{\sqrt{x^2+1}}$ *and* $h''(x) = \frac{1+x^2}{\left(x^2+1\right)^{3/2}}$

3. $\frac{dy}{dx} = \frac{2x^2}{\left(x^3+1\right)^{1/3}}$ *and* $\frac{d^2y}{dx^2} = \frac{2x\left(x^3+2\right)}{\left(x^3+1\right)^{4/3}}$

4. $H'(t) = 3\sec^2 3t$ *and* $H''(t) = 18\sec^2 3t\tan 3t$

5. $g'(t) = t^2e^{5t}(5t+3)$ *and* $g''(t) = te^{5t}\left(25t^2+30t+6\right)$

6. $y'' = 6e^{3x^2}\left(6x^2+1\right)$.

7. $y'' = 3\sin x\left(2\cos^2 x - \sin^2 x\right)$.

8. $f'(t) = t\cdot -\sin t + \cos t$; $f''(t) = -t\cos t - 2\sin t$

9. $60x^2 + 54x - 8$

10. $168x^5 - 60x^3 + 18x$

11. $y'' = -2\left(\sin x^2 + 2x^2 \cos x^2\right)$

12. $y'' = 2\sec^2 x\left(2\tan^2 x + \sec^2 x\right)$

13. $\frac{d^2y}{dx^2} = 9\sec 3x\left(\sec^2 3x + \tan^2 3x\right)$

14. $\frac{d^2y}{dx^2} = 4e^{2x}(x+1)$

15. $f''(x) = \frac{-2\left(x^2-3\right)}{\left(x^2+3\right)^2}$

16. $g''(x) = \frac{-2}{(x-2)^2}$

17. $h''(x) = \frac{5}{\left(x^2+5\right)^{3/2}}$

18. $F''(x) = \frac{2}{\left(3x^2-2x+1\right)^{3/2}}$

19. $\frac{d^2y}{dx^2} = \frac{14\left(3x^2+10\right)}{\left(x^2-10\right)^3}$

20. $\frac{d^2y}{dx^2} = \frac{6\left(7x^2-7x+2\right)}{\left(x^2-x+1\right)^3}$

21. $6x + 2$

22. $36x^2 - 120x^2 + 84$

23. $y'' = \frac{-8x\left(x^2-12\right)}{\left(x^2+4\right)^3}$

24. $y'' = \dfrac{-18x^2 + 24}{(x^2 - 4)^3}$

25. $y'' = \dfrac{(2x)(x^2 - 24)}{(8 - x^2)^{3/2}}$

26. $\dfrac{d^2y}{dx^2} = -\cos x$

27. $e^{-x}(x - 2)$

28. $e^{-x^2}(4x^2 - 2)$

29. $y'' = \dfrac{2x^3 + 54x}{(x^2 - 9)^3}$

30. $\dfrac{2}{9}x^{-4/3}$

1.7 Multiple Choice Answers

1.	E	2.	C	3.	B	4.	E	5.	B

1.8 Free Response Homework

1a.	21	1b.	-36	2.	-8	3.	-12	4.	-6	5.	-16
6.	30	7.	-56	8.	-18	9.	-8	10.	$-\frac{5}{16}$	11.	$\frac{5}{4}$
12.	$\frac{9}{2}$	13.	$\frac{49}{16}$	14.	0	15.	$-\frac{4}{9}$	16.	$-\frac{2}{3}$	17.	DNE

18. $\frac{4}{3}$ 19. dne 20. $\frac{8}{3}$ 21. Dne 22. $\frac{1}{3}$ 23. dne

24. dne 25. dne

26a. $y-0=-\frac{2}{3}(x-4)$ 26b. dne 26c. -2 26d. $-\frac{4}{3}$

27a. $y-3=6(x-4)$ 27b. 48 27c. 9 27d. -6

28a. $y-3=6(x-4)$ 28b. 0 28c. -1928 28d. 24

29a. $y-2=3\left(x-\frac{\pi}{2}\right)$ 29b. 2 29c. 14/3 29d. 3/4

30a. $y-1=1(x-4)$ 30b. 0 30c. 15 30d. 2.492

31. See AP Central

<u>1.8 Multiple Choice Homework</u>

1. A 2. A 3. D 4. E 5. A 6. E

7. B 8. D

<u>Derivative Review Practice Test Solutions</u>

1. D 2. A 3. D 4. B 5. D 6. D

7. D

8. $=7x^6-\frac{7}{2}x^{-1/8}+7^x\ln 7+\frac{4}{7}x^{-11/7}-\frac{1}{5}x^{-2}$

9. $D_x\left[e^{3x^2}\cos 4x\right]=e^{3x^2}(-\sin 4x)(4)+\cos 4xe^{3x^2}(6x)=2e^{3x^2}(3\cos 4x-2\sin 4x)$

10. 16

11a. k^2+12

11b. Show that the fourth derivative of f is $k^4e^{k(x-1)}+16g^{IV}(2x)$

$f''(x)=k^2e^{k(x-1)}+4g''(2x)\rightarrow f'''(x)=k^2e^{k(x-1)}(k)+4g'''(2x)(2)=k^3e^{k(x-1)}+8g'''(2x)$

$f'''(x)=k^3e^{k(x-1)}+8g'''(2x)\rightarrow f^{IV}(x)=k^3x^{k(x-1)}(k)+8g^{IV}(2x)\cdot(2)=k^4x^{k(x-1)}+16g^{IV}(2x)$